State Feedback Control of Linear Systems in the Presence of Devices with Finite Signal-to-Noise Ratio[1]

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Abstract

This paper provides necessary and sufficient conditions for mean-square state-feedback stabilization of linear systems whose white noise sources have intensities affinely related to the variance of the signal they corrupt. System with such noise sources have been called FSN (finite signal-to-noise) models, and the stability results provided in prior work were only sufficient conditions. Upper bounded $\mathcal{L}_2$ performance is also guaranteed herein by solving a control problem which is nonconvex only due to a certain scaling parameter. By fixing this parameter convex programming algorithms provide controllers.

1 Introduction

The traditional model of white noise involves a noise intensity (continuous-time case) or variance (discrete-time case) that is specified as a given or unknown constant or time-varying parameter. Conversely, the white noise model proposed in [1, 2] allows the noise variance to be affinely related to the variance of the signal corrupted by the noise. This model fits many practical situations, such as the roundoff error in both floating-point and fixed-point arithmetic. Hence, any research to improve performance in the presence of a finite-precision computing environment would benefit from the so-called FSN (finite signal-to-noise) models introduced in [1, 2]. Two kinds of problems are subject to the concerns about finite precision computing. The first is the task of building large-scale simulations which approximate the solution of ordinary or partial differential equations. The second is the task of building feedback controllers implemented in digital computers. In physics and engineering some recent papers and proposals for research have pursued the solution of literally millions of differential equations without any treatment of the finite precision computational issue. Yet it is known [3] that there is no upperbound on computational error, because the errors are realization dependent (reference [3] shows some arbitrarily bad realizations). Hence, increasing the number of bits in the computation does not solve the finite precision computing problem. One must pay attention to the realization. Otherwise, the errors can still be arbitrarily large, even with 80 bit arithmetic. Articles on robust control have pointed out that current robust control theory often produces "fragile" controllers which are extremely sensitive to controller errors [4, 5].

The argument for FSN models in continuous-time systems has been more recent. Manufacturing tolerances on certain components, such as the surface precision of a wing, where surface errors lead to turbulence generators, yield FSN models. Sensor and actuator noise can also be modeled as FSN noise sources, since the amplitude of the noise is related to the power required, and this power requirement depends on the magnitude of the signal that must pass through the device. By using variance as a measure of the "magnitude" of the signal, the FSN model appears again. In the classical definition of control design, where sensors and actuators have already been selected, the FSN model is less urgent. But, the more modern task of creating a system design theory motivates this paper. A system design theory must provide guidance for choosing the precision of components in a system, given only a performance requirement. This will require models that relate the precision of the component to the "size" of the signals in the component. If the cost of a component is proportional to its precision, then it is reasonable to minimize the sum of the cost of all components subject to a performance constraint. This use of FSN models appear in [6].

FSN models allow noise to destabilize a linear system, whereas traditional white noise models cannot destabilize. The traditional noise can only degrade performance. This explains why FSN models yield maximal accuracy at finite control gains, while the conventional noise models used in LQG control theory yield maximal accuracy at infinite control gain. In the FSN model, larger signals generate larger noises. Hence, maximal accuracy occurs at finite gain.

The mathematical structure of the FSN models also generates other control problems as special cases. FSN models introduce additional terms in the Lyapunov equation that describes the state covariance. More-

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over, as a special case of FSN noise, these terms produce the same terms introduced by multiplicative noise problems [7, 8]. In other special cases, these additional terms take on the structure of the terms introduced in pole assignment problems [9], where they can force the poles to a specified region of the complex plane. This region can be square, round, parabolic, or other shapes, depending on free parameters in the additional terms in the Lyapunov equation.

A complete theory for FSN models would therefore serve to advance theories for system design, for pole assignment, and for multiplicative noise problems. With sufficient motivation from both the physical and mathematical side, the paper continues the search for a complete theory for the control of FSN models of linear systems.

