DISCRETISATION OF FEEDBACK CONTROLLERS IN A POINTWISE GAP METRIC

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Abstract

In this paper, a new technique for discretising linear time-invariant (LTI) feedback controllers is proposed. The resulting sampled-data (SD) approximation is guaranteed to lie within a pointwise gap distance, closely related to the ν-gap metric, from the original LTI controller. Importantly, this permits explicit characterisation of possible degradation in closed-loop performance (accounting for stability and inter-sample behaviour.) It is shown that SD approximation, to within a bound on the pointwise gap metric defined, can be posed as a standard H^∞ SD synthesis problem, which may be solved via existing methods.

1 Introduction

The dynamics of many engineering systems naturally evolve in continuous time. Correspondingly, control system design is typically carried out in the continuous-time domain, resulting in control laws with continuous-time dynamics. It is often the case, however, that practical implementation of the control scheme will involve a computer, which can only process discrete information. In view of this, system discretisation has been studied for many years [1, 2, 3, 4, 5].

In this paper, a new technique for discretising linear time-invariant (LTI) feedback controllers is proposed. The technique is motivated by the rather tight robustness results of Vinnicombe [6, 7], for LTI feedback systems in terms of the ν-gap metric. It can be shown that given two LTI systems C and C_1, and a number β, there exists an LTI system R (dependent on C and β only) such that $\delta_\nu(C, C_1) \le \beta \Leftrightarrow \mathscr{F}_\ell(R^{-1}, C_1) \in \mathrm{H}^\infty$ and $\|\mathscr{F}_\ell(R^{-1}, C_1)\|_\infty \le 1$, where $\delta_\nu(C, C_1)$ denotes the ν-gap distance between C and C_1. This is suggestive of a procedure for obtaining a SD approximation C_{sd}, which is close to C (in some sense); i.e. synthesise C_{sd} so that the lower linear fractional transformation (LFT) $\mathscr{F}_\ell(R^{-1}, C_{\mathrm{sd}})$ is contractive (for some small β.) In what follows, it is shown that discretising in this way results in an SD approximation which lies within a certain pointwise gap distance from the original LTI controller. This pointwise metric is closely related to the ν-gap metric, and importantly, given an LTI plant for which a given LTI controller is known to perform satisfactorily, it permits an explicit characterisation of the possible degradation in closed-loop performance (accounting for stability and inter-sample behaviour) when the LTI controller is replaced by the SD approximation.

The paper is organised as follows. First a frequency domain framework for studying the SD approximation problem is established. Central to this is the "time-lifting" isomorphism [8, 9], by which periodic systems are equivalent to shift invariant systems. The pointwise gap metric described above is then defined, and several properties are established. This includes a sufficient condition for the pointwise gap distance between two systems to be less than a specified number. The subsequent section is devoted to capturing this sufficient condition for a given LTI controller and signals in the graph of another (possibly periodic) system. To this end, a "J-inner J-co-inner outer" factorisation (implicit in the DGKF solution to H^∞ synthesis problems) of an augmented system constructed from the given LTI controller, is employed. Finally, the new discretisation procedure is formalised in terms of a standard H^∞ SD synthesis problem.

2 Signals and Systems

In this section a frequency-domain setting, abstracted from the standard one for finite-dimensional LTI systems [7, 10, 11], is established for the rest of the paper. Throughout, the symbols $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{C}$, $\mathbb{C}_+$, $\mathbb{C}_-$, $\mathbb{T}$, $\mathbb{D}$, $j\mathbb{R}$ and $\mathbb{H}$ denote the real, integer and complex numbers, the open right-half plane, open left-half plane, unit circle and open unit disc in the complex plane, the imaginary axis, and the interval $[0, h) \subset \mathbb{R}$ for some $h > 0$, respectively.

