# CONTINUOUS-, DISCRETE- AND SAMPLED-DATA- $\mathcal{H}_\infty$ CONTROL - A UNIFIED FRAMEWORK

A-K Christiansson, B. Lennartson, H. Toivonen
University of Trollhättan/Uddevalla, Box 957, SE-461 29 Trollhättan, Sweden
Control Engineering Lab, Chalmers University of Technology, SE-412 96 Göteborg, Sweden
Department of Chemical Engineering, Åbo Akademi University, FIN-20500 Åbo, Finland
`akc@htu.se, bl@s2.chalmers.se, htoivone@abo.fi`

**Output feedback, $\mathcal{H}_\infty$-control, Periodic systems, Sampled-data, Lifting, Hold-states, Riccati equation**

## Abstract

We present a unified and general framework for $\mathcal{H}_\infty$-control in both continuous time, discrete time and combinations of these. The general result is a hybrid continuous-/discrete-time $\mathcal{H}_\infty$-controller. Using a compact hybrid notation, the work shows a close relationship between the continuous- and discrete-time solutions. In fact, the pure continuous and discrete time equations may be obtained as two similar interpretations of the general result. There are no assumptions made on certain system matrices being zero or normalised, e.g. $D_{11} = 0$. The method is Riccati equation (RE) based, and it is shown how the continuous REs can be "lifted" into discrete ones reflecting the system behaviour during the period.

Typical applications are control of continuous-time or discrete-time periodic systems, as well as multirate and sampled-data control, including mixed continuous and sampled-data measurements.

## 1 Introduction

$\mathcal{H}_\infty$-control for pure continuous-time and discrete-time problems are penetrated by many authors, however most of them consider simplified system assumptions, e.g. $D_{11} = 0$, e.g. [1]. Furthermore, continuous- and discrete-time cases are treated one at a time. When controlling a continuous-time plant with a digital controller, the system is hybrid in nature. The most well known example of such control is sampled-data control, which means that a discrete-time controller is designed for a continuous-time plant such that a continuous performance measure is fulfilled. The intersample behaviour during the sampling periods is thus taken into account when the controller is designed; for $\mathcal{H}_\infty$-applications see e.g. [2, 3, 4].

An obvious extension of this control problem is multirate sampled-data control where control signals and/or process measurements are sampled at different rates. If the multirate sampled-data scheme is repeated after a period of time, such a control system is an example of a periodic system.

For Linear Quadratic control, or rather the dual Kalman filter problem, a solution to this hybrid control problem was presented already in [5]. More recently the $\mathcal{L}_2$-induced norm for systems with jumps was considered in [6]. However, no control design was included and no periodic or multirate behaviour was assumed. The $\mathcal{H}_\infty$ control for systems with state jumps has also been considered in [7, 8, 9], mainly focused on single-rate sampled-data control and with a significant discrepancy between the pure continuous-time and the quite complex discrete-time part of the solution. In [10] systems with mixed continuous- and discrete measurements have been treated.

The contribution of this paper is a generalised and unified framework for both continuous-time, discrete-time and sampled-data $\mathcal{H}_\infty$-controller design, including a general criterion $D_{11} \neq 0$. It is worth noting that this is often the case when sensitivity is considered in the performance criterion. Most results in the literature handle $D_{11} \neq 0$ by some system transformation, e.g. [1], but this is not needed here. The solution is based on continuous REs with jumps, expressed by discrete REs. In the infinite-horizon case a periodic behaviour is assumed, and it is shown how the related continuous RE with jumps can be replaced by an equivalent discrete "lifted" periodic RE. The suggested approach is a flexible and efficient alternative to standard lifting methods, especially when the controller includes continuous behaviour and a number of discrete updates during one period, as in the case of multirate sampling. The results presented in [11, 12] are extended with the sampled-data application. Simpler results are achieved, especially in infinite-time horizon, compared to standard $\mathcal{H}_\infty$-theory.

The paper starts with notations, definitions and problem formulation in Sections 2-3 and Sections 4-6 give the general hybrid $\mathcal{H}_\infty$-design procedure. These are applied on the special sampled-data case in Section 7. Due to limited space we refer to [13] for proofs of the theorems. They follow to a great extent ideas in [14].

## 2 Hybrid notations

In this section, a compact hybrid notation is introduced that reflects both continuous-time and discrete-time properties. Consider times $t \in [0, T_f]$. The sampling instants are $t_k \in [0, T_f]$ and the time-limits $t_k^+$ and $t_k^-$ are defined as $\lim_{\epsilon \to 0}(t_k \pm \epsilon)$ for $\epsilon \in \mathbb{R}^+$.

Variables, that are solutions to differential- and/or difference equations, appear as piece-wise continuous variables with jumps. To describe differential- and difference-equations for such variables similarly, a forwards/backwards notation is introduced according to, "forwards":

$$x^+ = \left\{ \begin{array}{ll} \lim\limits_{\epsilon \to 0} \frac{x(t+\epsilon)-x(t)}{\epsilon} = \dot{x}(t), & t \neq t_k \\ \lim\limits_{\epsilon \to 0} x(t_k + \epsilon) = x(t_k^+), & k = 1, 2, \ldots \end{array} \right. \tag{1a}$$

and "backwards":

$$x^- = \left\{ \begin{array}{ll} \lim\limits_{\epsilon \to 0} \frac{x(t-\epsilon)-x(t)}{\epsilon} = -\dot{x}(t), & t \neq t_k \\ \lim\limits_{\epsilon \to 0} x(t_k - \epsilon) = x(t_k^-), & k = 1, 2, \ldots \end{array} \right. \tag{1b}$$

where $\epsilon \in \mathbb{R}^+$. The length of the sampling interval from time $t_k$ to $t_{k+1}$, is denoted $h_k$.