In this paper, we discuss the state feedback problem with FSN models. Section 2 formulates the problem. Section 3 characterizes mean-square stability and some useful new theorems and identities. Section 4 presents guarantees on $\mathcal{L}_2$ performance. Section 5 provides some examples and Section 6 draws some conclusions.

2 Problem formulation

Let a linear system be described by the state space equations

$$\dot{x} = Ax + B_u\left(u + w_a\right) + w_x \tag{1}$$
$$z = C_z x \tag{2}$$
$$u = K\left(x + w_s\right) \tag{3}$$

where w_i, $i = a, x, s$ are zero mean white noise sources, and all matrices and vectors are assumed to have proper dimensions. In this paper we consider that each noise source has two independent white noise components

$$w_i = \overline{w}_i + \underline{w}_i, \quad i = a, x, s \tag{4}$$

where the first component $\overline{w}_i$ is called *ambient noise* and is modeled according to the traditional assumptions. The second component $\underline{w}_i$ is related to the NSR (noise-to-signal ratio) σ_i^2 of the hardware devices. That is, we assume that

$$\begin{aligned} &\mathrm{E}_\infty\left\{w_i(t)\right\} = 0, \\ &\mathrm{E}_\infty\left\{w_i(t)w_i(\tau)^T\right\} = \left(\overline{W}_i + \underline{W}_i\right)\delta\left(t - \tau\right), \end{aligned} \tag{5}$$

for $i = a, x, s$. The $\overline{W}_i$ is a given constant positive definite matrix and

$$\underline{W}_i := \mathrm{E}_\infty\left\{\sigma_i^2 v_i(t) v_i(t)^T\right\}, \quad i = a, x, s \tag{6}$$

where the vector v_i is the vector the noise source w_i is corrupting. For model (1–6) these signals are $v_a = u$, $v_x = x$ and $v_s = x$.

Associated with the above model we define a quantity called *information quality* [1] that plays a fundamental role in the stabilizability properties of the FSN system (1–6).

Definition 1 *The inverse of the sum of the actuator and sensor NSR*

$$\sigma_{sa}^{-2} := \left(\sigma_s^2 + \sigma_a^2\right)^{-1} \tag{7}$$

is called the information quality *of system (1–6).*

This figure of merit is clearly related to the overall SNR (signal-to-noise ratio) of all measurement and control devices.

Assuming that the closed loop covariance $P := \mathrm{E}_\infty\left\{x(t)x(t)^T\right\}$ associated with the system (1–6) exists, matrices $\underline{W}_i$, $i = a, x, s$ can be calculated as

$$\underline{W}_x := \mathrm{E}_\infty\left\{\sigma_x^2 x(t)x(t)^T\right\} = \sigma_x^2 P, \tag{8}$$
$$\underline{W}_s := \mathrm{E}_\infty\left\{\sigma_s^2 x(t)x(t)^T\right\} = \sigma_s^2 P, \tag{9}$$
$$\underline{W}_a := \mathrm{E}_\infty\left\{\sigma_a^2 u(t)u(t)^T\right\} = \sigma_a^2 KPK^T. \tag{10}$$

Furthermore, one can show that the covariance matrix P should satisfy the Lyapunov equation

$$A_{\mathrm{cl}}P + PA_{\mathrm{cl}}^T + \sum_{i=1}^{2} D_{\mathrm{cl}_i} P D_{\mathrm{cl}_i}^T + \sum_{i=1}^{3} Q_{\mathrm{cl}_i} = \mathbf{0} \tag{11}$$

where

$$\begin{aligned} A_{\mathrm{cl}} &:= A + B_u K, & D_{\mathrm{cl}_1} &:= \sigma_x \mathbf{I}, \\ D_{\mathrm{cl}_2} &:= \sigma_{sa} B_u K, & Q_{\mathrm{cl}_1} &:= \overline{W}_x, \\ Q_{\mathrm{cl}_2} &:= B_u K \overline{W}_s K^T B_u^T, & Q_{\mathrm{cl}_3} &:= B_u \overline{W}_a B_u^T. \end{aligned}$$

From these, stability of the closed loop system (1–3) can be characterized by the following definition.