A signal is simply considered to be a function mapping from some domain of definition into a Hilbert space. Of particular interest here, are the following (frequency domain) function (signal) spaces:

$$\mathrm{L}^2_{j\mathbb{R}}(\mathcal{H}) := \{ f : j\mathbb{R} \to \mathcal{H} \;:\; \textstyle\int_{\mathbb{R}} \langle f(j\omega),\, f(j\omega)\rangle \, d\omega < \infty\};$$

$$\mathrm{L}^2_{\mathbb{T}}(\mathcal{H}) := \{ f : \mathbb{T} \to \mathcal{H} \;:\; \textstyle\int_{[0,2\pi)} \langle f(e^{j\theta}),\, f(e^{j\theta})\rangle \, d\theta < \infty\};$$

$$\mathrm{L}^2_{\mathbb{H}}(\mathcal{H}) := \{ f : \mathbb{H} \to \mathcal{H} \;:\; \textstyle\int_{\mathbb{H}} \langle f(t),\, f(t)\rangle \, dt < \infty\},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product (on $\mathcal{H}$.) Also of interest are $\mathrm{H}^2_{\mathbb{C}_+}(\mathcal{H})$, the subspace of all $f \in \mathrm{L}^2_{j\mathbb{R}}(\mathcal{H})$ which can be continued analytically into $\mathbb{C}_+$ so that $\int_{\mathbb{R}} \langle f(a + j\omega),\, f(a + j\omega)\rangle \, d\omega$ is uniformly bounded for $a > 0$, and $\mathrm{H}^2_{\mathbb{D}}(\mathcal{H})$, the subspace of all $f \in \mathrm{L}^2_{\mathbb{T}}(\mathcal{H})$ which can be continued analytically into $\mathbb{D}$ so that $\int_{[0,2\pi)} \langle f(re^{j\theta}),\, f(re^{j\theta})\rangle \, d\theta$ is uniformly bounded for $0 \le r < 1$. For convenience the underlying Hilbert space of a signal space (e.g. $\mathcal{H}$ above) is often suppressed.

Throughout systems are considered to be multiplication operators with frequency-domain symbols (transfer functions) in either:

(i) $\mathscr{R}^{p,m}$ – the set of functions $P : j\mathbb{R} \to \mathscr{L}(\mathbb{C}^m, \mathbb{C}^p)$ (a.e.) for which there exists a realisation of the form $P(\varphi) = \mathcal{C}(\varphi I - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}$, with $\mathcal{A} \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$, $\mathcal{B} \in \mathscr{L}(\mathbb{R}^m, \mathbb{R}^n)$, $\mathcal{C} \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^p)$ and $\mathcal{D} \in \mathscr{L}(\mathbb{R}^m, \mathbb{R}^p)$;

(ii) $\mathscr{D}^{p,m}$ – the set of functions $P : \mathbb{T} \to \mathscr{L}(\mathrm{L}^{2,m}_{\mathbb{H}}, \mathrm{L}^{2,p}_{\mathbb{H}})$ (a.e.) for which there exists a realisation of the form $P(\varphi) = \varphi\mathcal{C}(I - \varphi\mathcal{A})^{-1}\mathcal{B} + \mathcal{D}$, with $\mathcal{A} \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$, $\mathcal{B} \in \mathscr{L}(\mathrm{L}^{2,m}_{\mathbb{H}}, \mathbb{R}^n)$, $\mathcal{C} \in \mathscr{L}(\mathbb{R}^n, \mathrm{L}^{2,p}_{\mathbb{H}})$ and $\mathcal{D} \in \mathscr{L}(\mathrm{L}^{2,m}_{\mathbb{H}}, \mathrm{L}^{2,p}_{\mathbb{H}})$,

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where $\mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$ denotes the space of bounded linear operators mapping $\mathcal{H}_1$ into $\mathcal{H}_2$.[1] In what follows a realisation of a specific frequency domain symbol (transfer function) is denoted by either $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ or $\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right)$.