A hybrid signal $z$ and a hybrid matrix $A$ are defined as

$$z(t) = \left\{ \begin{array}{ll} z_c(t), & t \neq t_k \\ z_d(t_k), & k = 1, 2, .. \end{array} \right. \quad A(t) = \left\{ \begin{array}{ll} A_c(t), & t \neq t_k \\ A_d(t_k), & k = 1, 2, .. \end{array} \right.$$

Thus, when no subscripts $_c$ or $_d$ are used, signals and systems are in their hybrid form. When continuous time is considered in the following, times $t \neq t_k$, whereas when discrete-time is considered, time points are the discrete time updates $t = t_k, k = 1, 2, \ldots$. The size of the hybrid signal is composed of both discrete-time and continuous-time contributions as:

$$\|z\|^2_{[0,T_f]} = \int_0^{T_f} z_c'(t) z_c(t) dt + \sum_{t_k \in [0, T_f]} z_d'(t_k) z_d(t_k) \tag{2}$$

Note that $z_c$ in (2) is not formally defined at times $t_k$. However, the time limits exist, and the integral can be considered as a sum of integrals. A hybrid matrix $A$ is positive definite, if both $A_c$ and $A_d$ are positive definite.

**Hybrid system model:** Consider the continuous- and discrete-time system parts:

$$\mathcal{G}_c : \left\{ \begin{array}{l} \dot{x}(t) = A_c(t)x(t) + B_c(t)\mu_c(t) \\ \eta_c(t) = C_c(t)x(t) + D_c(t)\mu_c(t) \end{array} \right. , \qquad t \neq t_k$$

$$\mathcal{G}_d : \left\{ \begin{array}{l} x(t_k^+) = A_d(t_k)x(t_k^-) + B_d(t_k)\mu_d(t_k) \\ \eta_d(t_k) = C_d(t_k)x(t_k^-) + D_d(t_k)\mu_d(t_k) \end{array} \right. , k = 1, 2, \ldots$$

These can be described uniformed as the hybrid system $\mathcal{G}$ in compact form as

$$\mathcal{G} : \left[ \begin{array}{c} x^+ \\ \eta \end{array} \right] = \left[ \begin{array}{cc} A & B \\ C & D \end{array} \right] \left[ \begin{array}{c} x \\ \mu \end{array} \right] \tag{4}$$

With the augmented hybrid input/output signals $\mu = [w' \;\; u']'$ and $\eta = [z' \;\; y']'$ the matrices are partitioned according to:

$$\mathcal{G} : \left[ \begin{array}{c} x^+ \\ z \\ y \end{array} \right] = \left[ \begin{array}{ccc} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{array} \right] \left[ \begin{array}{c} x \\ w \\ u \end{array} \right] \tag{5}$$

Note that (4)-(5) can be interpreted as a pure continuous or pure discrete model, if $x^+$ is interpreted as $\dot{x}$ or $x_{k+1}$ respectively. The state transition from times $s$ to $t$, $\Pi_A(t, s)$, for the hybrid matrix $A$ is found from:

$$\left\{ \begin{array}{lll} \dot{\Pi}_A(t, s) & = A_c(t)\Pi_A(t, s), & t \neq t_k \\ \Pi_A(t_k^+, s) & = A_d(t_k)\Pi_A(t_k^-, s), & k = 1, 2, \ldots \\ \Pi_A(s, s) & = I & \end{array} \right.$$

**Hybrid Lyapunov equation and stability:** Riccati equations often occur in $\mathcal{H}_\infty$derivations, and a special case of these are Lyapunov equations. These are in continuous time, $t \neq t_k$, *differential* equations:

$$-\dot{S}(t) = A_c'(t)S(t) + S(t)A_c(t) + C_c'(t)C_c(t) \tag{6a}$$

$$\dot{P}(t) = A_c(t)P(t) + P(t)A_c'(t) + B_c(t)B_c'(t) \tag{6b}$$

and in discrete time, $t = t_k$, *difference* equations:

$$S(t_k^-) = A_d'(t_k)S(t_k^+)A_d(t_k) + C_d'(t_k)C_d(t_k) \tag{6c}$$

$$P(t_k^+) = A_d(t_k)P(t_k^-)A_d(t_k) + B_d(t_k)B_d'(t_k) \tag{6d}$$

Here $S$ and $P$ are symmetric quadratic piece-wise continuous matrices with jumps. Introduce the formal notation "$\diamond$" according to

$$A \diamond P = \left\{ \begin{array}{ll} A_c P + P A_c', & t \neq t_k \\ A_d P A_d', & t = t_k, k = 1, 2, \ldots \end{array} \right. \tag{7}$$

This implies that the Lyapunov equations can be written in the following hybrid way, to indicate the close formal structure between continuous- and discrete time:

$$S^- = A' \diamond S + C'C, \;\; P^+ = A \diamond P + BB' \tag{8}$$

In the following, when time instants in the matrices in discrete-time are not shown, they are "opposite" to the ones explicitly shown, e.g. in (8) $S$ means $S(t_k^+)$ whereas $P$ means $P(t_k^-)$.