Definition 2 (Mean-square stability) *The closed loop system (1–3) with FSN noise inputs characterized as in (4–6) is* mean-square stable *if there exists a positive definite covariance matrix P satisfying the Lyapunov equation (11).*

We are now ready to state the two problems to be analyzed in this paper.

Problem 1 (FSN stabilization) *Determine if there exists a controller gain K such that the closed loop FSN system (1–6) is mean-square stable.*

In the sequel we will be able to provide an LMI necessary and sufficient condition for *FSN stabilization*. We will be also able to characterize the *FSN stabilizability* in terms of the *information quality* of the systems and the familiar concept of stabilizability. The second problem of interest is the same one analyzed in [2].

Problem 2 (FSN performance) *If the FSN system (1–6) is stabilizable determine if there exists a gain K such that $\mathrm{E}_\infty\left\{z(t)^T z(t)\right\} < \mu$ for a given a scalar $\mu > 0$.*

In [2] the authors show that the *FSN performance* problem is a nonconvex optimization problem. They have characterized necessary optimality conditions for its solvability that, due to the absence of convexity, turns out to be a difficult calculation. In this paper we turn our attention to the properties of the equation (11) and provide a sufficient condition for FSN performance. We also show how this condition can be used as a guaranteed cost problem.

3 FSN stabilization

Before proceeding to the study of the FSN stabilization problem we need the following lemma on Lyapunov equations of the form (11).

Lemma 1 *The following are equivalent:*

i) The Lyapunov equation

$$AP + PA^T + DPD^T + Q = \mathbf{0} \qquad (12)$$

admits a symmetric and positive definite solution P for some *given symmetric and positive definite matrix Q.*

ii) The Lyapunov equation (12) admits a symmetric and positive definite solution P for any *given symmetric and positive definite matrix Q.*

iii) The Lyapunov inequality

$$AX + XA^T + DXD^T < 0 \qquad (13)$$

admits a symmetric and positive definite solution X.

Proof: We start noticing that ii) immediately implies i) and iii). Also iii) implies i) since there exists a symmetric and positive definite matrix Q that can be added to (13) to establish equality (12). So, it suffices to prove that i) implies ii). We do that using a contradiction argument. Let us suppose that there exists a symmetric and positive definite matrix $\bar{P}$ that satisfies (12) for some symmetric and positive matrix $\bar{Q}$. Additionally, suppose that a symmetric matrix $\tilde{P}$ that is not positive definite also solves (12) for some distinct symmetric and positive matrix $\tilde{Q}$. By rewriting (12) using Kronecker products

$$(\mathbf{I} \otimes A + A \otimes \mathbf{I} + D \otimes D)\,\mathrm{vec}(P) = -\,\mathrm{vec}(Q)$$

we see that the solution P depends continuously on matrix Q. Moreover, matrix P should have full rank whenever matrix Q has full rank. Therefore, as there exists a path linking matrix $\bar{Q}$ to matrix $\tilde{Q}$ constituted of symmetric and positive definite matrices, hence full rank matrices, there also exists a path of full rank matrices linking $\bar{P}$ to $\tilde{P}$ which implies that $\tilde{P}$ should not become singular that is, $\tilde{P}$ can not loose its positive definiteness, which establishes a contradiction. ■

The importance of the above lemma is to establish the inequality test (13) for the FSN stabilization problem without assuming any property on matrices A and D such as stability or controllability. The next theorem provides a complete answer to the FSN stabilization problem.