Given a $P \in \mathscr{R}$ (resp. $\mathscr{D}$), let $\mathbf{L}_P$ denote the Laurant operator defined by $(\mathbf{L}_P u)(\varphi) := P(\varphi)u(\varphi)$ for all $u \in \mathrm{dom}(\mathbf{L}_P) \subset \mathrm{L}^2_{j\mathbb{R}}$ (resp. $\mathrm{L}^2_{\mathbb{T}}$) and $\varphi \in j\mathbb{R}$ (resp. $\mathbb{T}$), and the multiplication operator $\mathbf{M}_P$ be defined by $(\mathbf{M}_P u)(\varphi) := P(\varphi)u(\varphi)$ for all $u \in \mathrm{dom}(\mathbf{M}_P) \subset \mathrm{H}^2_{\mathbb{C}_+}$ (resp. $\mathrm{H}^2_{\mathbb{D}}$) and $\varphi \in j\mathbb{R}$ (resp. $\mathbb{T}$.) The space of functions $P \in \mathscr{R}$ that satisfy $\|P\|_\infty := \sup_{\varphi \in j\mathbb{R}} \boldsymbol{\mu}(P(\varphi)) < \infty$ is denoted by the $\mathscr{R}\mathrm{L}^\infty_{j\mathbb{R}}$, where $\boldsymbol{\mu}(\mathbf{X}) := \sup_{\|u\|=1} \|\mathbf{X}u\|$ for $\mathbf{X} \in \mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$. This corresponds to those $P \in \mathscr{R}$ with a realisation $(\mathcal{A}, \cdot, \cdot, \cdot)$ that satisfies $\mathrm{spec}(\mathcal{A}) \cap j\mathbb{R} = \emptyset$. The symbol $\mathscr{R}\mathrm{H}^\infty_{\mathbb{C}_+}$ denotes the Hardy space of all $P \in \mathscr{R}\mathrm{L}^\infty_{j\mathbb{R}}$ that can be continued analytically into $\mathbb{C}_+$. This corresponds to those $P \in \mathscr{R}$ with a realisation $(\mathcal{A}, \cdot, \cdot, \cdot)$ that satisfies $\mathrm{spec}(\mathcal{A}) \subset \mathbb{C}_-$. Similarly, the space of functions $P \in \mathscr{D}$ that satisfy $\|P\|_\infty := \sup_{\varphi \in \mathbb{T}} \boldsymbol{\mu}(P(\varphi)) < \infty$, is denoted by $\mathscr{D}\mathrm{L}^\infty_{\mathbb{T}}$, and $\mathscr{D}\mathrm{H}^\infty_{\mathbb{D}}$ denotes the Hardy space of all $P \in \mathscr{D}\mathrm{L}^\infty_{\mathbb{T}}$ that can be continued analytically $\mathbb{D}$. The space $\mathscr{D}\mathrm{L}^\infty_{\mathbb{T}}$ (resp. $\mathscr{D}\mathrm{H}^\infty_{\mathbb{D}}$) corresponds to those $P \in \mathscr{D}$ with a realisation $(\mathcal{A}, \cdot, \cdot, \cdot)$ that satisfies $\mathrm{spec}(\mathcal{A}) \cap \mathbb{T} = \emptyset$ (resp. $\mathrm{spec}(\mathcal{A}) \subset \mathbb{D}$.)

When $P \in \mathscr{R}\mathrm{L}^\infty_{j\mathbb{R}}$ (resp. $\mathscr{D}\mathrm{L}^\infty_{\mathbb{T}}$), $\mathbf{L}_P$ is an element of $\mathscr{L}(\mathrm{L}^2_{j\mathbb{R}}, \mathrm{L}^2_{j\mathbb{R}})$ (resp. $\mathscr{L}(\mathrm{L}^2_{\mathbb{T}}, \mathrm{L}^2_{\mathbb{T}})$) and $\|\mathbf{L}_P\| = \|P\|_\infty$. Moreover, when $P \in \mathscr{R}\mathrm{H}^\infty_{\mathbb{C}_+}$ (resp. $\mathscr{D}\mathrm{H}^\infty_{\mathbb{D}}$), $\mathbf{M}_P \in \mathscr{L}(\mathrm{H}^2_{\mathbb{C}_+}, \mathrm{H}^2_{\mathbb{C}_+})$ (resp. $\mathscr{L}(\mathrm{H}^2_{\mathbb{D}}, \mathrm{H}^2_{\mathbb{D}})$) and $\|\mathbf{M}_P\| = \|P\|_\infty$ [12, 13]. If a $P \in \mathscr{R}$ (resp. $\mathscr{D}$) satisfies $P(\varphi)^* P(\varphi) = I$ for all $\varphi \in j\mathbb{R}$ (resp. $\mathbb{T}$), then P is called rigid and $\mathbf{L}_P$ is an isometry. If $P \in \mathscr{R}\mathrm{H}^\infty_{\mathbb{C}_+}$ (resp. $\mathscr{D}\mathrm{H}^\infty_{\mathbb{D}}$) also holds, then P is called inner. On the other hand, if $P \in \mathscr{R}$ (resp. $\mathscr{D}$) satisfies $P(\varphi)P(\varphi)^* = I$ for all $\varphi \in j\mathbb{R}$ (resp. $\mathbb{T}$), then P is called co-rigid and $\mathbf{L}_P$ is a co-isometry. If $P \in \mathscr{R}\mathrm{H}^\infty_{\mathbb{C}_+}$ (resp. $\mathscr{D}\mathrm{H}^\infty_{\mathbb{D}}$) also holds, then P is called co-inner. Finally, if $P(\varphi)^* P(\varphi) = P(\varphi)P(\varphi)^* = I$ for all $\varphi \in j\mathbb{R}$ (resp. $\mathbb{T}$), then P is called unitary. Henceforth, the notation P^* is used to denote the function that satisfies $P^*(\varphi) = P(\varphi)^*$ for (almost) all $\varphi \in j\mathbb{R}$ (resp. $\mathbb{T}$.) If $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ is a realisation of a $P \in \mathscr{R}$, then $(-\mathcal{A}^*, -\mathcal{C}^*, \mathcal{B}^*, \mathcal{D}^*)$ is a realisation of $P^* \in \mathscr{R}$. Also, if $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ is a realisation of $P \in \mathscr{D}$, such that $0 \notin \mathrm{spec}(\mathcal{A})$, then $P^* \in \mathscr{D}$ and $((\mathcal{A}^*)^{-1}, -(\mathcal{A}^*)^{-1}\mathcal{C}^*, \mathcal{B}^*(\mathcal{A}^*)^{-1}, \mathcal{D}^* - \mathcal{B}^*(\mathcal{A}^*)^{-1}\mathcal{C}^*)$ is a realisation of P^*.