A hybrid system, with system matrix $A$, is *exponentially stable* if there exist two positive real numbers $c_1, c_2$ such that $\|\Pi_A(t, s)\| \leq c_1 e^{-c_2(t-s)} \;\; \forall t \geq s \geq 0$. This is also denoted as the system $A$ being exponentially stable. Next follows definitions of stabilisability and detectability for a hybrid system, cf. [15]:

$\mathcal{G}$ is *stabilisable* if there exists a bounded hybrid matrix $L$ such that $A - BL$ is exponentially stable. This is also denoted as $(A, B)$ being stabilisable. Similarly, $\mathcal{G}$ is *detectable* if there exists a bounded hybrid matrix $K$ such that $A - KC$ is exponentially stable. This is also denoted as $(C, A)$ being detectable.

## 3 Control problem

Our control problem may be expressed as: *Find a hybrid controller such that the hybrid performance bound*

$$\|\mathcal{G}_{zw}\|_\infty = \sup_{\|w\|_{[0,T_f]} \neq 0} \frac{\|z\|_{[0,T_f]}}{\|w\|_{[0,T_f]}} < \gamma \tag{9}$$

holds for a specified positive constant $\gamma$. This is preferably reformulated when also a penalty $S(T_f)$ on the final states $x(T_f)$ is included, as

$$\begin{aligned} &J(z, w, T_f) = &(10)\\ &= \|z\|^2_{[0,T_f]} - \gamma^2 \|w\|^2_{[0,T_f]} + x'(T_f)S(T_f)x(T_f) < 0 \end{aligned}$$

To obtain $J < 0$, the well known "$\gamma$-iteration" has to be performed, and the smallest $\gamma$-value is finally chosen. The problem will be solved in three major steps, first the static feedback full information case is treated, followed by the adjoint filter and output-feedback cases. Finally, the sampled-data case is a simplified special case of these.

## 4 The static feedback solution

In this section we give a fairly complete presentation of the static feedback solution, and rely on these details when we later discuss the filter and the output feedback cases. In the static feedback full information case, the controller has access to all plant states $x$ and disturbances $w$, but the measured outputs $y$ are of no interest. Therefore the hybrid plant to consider is (part of (5))

$$\begin{bmatrix} x^+ \\ z \end{bmatrix} = \begin{bmatrix} A & B \\ C_z & D_z \end{bmatrix} \begin{bmatrix} x \\ \mu \end{bmatrix} \tag{11}$$

where $B = \begin{bmatrix} B_w & B_u \end{bmatrix}$ and $D_z = \begin{bmatrix} D_{zw} & D_{zu} \end{bmatrix}$. Introduce the hybrid matrices

$$\begin{aligned} Q_{\mu_c} &= D'_{z_c} D_{z_c} - \gamma^2 Q_\gamma = \begin{bmatrix} Q_{w_c} & Q_{wu_c} \\ Q_{uw_c} & Q_{u_c} \end{bmatrix} &(12a)\\ Q_{\mu x_c} &= B'_c S + D'_{z_c} C_{z_c} = \begin{bmatrix} Q_{wx_c} \\ Q_{ux_c} \end{bmatrix}, \quad t \neq t_k &(12b)\\ Q_{\mu_d} &= B'_d S(t_k^+) B_d + D'_{z_d} D_{z_d} - \gamma^2 Q_\gamma &(12c)\\ Q_{\mu x_d} &= B'_d S(t_k^+) A_d + D'_{z_d} C_{z_d}, \qquad t = t_k &(12d)\\ \hat{Q}_w &= Q_{wu} Q_u^{-1} Q_{uw} - Q_w &(12e) \end{aligned}$$

with $Q_\gamma = \operatorname{diag}(I_{n_w}, 0_{n_u})$. Matrix partitions in discrete time $Q_{w_d}, Q_{wu_d}, Q_{uw_d}, Q_{u_d}, Q_{ux_d}, Q_{\mu x_d}$ are defined similar to continuous time. Now we are ready to present a theorem, which shows that the similarity between the continuous- and discrete-time parts of the solution is striking compared to standard results in $\mathcal{H}_\infty$ control, see e.g. [1]. A related formulation of the discrete part in our theorems is given in [1]. For a corresponding time-invariant continuous-time solution, a related but not such compact description can be found in [16, 17].