Theorem 2 *There exists a controller gain K such that the closed loop FSN system (1–6) is mean-square stable if, and only if, there exists a symmetric matrix X such that*

$$\begin{bmatrix} AX + XA^T + B_u L + L^T B_u^T + \sigma_x^2 X & \sigma_{sa} B_u L \\ \sigma_{sa} L^T B_u^T & -X \end{bmatrix} < 0. \qquad (14)$$

In the affirmative case, a stabilizing gain is given by $K = LX^{-1}$.

Proof: According to Lemma 1, there exists a stabilizing gain K if, and only if, there exists a symmetric matrix X such that

$$A_{\mathrm{cl}} X + X A_{\mathrm{cl}}^T + \sum_{i=1}^{2} D_{\mathrm{cl}_i} X {D_{\mathrm{cl}_i}}^T < 0, \quad X > 0.$$

With the change of variable [10] $L := KX$ and the use of Schur complement the above condition can be equivalently written as (14). ■

The inequality (14) is an LMI, hence the existence of a feasible solution is a convex problem that can be solved with the use of available interior-point algorithms. In [2], no computable necessary and sufficient condition for FSN stabilization has been obtained. However, it has been shown that the information quality σ_{sa}^{-1} is associated with a necessary condition for FSN stability. The next theorem recover and strengthen this condition by adding sufficiency.

Theorem 3 *There exists a controller gain K such that the closed loop FSN system (1–6) is mean-square stable if, and only if, the pair $(A + (\sigma_x^2/2)\mathbf{I}, B_u)$ is stabilizable and $\sigma_{sa}^{-2} > \sigma_x^2 + 2\max \mathrm{Re}\,\lambda(A)$.*

Proof: We start by defining $\bar{A} := A + (\sigma_x^2/2)\mathbf{I}$ and rewriting (14) in the form (15). From this inequality

$$\begin{bmatrix} \bar{A}X + X\bar{A}^T & \mathbf{0} \\ \mathbf{0} & -X \end{bmatrix} + \begin{bmatrix} B_u \\ \mathbf{0} \end{bmatrix} L \begin{bmatrix} \mathbf{I} & \sigma_{sa}\mathbf{I} \end{bmatrix} + \begin{bmatrix} \mathbf{I} \\ \sigma_{sa}\mathbf{I} \end{bmatrix} L^T \begin{bmatrix} B_u^T & \mathbf{0} \end{bmatrix} < 0. \tag{15}$$

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and using Finsler's Lemma [11] we obtain that there exists a stabilizing matrix L satisfying (14) if, and only if, there exists a symmetric and positive definite matrix X that simultaneously satisfies

$$\begin{aligned} &\mathcal{N}_{B_u} \left(\bar{A}X + X\bar{A}^T\right) \mathcal{N}_{B_u}^T < 0, \\ &\left(\bar{A} - (\sigma_{sa}^{-2}/2)\mathbf{I}\right) X + X \left(\bar{A} - (\sigma_{sa}^{-2}/2)\mathbf{I}\right)^T < 0. \end{aligned} \tag{16}$$

One can verify (see [11] for details) that the first condition in (16) might be satisfied only if the pair $(\bar{A}, B_u)$ is stabilizable and that the second might be satisfied only if matrix $\bar{A} - (\sigma_{sa}^{-2}/2)\mathbf{I}$ is stable, i.e., $\sigma_{sa}^{-2} > \sigma_x^2 + 2\max \mathrm{Re}\,\lambda(A)$. Notice that these two conditions should be simultaneously satisfied by the same matrix X, which proves only the necessity part of the statement.