Importantly, characterising systems as multiplication operators with frequency-domain symbols of the form described above is adequate to address the SD approximation problem outlined in the introduction. Most would be familiar with the equivalence (via the Fourier transform isomorphism) between a time-domain representation of a finite-dimensional LTI state-space system and multiplication by a corresponding transfer function in $\mathscr{R}$. But it is also true that a time-domain representation of a finite-dimensional linear *periodically time-varying* state-space system (including LTI systems) is equivalent to multiplication by a corresponding frequency-domain symbol in $\mathscr{D}$. Central to this equivalence is the the "time-lifting" transform defined for $f : \mathbb{R} \to \mathcal{H}$ by [8, 9]

$$\mathrm{f}(k) := (\mathbf{W} f)(k) := f(\theta + kh) \quad (\theta \in \mathbb{H}).$$

This is an isomorphism between $\mathrm{L}^2_{\mathbb{R}} := \{f : \int_{\mathbb{R}} \langle f(t), f(t)\rangle dt < \infty\}$ (resp. $\mathrm{L}^{2+}_{\mathbb{R}} := \{f \in \mathrm{L}^2_{\mathbb{R}} : f(t) = 0 \text{ for } t < 0\}$) and $\ell^2_{\mathbb{Z}}(\mathrm{L}^2_{\mathbb{H}}) := \{\mathrm{f} : \sum_{\mathbb{Z}} \langle \mathrm{f}(k), \mathrm{f}(k)\rangle < \infty\}$ (resp. $\ell^{2+}_{\mathbb{Z}}(\mathrm{L}^2_{\mathbb{H}}) := \{\mathrm{f} \in \ell^2_{\mathbb{Z}}(\mathrm{L}^2_{\mathbb{H}}) : \mathrm{f}(k) = 0 \text{ for } k < 0\}$), which is in turn isomorphic to $\mathrm{L}^2_{\mathbb{T}}(\mathrm{L}^2_{\mathbb{H}})$ (resp. $\mathrm{H}^2_{\mathbb{D}}(\mathrm{L}^2_{\mathbb{H}})$) via the $\mathbf{Z}$-transform

$$(\mathbf{Z}\mathrm{f})(\varphi) := \sum_{k \in \mathbb{Z}} \varphi^k \mathrm{f}(k).$$

Now consider, for example, the finite-dimensional linear periodic system $\mathbf{P}$, governed by the differential equations $\dot{x} = \mathcal{A}(t)x + \mathcal{B}(t)u$; $y = \mathcal{C}(t)x + \mathcal{D}(t)u$, with $\mathcal{A}(t + kh) = \mathcal{A}(t)$, $\mathcal{B}(t + kh) = \mathcal{B}(t)$, $\mathcal{C}(t + kh) = \mathcal{C}(t)$ and $\mathcal{D}(t + kh) = \mathcal{D}(t)$ for $k \in \mathbb{Z}$ and some $h > 0$. It follows straightforwardly, that $\mathbf{WPW}^{-1}$ is governed by the system of difference equations $x((k+1)h) = \grave{\mathcal{A}}x(kh) + \grave{\mathcal{B}}u(kh+\theta)$; $y(kh + \theta) = \grave{\mathcal{C}}x(kh) + \grave{\mathcal{D}}u(kh + \theta)$, where