**Theorem 1** *Consider the hybrid plant in (11) on $t \in [0, T_f]$ with $\|w\|_{[0,T_f]} \neq 0$. Then there exists a hybrid full information controller $u = \mathcal{K}(x, w)$ which achieves the performance bound $J < 0$, if and only if there exists a bounded piece-wise continuous matrix function $S \geq 0$, which satisfies the hybrid RE*

$$S^- = A' \diamond S + C'_z C_z - L' Q_\mu L, \quad S(T_f) = 0 \tag{13}$$

*where the hybrid static feedback gain $L = Q_\mu^{-1} Q_{\mu x}$ and $Q_\mu, Q_{\mu x}$ are defined in (12). Necessary conditions for the existence of bounded solutions $S$ are $\hat{Q}_w > 0, Q_u > 0$. The hybrid optimal control signal $u^* = -L_u x - L_{uw} w$ and the worst disturbance $w^* = L_w x$, where $L_u = Q_u^{-1} Q_{ux}, L_{uw} = Q_u^{-1} Q_{uw}, L_w = \hat{Q}_w^{-1}(Q_{wx} - Q_{wu} Q_u^{-1} Q_{ux})$.*

With worst disturbance $w^*$, the optimal control is $u^* = -\bar{L}x$, where $\bar{L} = L_u + L_{uw} L_w$. In infinite-time horizon this theorem holds with some additional demands on the plant. Let the solution to the RE in this case, when it exists, be denoted $S_\infty$. Define $A_u = A - B_u (D'_{zu} D_{zu})^{-1} D'_{zu} C_z$, $C_u = \hat{D}_{zu} C_z$

**Theorem 2** *Consider the situation in Theorem 1. Assume further that the system $(A, B_u)$ is stabilisable, and the system $(C_u, A_u)$ is detectable. Then there exists a hybrid static feedback full information controller $\mathcal{K}$ that exponentially internally stabilises the system (11) and achieves the performance bound $J < 0$ when $T_f \to \infty$, if and only if there exists a bounded symmetric piece-wise continuous matrix function $S_\infty \geq 0$, which satisfies (13), such that the hybrid system matrix $A - BL$ is exponentially stable. One controller that achieves the bound is then given as in Theorem 1.*

Note that the hybrid RE (13) and the feedback gain $L$ indicate a formal close relationship between continuous time and discrete time, but the matrices in (12) differ in this respect. The matrices $Q_\mu$ are indefinite due to the min-max nature of the problem; the control signal $u$ is to be chosen to minimise the cost $J$, while the disturbance $w$ tries to maximise it.

**"Lifting" the continuous-time solution:** In the sampled-data case, the continuous-time RE shall be solved between the sampling instants, while the discrete-time RE shall be solved at the sampling instants, all iterated into convergence. However, the continuous-time part of (13) may be replaced by a "lifted", discrete RE. To achieve this, study a reformulation of the continuous-time part of the RE as

$$-\dot{S} = A'_{\mu_c} S + S A_{\mu_c} - S B_{\mu_c} B'_{\mu_c} S + C'_{\mu_c} C_{\mu_c} \tag{14}$$

representing the continuous behaviour during the period. Here

$$\begin{aligned} A_{\mu_c} &= A_c - B_c Q_{\mu_c}^{-1} D'_{z_c} C_{z_c} &(15a)\\ C'_{\mu_c} C_{\mu_c} &= C'_{z_c}(I - D_{z_c} Q_{\mu_c}^{-1} D'_{z_c}) C_{z_c} &(15b)\\ B_{\mu_c} B'_{\mu_c} &= B_c Q_{\mu_c}^{-1} B'_c &(15c) \end{aligned}$$

The Hamiltonian matrix associated with this problem is

$$\mathcal{H}_\mu = \begin{bmatrix} A_{\mu_c} & -B_{\mu_c}B'_{\mu_c} \\ -C'_{\mu_c}C_{\mu_c} & -A'_{\mu_c} \end{bmatrix} \tag{16}$$

and its transition matrix from $t^-_{k+1}$ to $t$ is $\Pi(t, t^-_{k+1})$ is defined by

$$\dot{\Pi}(t, t^-_{k+1}) = \mathcal{H}_\mu(t)\Pi(t, t^-_{k+1}); \quad \Pi(t^-_{k+1}, t^-_{k+1}) = I \tag{17}$$

$\Pi$ is partitioned into sub-matrices of size $n_x \times n_x$ according to $\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix}$. Now a “lifting” theorem is presented.

**Theorem 3** *The solution to the continuous-time RE (14) based on $S(t^-_{k+1})$ can be obtained by the following discrete-time RE*

$$S(t^+_k) = \tilde{A}'_\mu S(t^-_{k+1})\tilde{\Psi}_\mu \tilde{A}_\mu + \tilde{C}'_z\tilde{C}_z \tag{18}$$

*where time arguments in the discretised matrices are as in the definitions below:*

$$\begin{aligned} \tilde{A}_\mu(t_k, h_k) &= \Pi^{-1}_{11}(t^+_k, t^-_{k+1}) \qquad (19) \\ \tilde{B}_{\tilde{\mu}}(t_k, h_k)\tilde{E}^{-1}_\mu \tilde{B}'_{\tilde{\mu}}(t_k, h_k) &= \Pi^{-1}_{11}(t^+_k, t^-_{k+1})\Pi_{12}(t^+_k, t^-_{k+1}) \\ \tilde{C}'_z(t_k, h_k)\tilde{C}_z(t_k, h_k) &= \Pi_{21}(t^+_k, t^-_{k+1})\Pi^{-1}_{11}(t^+_k, t^-_{k+1}) \\ \tilde{E}_\mu &= \mathrm{diag}(-\gamma^2 I_{n_w}, I_{n_u}) \\ \tilde{\Psi}_\mu(t_k, h_k) &= (I + \tilde{B}_{\tilde{\mu}}\tilde{E}^{-1}_\mu \tilde{B}'_{\tilde{\mu}} S(t^-_{k+1}))^{-1} \end{aligned}$$