In order to prove sufficiency, let us suppose, without loss of generality, that the pair $(\bar{A}, B_u)$ is given in Kalman's controllable decomposition form, that is

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \mathbf{0} & \bar{A}_{22} \end{bmatrix}, \quad B_u = \begin{bmatrix} B_{u1} \\ \mathbf{0} \end{bmatrix}.$$

Assuming that matrix $\bar{A} - (\sigma_{sa}^{-2}/2)\mathbf{I}$ is stable and that the pair $(\bar{A}, B_u)$ is stabilizable, the matrix

$$\tilde{A} = \begin{bmatrix} \bar{A}_{11} - (\sigma_{sa}^{-2}/2)\mathbf{I} & \bar{A}_{12} \\ \mathbf{0} & \bar{A}_{22} \end{bmatrix}$$

is surely stable. Furthermore, it can be shown that there exists a symmetric and positive definite matrix

$$\tilde{X} := \begin{bmatrix} \tilde{X}_{11} & \mathbf{0} \\ \mathbf{0} & \tilde{X}_{22} \end{bmatrix}$$

such that $\tilde{A}\tilde{X} + \tilde{X}\tilde{A}^T < 0$. From this inequality we have that

$$\begin{aligned} &\left(\bar{A} - (\sigma_{sa}^{-2}/2)\mathbf{I}\right) \tilde{X} + \tilde{X} \left(\bar{A} - (\sigma_{sa}^{-2}/2)\mathbf{I}\right)^T \\ &\qquad < -\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sigma_{sa}^{-2}\tilde{X}_{22} \end{bmatrix} \le 0 \end{aligned}$$

and

$$\bar{A}_{22}\tilde{X}_{22} + \tilde{X}_{22}\bar{A}_{22}^T = \mathcal{N}_{B_u} \left(\bar{A}\tilde{X} + \tilde{X}\bar{A}^T\right) \mathcal{N}_{B_u}^T < 0,$$

which shows that the symmetric and positive definite matrix $\tilde{X}$ simultaneously satisfies both inequalities in (16), proving sufficiency. ■

The following corollary can be seen as a weaker version of the above theorem. It is an immediate consequence of the fact that the controllability of the pair (A, B_u) is not affected by a shift in the real axis.

Corollary 4 *If the pair (A, B_u) is controllable and $\sigma_{sa}^{-2} > \sigma_x^2 + 2\max \mathrm{Re}\,\lambda(A)$ then there exists a controller gain K such that the closed loop FSN system (1–6) is mean-square stable.*

4 FSN performance

In the previous section, a necessary and sufficient LMI condition to check the existence of a stabilizing FSN gain has been given. This section is dedicated to provide conditions that additionally guarantee some closed loop performance. As shown in [2], this is a nonconvex problem, and we have no hope to provide a global optimal solution. Instead, we will look for ways to overcome the nonconvexity of the problem by providing alternative conditions that can be easily checked. A fundamental result that enables us to pursue the search of upper-bounds to the solution to the Lyapunov equation (11) is giving in the following lemma.

Lemma 5 *If there exists a symmetric and positive definite matrix P that satisfies (11) then any symmetric and positive definite matrix X such that*

$$A_{\mathrm{cl}}X + XA_{\mathrm{cl}}^T + \sum_{i=1}^{2} D_{\mathrm{cl}_i} X D_{\mathrm{cl}_i}^T + \sum_{i=1}^{3} Q_{\mathrm{cl}_i} < 0 \tag{17}$$

is an upper bound for P that is, $X > P$.

This lemma can be proved using a monotonicity argument based on the properties of Lemma 1. The next theorem provides an easily computable test for the FSN performance problem.

Theorem 6 *For a given a scalar $\mu > 0$, if there exist a matrix L, a symmetric matrix X and a scalar $\alpha > 0$ such that (18) and*

$$\mathrm{trace}\left[C_z X C_z^T\right] < \mu, \quad \alpha \overline{W}_s < X \tag{19}$$

holds then there exists a controller gain K such that the closed loop FSN system (1–6) is mean-square stable and $\mathrm{E}_\infty\left\{z(t)^T z(t)\right\} < \mu$. In the affirmative case, a corresponding gain is given by $K = LX^{-1}$.