$$\begin{aligned}
&\grave{\mathcal{A}} : x(kh) \mapsto \mathbf{\Phi}_{\mathcal{A}}(h, 0)x(kh),\\
&\grave{\mathcal{B}} : u(kh + \theta) \mapsto \textstyle\int_0^h \mathbf{\Phi}_{\mathcal{A}}(h, \tau)\mathcal{B}(\tau)u(kh + \tau)d\tau,\\
&\grave{\mathcal{C}} : x(kh) \mapsto \mathcal{C}(\theta)\mathbf{\Phi}_{\mathcal{A}}(\theta, 0)x(kh),\\
&\grave{\mathcal{D}} : u(kh + \theta) \mapsto \mathcal{D}(\theta)u(kh + \theta)\\
&\qquad\qquad + \textstyle\int_0^\theta \mathcal{C}(\theta)\mathbf{\Phi}_{\mathcal{A}}(\theta, \tau)\mathcal{B}(\tau)u(kh + \tau)d\tau,
\end{aligned}$$

and $\mathbf{\Phi}_{\mathcal{A}}(t, \tau)$ denotes the state transition matrix satisfying $\frac{d}{dt}\mathbf{\Phi}_{\mathcal{A}}(t, \tau) = \mathcal{A}(t)\mathbf{\Phi}_{\mathcal{A}}(t, \tau)$; $\mathbf{\Phi}_{\mathcal{A}}(\tau, \tau) = I$. This is clearly a shift-invariant representation and hence, it follows (applying the $\mathbf{Z}$-transform) that the $\mathrm{L}^2_{\mathbb{R}}$(resp. $\mathrm{L}^{2+}_{\mathbb{R}}$)-graph (all $\mathrm{L}^2_{\mathbb{R}}$ (resp. $\mathrm{L}^{2+}_{\mathbb{R}}$) input-output pairs) of $\mathbf{P}$ is isomorphic (via $\mathbf{W}$ and $\mathbf{Z}$) to the $\mathrm{L}^2_{\mathbb{T}}(\mathrm{L}^2_{\mathbb{H}})$(resp. $\mathrm{H}^2_{\mathbb{D}}(\mathrm{L}^2_{\mathbb{H}})$)-graph of multiplication by $\varphi\grave{\mathcal{C}}(I - \varphi\grave{\mathcal{A}})^{-1}\grave{\mathcal{B}} + \grave{\mathcal{D}} \in \mathscr{D}$. Note that if $\mathcal{A}(t) = \mathcal{A}(t + \tau)$, $\mathcal{B}(t) = \mathcal{B}(t + \tau)$, $\mathcal{C}(t) = \mathcal{C}(t + \tau)$ and $\mathcal{D}(t) = \mathcal{D}(t+\tau)$ for all $\tau \in \mathbb{R}$, then $\mathbf{P}$ is also equivalent (via the Fourier transform) to mulitplication by $\mathcal{C}(\varphi I - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D} \in \mathscr{R}$ (on $\mathrm{L}^2_{j\mathbb{R}}$ (resp. $\mathrm{H}^2_{\mathbb{C}_+}$).) Indeed, in this way (i.e. via the Fourier transform, $\mathbf{Z}$-transform and $\mathbf{W}$-transform isomorphism), multiplication by any $P \in \mathscr{R}$ is equivalent to multiplication by a $\mathrm{P} \in \mathscr{D}$, called the "time-lifted" equivalent. Throughout, the upright roman font is used to denote the "time-lifted" equivalent.

Finally, consider a finite-dimensional LTI filter $\mathbf{F}$, governed by the system of differential equations $\dot{x}_F = \mathcal{A}_F x_F + \mathcal{B}_F u$; $u_F = \mathcal{C}_F x_F$ with $\mathrm{spec}(\mathcal{A}_F) \subset \mathbb{C}_-$, and an h-periodic SD system $\mathbf{S}$, governed by the difference equations $x_S(k + 1) = \mathcal{A}_S x_S(k) + \mathcal{B}_S u_F(kh)$; $y(kh + \theta) = \mathcal{C}_S x_S(k) + \mathcal{D}_S u_F(kh)$ ($\theta \in \mathbb{H}$). The $\mathrm{L}^2_{\mathbb{R}}$(resp. $\mathrm{L}^{2+}_{\mathbb{R}}$)-graph of the pre-filtered periodic SD system $\mathbf{C}_{\mathrm{sd}} := \mathbf{SF}$ is isomorphic (via $\mathbf{W}$ and $\mathbf{Z}$) to the $\mathrm{L}^2_{\mathbb{T}}(\mathrm{L}^2_{\mathbb{H}})$(resp. $\mathrm{H}^2_{\mathbb{D}}(\mathrm{L}^2_{\mathbb{T}})$)-graph of multiplication by