*and $\Pi(t^+_k, t^-_{k+1})$ is the transition matrix defined in (17). Furthermore, the discretised system model*

$$\begin{bmatrix} x(t^-_{k+1}) \\ \tilde{z}(t_k, h_k) \end{bmatrix} = \begin{bmatrix} \tilde{A}_\mu & \tilde{B}_{\tilde{\mu}} \\ \tilde{C}_{\tilde{z}} & \tilde{D}_{\tilde{z}\tilde{\mu}} \end{bmatrix} \begin{bmatrix} x(t^+_k) \\ \tilde{\mu}(t_k, h_k) \end{bmatrix} \tag{20}$$

$$\textit{where } \tilde{D}'_{\tilde{z}\tilde{\mu}}\tilde{D}_{\tilde{z}\tilde{\mu}} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, \ \tilde{D}'_{\tilde{z}\tilde{\mu}}\tilde{C}_{\tilde{z}} = 0$$

*and a corresponding $\mathcal{H}_\infty$-criterion, cf. (10), generates the discrete-time RE (18) between times $t^+_k$ and $t^-_{k+1}$.*

When the control is optimal, $u^*$, and the disturbance is the worst one, $w^*$, then the closed loop system matrix is $A - BL$, in continuous time also expressed as $A_{\mu_c} - B_{\mu_c}B'_{\mu_c}S$. It can be shown, see e.g. [18], that the state transition from $t^+_k$ to $t^-_{k+1}$ equals

$$\begin{aligned} &\Pi_{A-BL}(t^-_{k+1}, t^+_k) = \\ &[\Pi_{11}(t^+_k, t^-_{k+1}) + \Pi_{12}(t^+_k, t^-_{k+1})S(t^-_{k+1})]^{-1} \end{aligned} \tag{21}$$

In discrete-time the state transition from times $t^-_k$ to $t^+_k$ is $A_d - B_dL_d$. Thus the state transition between times $t^-_k$ to $t^-_{k+1}$ is achieved as

$$\Pi_{A-BL}(t^-_{k+1}, t^-_k) = \Pi_{A-BL}(t^-_{k+1}, t^+_k)(A_d - B_dL_d)$$

## 5 The filter solution

The problem considered in this section is the estimation of the performance $z$ as a function of the measured output $y$. We do not treat any control signals $u$ in the first step, since these will not contribute to the estimation error and therefore do not affect the optimality. Thus consider

$$\begin{bmatrix} x^+ \\ z - \hat{z} \\ y \end{bmatrix} = \begin{bmatrix} A & B_w & 0 \\ C_z & D_{zw} & -I \\ C_y & D_{yw} & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ \hat{z} \end{bmatrix} \tag{22}$$

Introduce $C = [C'_z \ C'_y]'$ and $D_w = [D'_{zw} \ D'_{yw}]'$. The problem is to search a *filter* such that the performance bound, cf. (9)

$$\sup_{\|w\|_{[0,T_f]} \neq 0} \frac{\|z - \hat{z}\|_{[0,T_f]}}{\|w\|_{[0,T_f]}} < \gamma \tag{23}$$

holds for a specified positive constant $\gamma$. This filter problem will be treated as an adjoint static feedback disturbance feedforward controller problem as in [12, 1]. Only the resulting expressions are listed here. Introduce the hybrid matrices, cf. (12)

$$\begin{aligned} R_{\eta_c} &= D_{w_c}D'_{w_c} - \gamma^2 R_\gamma = \begin{bmatrix} R_{z_c} & R_{zy_c} \\ R_{yz_c} & R_{y_c} \end{bmatrix} && \text{(24a)} \\ R_{x\eta_c} &= B_{w_c}D'_{w_c} + PC'_c = \begin{bmatrix} R'_{xz_c} \\ R'_{xy_c} \end{bmatrix}', \ \ t \neq t_k && \text{(24b)} \\ R_{\eta_d} &= D_{w_d}D'_{w_d} + C_dP(t^-_k)C'_d - \gamma^2 R_\gamma && \text{(24c)} \\ R_{x\eta_d} &= B_{w_d}D'_{w_d} + A_dP(t^-_k)C'_d, \qquad t = t_k && \text{(24d)} \\ \hat{R}_z &= R_{zy}R^{-1}_y R_{yz} - R_z && \text{(24e)} \end{aligned}$$

with $R_\gamma = \mathrm{diag}(I_{n_z}, 0_{n_y})$. We now summarise the results in a theorem, dual to Theorem 1.