Proof: Assuming that (18–19) holds, the Schur complement of (18) and the second inequality in (19) pro-

$$\begin{bmatrix} AX + XA^T + B_uL + L^TB_u^T + \sigma_x^2 X + \overline{W}_x + B_u\overline{W}_aB_u^T & \sigma_{sa}B_uL & B_uL \\ \sigma_{sa}L^TB_u^T & -X & \mathbf{0} \\ L^TB_u^T & \mathbf{0} & -\alpha X \end{bmatrix} < 0 \tag{18}$$

vide

$$\begin{aligned} 0 > &AX + XA^T + B_uL + L^TB_u^T + \sigma_x^2 X + B_u\overline{W}_aB_u^T \\ &+ \sigma_{sa}^2 B_uLX^{-1}L^TB_u^T + \alpha^{-1}B_uLX^{-1}L^TB_u^T + \overline{W}_x \\ > &AX + XA^T + B_uL + L^TB_u^T + \sigma_x^2 X + B_u\overline{W}_aB_u^T \\ &+ \sigma_{sa}^2 B_uLX^{-1}L^TB_u^T + B_uLX^{-1}\overline{W}_sX^{-1}L^TB_u^T + \overline{W}_x. \end{aligned}$$

Therefore, the change of variable $K = LX^{-1}$ and Lemma 5 let us conclude that the symmetric and positive definite matrix X is an upperbound to the matrix P that solves (11) and that

$$\begin{aligned} \mathrm{E}_\infty\left\{z(t)^Tz(t)\right\} &:= \operatorname{trace}\left[C_zPC_z^T\right], \\ &< \operatorname{trace}\left[C_zXC_z^T\right] < \mu, \end{aligned}$$

which concludes this proof. ■

The above theorem provides a sufficient condition for checking FSN performance that can be easily computed. Even though it is not a convex problem, inequalities (18–19) are LMI for fixed values of the scalar α, and Theorem 6 can be verified trough an unidimensional search on α. In [2], the authors have provided problem labeled OCF_1, which has been shown to reduce the FSN performance problem to a convex program whenever $\overline{W}_s = \mathbf{0}$. The next corollary shows that Theorem 6 can be seen as a extension to this problem.

Corollary 7 (Problem OCF_1) *Assuming $\overline{W}_s = \mathbf{0}$ and given a scalar $\mu > 0$, there exist a matrix L, a symmetric matrix X and a scalar $\alpha > 0$ such that (20) and*

$$\operatorname{trace}\left[C_zXC_z^T\right] < \mu \tag{21}$$

holds if, and only if, there exists a controller gain K such that the closed loop FSN system (1–6) is mean-square stable and $\mathrm{E}_\infty\left\{z(t)^Tz(t)^T\right\} < \mu$. In the affirmative case, a corresponding gain is given by $K = LX^{-1}$.

Proof: As $\overline{W}_s$ is equal to zero, the second inequality in (19) is automatically satisfied for any value of $\alpha > 0$. Hence, one can assign to α an arbitrarily large value, eliminating its corresponding row and column in (18). Necessity can be proven by showing that, in this particular case, X can be taken to be arbitrarily close to the solution P of the Lyapunov equation (11). ■

One can show that the necessary and sufficient conditions for the existence of a stabilizing matrix L in Corollary 7 coincide with the ones defined in [2] for problem OCF_1. In fact, Theorem 6 can be seen as an extension to problem OCF_1 since it provides a controller whenever $\overline{W}_s < \alpha^{-1}X$ while problem OCF_1 can only deal with the case $\overline{W}_s = \mathbf{0}$.

The problem of minimizing the upper bound factor μ can be easily defined based on Theorem 6 or Corollary 7 since μ enters linearly in the inequalities. Indeed, as the example in the next section exemplify, this would be one of the best ways of using Theorem 6 since it allows the determination of the dependence of the cost with the auxiliary scaling parameter α.