$$C_{\mathrm{sd}}(\varphi) = \left(\begin{array}{c|c} \begin{pmatrix} \grave{\mathcal{A}}_F & 0 \\ \mathcal{B}_S\mathcal{C}_F & \mathcal{A}_S \end{pmatrix} & \begin{pmatrix} \grave{\mathcal{B}}_F \\ 0 \end{pmatrix} \\ \hline \begin{pmatrix} \grave{\mathcal{C}}_1 & \grave{\mathcal{C}}_2 \end{pmatrix} & 0 \end{array}\right) \in \mathscr{D},$$

where $\grave{\mathcal{A}}_F : x \mapsto \exp(h\mathcal{A}_F)x$, $\grave{\mathcal{B}}_F : u \mapsto \int_0^h \exp((h - \tau)\mathcal{A}_F)\mathcal{B}_F u(\tau)\, d\tau$, $\grave{\mathcal{C}}_1 : x \mapsto \mathcal{D}_S\mathcal{C}_F x$, and $\grave{\mathcal{C}}_2 : x \mapsto \mathcal{C}_S x$.

3 Graph Symbols and Feedback Systems

In the next two sections attention is directed towards systems represented by transfer functions in $\mathscr{D}$, noting that this also captures all transfer functions in $\mathscr{R}$ (see above.)

Given a function $C \in \mathscr{D}$, a factorisation $C = ND^{-1}$, with $N, D \in \mathscr{D}\mathrm{H}^\infty_{\mathbb{D}}$ and $D^{-1} \in \mathscr{D}$, is called right coprime over $\mathscr{D}\mathrm{H}^\infty_{\mathbb{D}}$ if there exist functions $\tilde{X}, \tilde{Y} \in \mathscr{D}\mathrm{H}^\infty_{\mathbb{D}}$ such that $(\tilde{X} \quad \tilde{Y})\binom{D}{N} = I$. Similarly, a factorisation $C = \tilde{D}^{-1}\tilde{N}$, where $\tilde{N}, \tilde{D} \in \mathscr{D}\mathrm{H}^\infty_{\mathbb{D}}$ and $\tilde{D}^{-1} \in \mathscr{D}$, is called left coprime over $\mathscr{D}\mathrm{H}^\infty_{\mathbb{D}}$ if there exist functions $X, Y \in \mathscr{D}\mathrm{H}^\infty_{\mathbb{D}}$ such that $(-\tilde{N} \quad \tilde{D})\binom{-Y}{X} = I$. The right and left coprime factorisations are said to be normalised if $N^*N + D^*D = I$

[1]The spatial dimensions of $\mathscr{R}^{p,m}$ and $\mathscr{D}^{p,m}$ are often ommitted.

and $\tilde{N}\tilde{N}^* + \tilde{D}\tilde{D}^* = I$, respectively. Importantly, such factorisations exist for any $P \in \mathscr{D}$ (see [14].)

Given right and left coprime factorisations $C = ND^{-1} = \tilde{D}^{-1}\tilde{N} \in \mathscr{D}$, define

$$K := \begin{pmatrix} N \\ D \end{pmatrix} \in \mathscr{D}\mathrm{H}^{\infty}_{\mathbb{D}} \quad \text{and } \tilde{K} := \begin{pmatrix} -\tilde{D} & \tilde{N} \end{pmatrix} \in \mathscr{D}\mathrm{H}^{\infty}_{\mathbb{D}}.$$