**Theorem 4** *Consider the hybrid system (22). Then there exists a hybrid filter $\hat{z} = \mathcal{F}(y)$ on $t \in [0, T_f]$, which achieves the hybrid performance bound (23), if and only if there exists a bounded symmetric piece-wise continuous matrix function $P \geq 0$, which satisfies the hybrid RE*

$$P^+ = A \diamond P + B_wB'_w - KR_\eta K', \quad P(0) = 0 \tag{25}$$

*where $K = R_{x\eta}R^{-1}_\eta$ and $R_{x\eta}, R_\eta$ are defined in (24). Necessary conditions for the existence of a bounded solution $P$ are $\hat{R}_z > 0, \ \ R_y > 0$. One hybrid filter that achieves the bound is, with $\hat{x}(0) = 0$,*

$$\begin{aligned} \hat{x}^+ &= A\hat{x} + B_uu + K_y(y - C_y\hat{x} - D_{yu}u) && \text{(26a)} \\ \hat{z} &= C_z\hat{x} + D_{zu}u + K_{zy}(y - C_y\hat{x} - D_{yu}u) && \text{(26b)} \end{aligned}$$

*where $K_y = R_{xy}R^{-1}_y, K_{zy} = R_{zy}R^{-1}_y$*

In (26) the known but arbitrary control signal $u$ is included. In infinite-time horizon, the filter must also be stabilising, which adds some demands on the plant. Furthermore, when a solution to the RE exists, it is denoted $P_\infty$. Define $A_y = A - B_wD'_{yw}(D_{yw}D'_{yw})^{-1}C_y, \ \ B_y = B_w\hat{D}_{yw}$.

**Theorem 5** *Consider the situation in Theorem 4. Assume further that the system $(C_y, A)$ is detectable and $(A_y, B_y)$ is stabilisable. Then there exists a hybrid filter that exponentially internally stabilises the system and achieves the performance bound (23) when $T_f \to \infty$, if and only if there exists a bounded symmetric piece-wise continuous matrix function $P_\infty \geq 0$, which satisfies the hybrid RE (25) such that the hybrid system matrix $A - KC$ is exponentially stable. One filter that achieves the bound is achieved as in Theorem 4.*

Dual discussions on "lifting" and periodic behaviour are valid also in the filter case with due modifications, [13].

## 6 The output feedback solution

Now the control problem is reorganised in order to find the output feedback controller as $u = \bar{\mathcal{K}}(y)$. Introduce signal transformations according to

$$\bar{w} = \gamma^{-1}\hat{Q}_w^{\frac{1}{2}}(w - w^*); \;\; \bar{u} = Q_u^{\frac{1}{2}}(u - u^*) \tag{27}$$

which can be considered as weighted deviations from the optimal ones. This transformation yields the following equality, which can be regarded as a $\gamma$-weighted isometric property, [1]:

$$J(T_f) = \|z\|_2^2 - \gamma^2\|w\|_2^2 = \|\bar{u}\|_2^2 - \gamma^2\|\bar{w}\|_2^2 \tag{28}$$

Now the system to consider is the transformed system with input $\bar{w}$ (instead of $w$) and output $\bar{u}$ (instead of $z$):

$$\begin{bmatrix} x^+ \\ \bar{u} \\ y \end{bmatrix} = \begin{bmatrix} A + B_w L_w & \gamma B_w \hat{Q}_w^{-\frac{1}{2}} & B_u \\ Q_u^{\frac{1}{2}}(L_u + L_{uw} L_w) & \gamma Q_u^{\frac{1}{2}} L_{uw} \hat{Q}_w^{-\frac{1}{2}} & Q_u^{\frac{1}{2}} \\ C_y + D_{yw} L_w & \gamma D_{yw} \hat{Q}_w^{-\frac{1}{2}} & D_{yu} \end{bmatrix} \begin{bmatrix} x \\ \bar{w} \\ u \end{bmatrix}$$

Introduce the shorthands according to

$$\begin{bmatrix} x^+ \\ \bar{u} \\ y \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B}_{\bar{w}} & B_u \\ \bar{C}_{\bar{u}} & \bar{D}_{\bar{u}\bar{w}} & Q_u^{\frac{1}{2}} \\ \bar{C}_y & \bar{D}_{y\bar{w}} & D_{yu} \end{bmatrix} \begin{bmatrix} x \\ \bar{w} \\ u \end{bmatrix} \tag{29}$$

When considering the estimated $\hat{\bar{u}}$, with $u = 0$,

$$\begin{bmatrix} \hat{x}^+ \\ \bar{u} - \hat{\bar{u}} \\ y \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B}_{\bar{w}} & 0 \\ \bar{C}_{\bar{u}} & \bar{D}_{\bar{u}\bar{w}} & -I \\ \bar{C}_y & \bar{D}_{y\bar{w}} & 0 \end{bmatrix} \begin{bmatrix} x \\ \bar{w} \\ \hat{\bar{u}} \end{bmatrix} \tag{30}$$

the problem is equivalent to the filter problem in Section 5, since (30) is similar to (22), and with $\hat{\tilde{u}} = 0$ the related criterion equals (28). The matrices needed for the output feedback solution are now given similar to the filter case, with $\bar{C} = [\bar{C}'_{\tilde{u}} \;\; \bar{C}'_y]'$, $\bar{D}_{\tilde{w}} = [\bar{D}'_{\tilde{u}\tilde{w}} \;\; \bar{D}'_{y\tilde{w}}]'$:

$$\begin{aligned}
\bar{R}_{\eta_c} &= \bar{D}_{\tilde{w}_c}\bar{D}'_{\tilde{w}_c} - \gamma^2 R_\gamma = \begin{bmatrix} \bar{R}_{\tilde{u}_c} & \bar{R}_{\tilde{u}y_c} \\ \bar{R}_{y\tilde{u}_c} & \bar{R}_{y_c} \end{bmatrix} && \text{(31a)} \\
\bar{R}_{x\eta_c} &= \bar{B}_{\tilde{w}_c}\bar{D}'_{\tilde{w}_c} + \bar{P}\bar{C}'_c = \begin{bmatrix} \bar{R}'_{x\tilde{u}_c} \\ \bar{R}'_{xy_c} \end{bmatrix}', t \neq t_k && \text{(31b)} \\
\bar{R}_{\eta_d} &= \bar{D}_{\tilde{w}_d}\bar{D}'_{\tilde{w}_d} + \bar{C}_d\bar{P}(t_k^-)\bar{C}'_d - \gamma^2 R_\gamma && \text{(31c)} \\
\bar{R}_{x\eta_d} &= \bar{B}_{\tilde{w}_d}\bar{D}'_{\tilde{w}_d} + \bar{A}_d\bar{P}(t_k^-)\bar{C}'_d, \qquad t = t_k && \text{(31d)}
\end{aligned}$$

We now summarise the output feedback case in finite-time horizon:

**Theorem 6** *Consider the hybrid plant $\mathcal{G}$ in (4). Then there exists a hybrid controller $u = \bar{\mathcal{K}}(y)$ on $t \in [0, T_f]$ which achieves the performance bound $J < 0$, if and only if there exist piece-wise continuous symmetric matrix functions $S \geq 0$ and $P \geq 0$ satisfying (13) and (25) respectively, and such that the spectral radius $\rho(SP) < \gamma^2$ for all times $t$. Necessary conditions for the existence of bounded solutions $S$ and $P$ are given in Theorems 1 and 4 respectively. Then $\bar{P} = P(I - \gamma^{-2}SP)^{-1}$ in (31), and one controller is given by*

$$\begin{aligned}
\hat{x}^+ &= \bar{A}\hat{x} + B_u u + \bar{K}_y(y - \bar{C}_y\hat{x} - D_{yu}u) && \text{(32a)} \\
u &= -\bar{L}\hat{x} - \bar{K}_{uy}(y - \bar{C}_y\hat{x} - D_{yu}u) && \text{(32b)}
\end{aligned}$$

*where $\bar{A}, \ldots$ are defined by (29), $\bar{L}$ is defined after Theorem 1 and $\bar{K}_y = \bar{R}_{xy}\bar{R}_y^{-1}, \bar{K}_{uy} = Q_u^{-\frac{1}{2}}\bar{R}_{\tilde{u}y}\bar{R}_y^{-1}$ with $\bar{R}_{xy}, \bar{R}_y$ and $\bar{R}_{\tilde{u}y}$ defined in (31).*

Note that all matrices in (30) are time-dependent and especially that the spectral radius $\rho(S(t)P(t))$ must be fulfilled on $t \in [0, T_f]$. Furthermore, of practical reasons $D_{yu} = 0$ is a realistic assumption, which implies avoiding algebraic loops. In infinite-time horizon the $\mathcal{H}_\infty$-solution is found from the following theorem. Let the RE-solution, when it exists, be denoted $\bar{P}_\infty$.

**Theorem 7** *Consider the situation in Theorem 6. Assume further that the system $(A, B_u, C_y)$ is stabilisable and detectable and the system $(A_y, B_y)$ is stabilisable and $(C_u, A_u)$ is detectable. Then there exists a stabilising hybrid controller $u = \bar{\mathcal{K}}(y)$ which achieves the performance bound $J < 0$ when final time $T_f \to \infty$, if and only if the following three items all hold:*

1. *there exists a symmetric piece-wise continuous matrix function $S_\infty \geq 0$ satisfying (13) such that the system $A - BL$ is exponentially stable*
2. *there exists a symmetric piece-wise continuous matrix function $P_\infty \geq 0$ satisfying (25) such that the system $A - KC$ is exponentially stable*
3. *the spectral radius $\rho(S_\infty P_\infty) < \gamma^2$*

*The controller is given as in Theorem 6*

## 7 The sampled-data solution

The sampled-data case is the hybrid case, when neither measurements nor control signals are in continuous time. The performance measure, however, is *only* in continuous time. This allows us to make significant simplifications in the hybrid results from Sections 4 to 6. The continuous-time evolution during the sampling interval is easily found from the "lifted" solutions, whereas the discrete-time characteristics give updates at the sampling instants. The corresponding discretised system model for the static feedback and filter cases turn out to be the same:

$$\begin{bmatrix} x(t_{k+1}^-) \\ \tilde{z}(t_k, h_k) \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B}_{\tilde{w}} \\ \tilde{C}_{\tilde{z}} & 0 \end{bmatrix} \begin{bmatrix} x(t_k^+) \\ \tilde{w}(t_k, h_k) \end{bmatrix}$$

where $\tilde{A}, \tilde{B}_u, \tilde{C}_{\tilde{z}}$ are found as in (19). Due to lack of space, only the final controller for the case with a hold circuit at controller output is given:

$$\begin{aligned} \hat{x}(t_{k+1}^-) &= \tilde{\Psi}_{\tilde{w}}(\tilde{A} - \tilde{B}_u L_{u_d})\hat{x}(t_k^+) && \text{(33a)} \\ \hat{x}(t_k^+) &= \hat{x}(t_k^-) + \bar{K}_{y_d}(y_d(t_k) - C_{y_d}\hat{x}(t_k^-)) && \text{(33b)} \\ u_d^*(t_k) &= -L_{u_d}\hat{x}(t_k^+) && \text{(33c)} \end{aligned}$$

where $\tilde{\Psi}_{\tilde{w}}$ is found as in (19), $L_{u_d}$ is found as in Theorem 1 and $\bar{K}_{y_d}$ as in Theorem 6.