5 Illustrative example

Consider the following illustrative example where

$$\begin{gathered} A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C_z = \begin{bmatrix} 1 & 0 \end{bmatrix}, \\ \overline{W}_x = 0.01\,\mathbf{I}, \qquad \overline{W}_s = 0.01\,\mathbf{I}, \qquad \overline{W}_a = 0.01, \\ \sigma_x^2 = 0.01, \qquad \sigma_s^2 = 0.1, \qquad \sigma_a^2 = 0.1. \end{gathered}$$

Although matrix A is unstable, in the traditional noise model, where the values of σ_i^2 for $i = a, x, s$ are zero, this example could be stabilized for any values of the ambient noise given that (A, B_u) is controllable. However, following Theorem 3, this FSN system can only be stabilized if the information quality is above the lowerbound

$$\sigma_{sa}^{-2} = 5 > 2.01 = \sigma_x^2 + 2\max\operatorname{Re}\lambda(A).$$

This value imposes a lowerbound on the precision required for sensing and actuating.

The minimum value of α that provides a feasible solution to the guaranteed cost problem in Theorem 6 is $\alpha_{\min} = 20.7$. Starting at this point we have minimized the guaranteed cost bound μ for several values of α obtaining the plot shown in Figure 1. In this figure the solid line shows the values of the guaranteed cost while the dotted line shows the actual cost (output variance). The controller

$$K^* = \begin{bmatrix} -5.2928 & -4.1987 \end{bmatrix}$$

minimizes the guaranteed cost at $\alpha^* \approx 26.2$. For this controller, the value of the guaranteed cost is $\mu^* = 0.4483$ and the output variance is $\mathrm{E}_\infty\left\{z(t)^Tz(t)\right\} = 0.3344$. Note from Figure 1 that the actual minimum of the output variance does not occur at α^*.

$$\begin{bmatrix} AX + XA^T + B_uL + L^TB_u^T + \sigma_x^2X + \overline{W}_x + B_u\overline{W}_aB_u^T & \sigma_{sa}B_uL \\ \sigma_{sa}L^TB_u^T & -X \end{bmatrix} < 0, \tag{20}$$

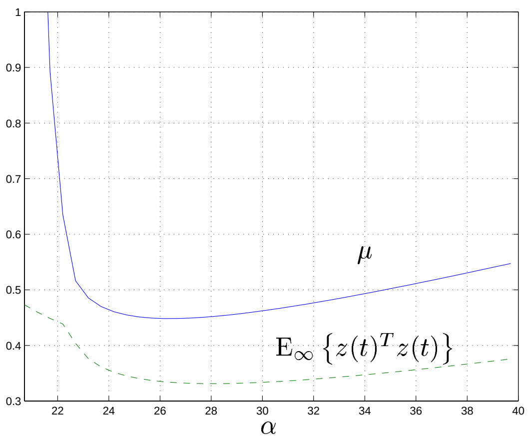

Figure 1: Performance as a function of α

6 Conclusion

FSN models have noise sources whose intensities are affinely related to the corrupted signals. FSN models appear in many practical control applications. Systems with FSN noises have properties that systems with traditional noise sources cannot possess. FSN noise is able to destabilize a linear system and maximum accuracy occurs at a finite gain. These are very welcome properties in the linear control theory. By providing easily computable necessary and sufficient conditions for mean-square state-feedback stabilization of systems with FSN noise sources, this paper represents a step toward a comprehensive FSN control theory.

In this paper, a practical test for FSN systems stabilization is given as a set of LMI conditions. These conditions are shown to be equivalent to the stabilizability of a shifted version of the plant dynamic matrix and the original input matrix, along with a certain bound on the system *information quality*. The information quality is a measure of the precision of the sensing and actuating functions of the system. A sufficient condition for guaranteed $\mathcal{L}_2$ performance is provided by introducing a scalar multiplicative parameter in a set of LMI conditions. Optimal guaranteed cost performance is obtained by performing a line search on the scaling variable.

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