It follows that $\mathrm{gr}(\mathbf{M}_C) = \mathrm{ran}(\mathbf{M}_K) = \ker(\mathbf{M}_{\tilde{K}}) \subset \mathrm{H}^2_{\mathbb{D}}$ and $\mathrm{gr}(\mathbf{L}_C) = \mathrm{ran}(\mathbf{L}_K) = \ker(\mathbf{L}_{\tilde{K}}) \subset \mathrm{L}^2_{\mathbb{T}}$, where $\mathrm{gr}(\cdot)$ denotes the graph (i.e. all input-output pairs.) Similarly, for almost all[2] frequencies $\varphi \in \mathbb{T}$, $\mathrm{ran}(K(\varphi)) = \ker(\tilde{K}(\varphi)) \subset \mathrm{L}^{2,m+p}_{\mathbb{H}}$. The functions K and $\tilde{K}$ are called right and left graph symbols, which are said to be normalised if $K^*K = I$ and $\tilde{K}\tilde{K}^* = I$. In this case

$$\begin{pmatrix} K^* \\ \tilde{K} \end{pmatrix} \begin{pmatrix} K & \tilde{K}^* \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{1}$$

Correspondingly, $\mathrm{ran}(\begin{pmatrix} K^*(\varphi) \\ \tilde{K}(\varphi) \end{pmatrix}) = \mathrm{L}^{2,m+p}_{\mathbb{H}}$ at any $\varphi \in \mathbb{T}$. Moreover, for almost all $\varphi \in \mathbb{T}$, $\ker(\begin{pmatrix} K^*(\varphi) \\ \tilde{K}(\varphi) \end{pmatrix}) = \ker(K(\varphi)^*) \cap \ker(\tilde{K}(\varphi)) = \mathrm{ran}(K(\varphi))^{\perp} \cap \mathrm{ran}(K(\varphi)) = \{0\}$.[3] So $\begin{pmatrix} K^*(\varphi) \\ \tilde{K}(\varphi) \end{pmatrix}$ is bijective for almost all $\varphi \in \mathbb{T}$, and hence, in light of (1) it follows that

$$\begin{pmatrix} K(\varphi) & \tilde{K}^*(\varphi) \end{pmatrix} \begin{pmatrix} K^*(\varphi) \\ \tilde{K}(\varphi) \end{pmatrix} = I. \tag{2}$$

In fact this is true for all $\varphi \in \mathbb{T}$, since $K \in \mathscr{D}\mathrm{H}^{\infty}_{\mathbb{D}}$ and $\tilde{K} \in \mathscr{D}\mathrm{H}^{\infty}_{\mathbb{D}}$ are continuous on $\mathbb{T}$.[4] As in [6, 7], the properties (1) and (2), of normalised graph symbols, play an important role in this paper.

Consider the standard feedback configuration, shown in Figure 1. When it exists (in an appropriate sense), the transfer function from (w_1, w_2) to (y, u, y_p, u_p) is denoted by $[P, C]$. Suppose

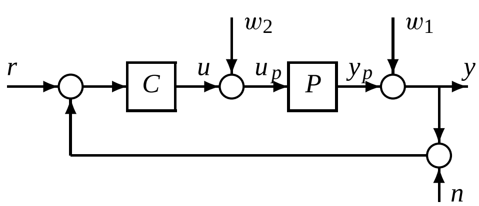


Figure 1: Standard Feedback Configuration

that $P, C \in \mathscr{D}$, so that the *normalised* coprime factorisations $C = ND^{-1} = \tilde{D}^{-1}\tilde{N}$ and $P = N_P D_P^{-1} = \tilde{D}_P^{-1}\tilde{N}_P$, and corresponding graph symbols

$$K := \begin{pmatrix} N \\ D \end{pmatrix},\ \tilde{K} := \begin{pmatrix} -\tilde{D} & \tilde{N} \end{pmatrix},\ G := \begin{pmatrix} D_P \\ N_P \end{pmatrix},\ \tilde{G} := \begin{pmatrix} -\tilde{N}_P & \tilde{D}_P \end{pmatrix},$$

exist.[5] When $(I - PC)$ and $(I - CP)$ are invertible in $\mathscr{D}$ (i.e. $[P, C]$ is well-posed), the transfer function from the signals $\begin{pmatrix} w_2 \\ w_1 \end{pmatrix}$ to $\begin{pmatrix} u \\ y \end{pmatrix}$ is

$$T_1(P, C) = \begin{pmatrix} C \\ I \end{pmatrix} (I - PC)^{-1} \begin{pmatrix} -P & I \end{pmatrix} = K(\tilde{G}K)^{-1}\tilde{G}, \tag{3}$$

[2] With possibly the exception of finitely many points at which $D(\varphi)$ (or $\tilde{D}(\varphi)$) may not be boundedly invertible; i.e. at poles of C on $\mathbb{T}$.