## Summary

We have presented a general and unified, RE based, framework for hybrid continuous-/discrete-time and sampled-data $\mathcal{H}_\infty$ output feedback controller design. Hybrid notations are introduced to indicate a very close relationship between the continuous-time and discrete-time solutions. The framework is easily adopted, to pure continuous- and discrete-time systems, by simply interpreting the equations accordingly, but also to (multi-rate) sampled-data systems and hybrid systems with a mixture of continuous-time and discrete-time parts.

The solution is based on continuous REs with jumps, expressed by discrete REs. In the infinite-horizon case, a periodic behaviour is assumed, and it is shown how the related continuous REs with jumps can be replaced by equivalent discrete periodic REs, which can be solved by standard methods. This means that both continuous-time and discrete-time performance, control, measurement and disturbance signals may be taken into account.

We treat a general system model without prerequisites like certain matrices being zero, and this is done without the often complicated transformations, that are found in the literature. The strategies shown in this paper have all successfully been applied on a MIMO-model of a jet-engine.

**Acknowledgements:** The work is partly financed by the Swedish NFFP-programme.

## References

[1] M. Green and D. Limebeer, *Linear Robust Control*. Prentice Hall, Englewood Cliffs, N. J., 1995.

[2] B. Bamieh and J. Pearson, "A general framework for linear periodic systems with applications to $\mathcal{H}_\infty$-sampled-data control," *IEEE Trans. on Automatic Control*, vol. 37, no. 4, pp. 418–435, 1992.

[3] P. Kabamba and S. Hara, "Worst-case analysis and design of sampled-data control systems," *IEEE Trans. on Automatic Control*, vol. 38, pp. 1337–1357, 1993.

[4] H. Toivonen, "Sampled-data control of continuous-time systems with an $\mathcal{H}_\infty$ optimality criterion," *Automatica*, vol. 28, no. 1, pp. 45–54, 1992.

[5] B. Lennartson, "Periodic solutions of Riccati equations applied to multirate sampling," *Int. J. Control*, vol. 48, no. 3, pp. 1025–1042, 1988.

[6] N. Sivashankar and P. Khargonekar, "Characterization of the $\mathcal{L}_2$-induced norm for linear systems with jumps with applications to sampled-data systems," *SIAM J. Control and Optimization*, vol. 32, no. 4, pp. 1128–1150, 1994.

[7] H. Katayama and A. Ichikawa, "$\mathcal{H}_\infty$-control with output feedback for time-varying discrete systems," *Int J. Control*, vol. 63, no. 6, pp. 1167–1178, 1996.

[8] A. Ichikawa and H. Katayama, "$\mathcal{H}_\infty$ control for a general jump system with application to sampled-data systems," in *CDC35*, (Kobe, Japan), pp. 446–451, dec 1996.

[9] T. Chen and B. Francis, "$\mathcal{H}_2$-optimal sampled-data control," *IEEE Trans. on Automatic Control*, vol. 36, no. 4, pp. 387–397, 1991.

[10] O. Lindgärde, *Frequency analysis of sampled-data systems applied to a lime slaking process*. Ph.d. thesis, Chalmers University of Technology, 1999.

[11] B. Lennartson, O. Lindgärde, H. Toivonen, and M. Sågfors, "Mixed continuous-time and discrete-time $\mathcal{H}_\infty$ control," in *CDC36*, (San Diego, CA), pp. 2169 – 2174, dec 1997.

[12] A. Christiansson, B. Lennartson, and H. Toivonen, "Mixed continuous/discrete-time output feedback $\mathcal{H}_\infty$- control, a unified approach," in *ECC*, (Karlsruhe), pp. F1040–4, 1999.

[13] A.-K. Christiansson, *A General Framework for Hybrid $\mathcal{H}_\infty$-Control*. Licentiate thesis, Chalmers University of Technology, Control and Automation Laboratory, 2000.

[14] R. Ravi, K. Nagpal, and P. Khargonekar, "$\mathcal{H}_\infty$ control of linear time-varying systems: A state-space approach," *SIAM J. Control and Optimization*, vol. 29, no. 6, pp. 1394–1413, 1991.

[15] W. Sun, K. Nagpal, and P. Khargonekar, "$\mathcal{H}_\infty$-control and filtering for sampled-data systems," *IEEE Trans. on Automatic Control*, vol. 38, no. 8, pp. 1162–1175, 1993.

[16] K. Glover and J. Doyle, "State-space formulae for all stabilizing controllers that satisfy an $\mathcal{H}_\infty$-norm bound and relations to risk sensitivity," *Systems and Control Letters*, no. 11, pp. 167–172, 1988.

[17] A. Stoorvogel, *The $\mathcal{H}_\infty$ Control Problem. A State Space Approach*. Prentice Hall International, 1992.

[18] H. Toivonen and M. Sågfors, "The sampled-data $\mathcal{H}_\infty$ problem: A unified framework for discretization based methods and Riccati-equation solution," *Int. J. Control*, vol. 66, pp. 289–309, 1997.