[3] Note that $\mathrm{cl}(\mathrm{ran}(K(\varphi))) = \mathrm{ran}(K(\varphi))$ since $K(\varphi)$ is left-invertible.

[4] In the $C \in \mathscr{R}$ case, such complicated arguments are not required, since at each frequency $\begin{pmatrix} K^*(\varphi) \\ \tilde{K}(\varphi) \end{pmatrix}$ is square and finite-dimensional, which combined with (1) is enough to imply that it is unitary.

[5] Note that the graph symbols G and $\tilde{G}$ are "inverted" (i.e. they correspond to inputs on top and outputs on the bottom.) This is simply for notational convenience in what follows.

and the transfer function from $\begin{pmatrix} w_2 \\ w_1 \end{pmatrix}$ to $\begin{pmatrix} u_p \\ y_p \end{pmatrix}$ is

$$T_2(P, C) = \begin{pmatrix} I \\ P \end{pmatrix} (I - CP)^{-1} \begin{pmatrix} I & -C \end{pmatrix} = G(\tilde{K}G)^{-1}\tilde{K}. \tag{4}$$

Note that $T_1(P, C)$ and $T_2(P, C)$ capture all closed-loop transfer functions commonly employed in robustness/performance analysis. Since $T_1(P, C) + T_2(P, C) = I$, it follows that $T_1(P, C) \in \mathscr{D}\mathrm{L}^{\infty}_{\mathbb{T}}$ (resp. $\mathscr{D}\mathrm{H}^{\infty}_{\mathbb{D}}$) $\Leftrightarrow T_2(P, C) \in \mathscr{D}\mathrm{L}^{\infty}_{\mathbb{T}}$ (resp. $\mathscr{D}\mathrm{H}^{\infty}_{\mathbb{D}}$), and as such either is equivalent to $[P, C] \in \mathscr{D}\mathrm{L}^{\infty}_{\mathbb{T}}$ (resp. $\mathscr{D}\mathrm{H}^{\infty}_{\mathbb{D}}$.) Furthermore, when $[P, C] \in \mathscr{R}\mathrm{L}^{\infty}_{j\mathbb{R}}$, it can be shown that the generic closed-loop performance index $\boldsymbol{b}(P, C) := \|T_1(P, C)\|^{-1}_{\infty} = \|T_2(P, C)\|^{-1}_{\infty}$ (if $[P, C] \notin \mathscr{R}\mathrm{L}^{\infty}_{j\mathbb{R}}$, $\boldsymbol{b}(P, C) := 0$.) Also, note from (3) and (4), that $[P, C] \in \mathscr{D}\mathrm{L}^{\infty}_{\mathbb{T}}$ (resp. $\mathscr{D}\mathrm{H}^{\infty}_{\mathbb{D}}$) $\Leftrightarrow (\tilde{G}K)^{-1} \in \mathscr{D}\mathrm{L}^{\infty}_{\mathbb{T}}$ (resp. $\mathscr{D}\mathrm{H}^{\infty}_{\mathbb{D}}$) $\Leftrightarrow (\tilde{K}G)^{-1} \in \mathscr{D}\mathrm{L}^{\infty}_{\mathbb{T}}$ (resp. $\mathscr{D}\mathrm{H}^{\infty}_{\mathbb{D}}$), since K and G are left invertible in $\mathscr{D}\mathrm{H}^{\infty}_{\mathbb{D}}$, and $\tilde{G}$ and $\tilde{K}$ are right invertible in $\mathscr{D}\mathrm{H}^{\infty}_{\mathbb{D}}$. Moreover,

$$\begin{aligned} \boldsymbol{b}(P, C) &= 1/\sup_{\varphi \in \mathbb{T}} \boldsymbol{\mu}(K(\tilde{G}K)^{-1}\tilde{G}(\varphi)) \\ &= \inf_{\varphi \in \mathbb{T}} 1/\boldsymbol{\mu}((\tilde{G}K)^{-1}(\varphi)) = \inf_{\varphi \in \mathbb{T}} \boldsymbol{\tau}(\tilde{G}K(\varphi)) \leq 1, \end{aligned}$$

where the second equality holds because $K(\varphi)^*K(\varphi) = I$ and $\tilde{G}(\varphi)\tilde{G}(\varphi)^* = I$ for all $\varphi \in \mathbb{T}$, and $\boldsymbol{\tau}(\mathbf{X}) := \inf_{\|u\|=1} \|\mathbf{X}u\|$ for $\mathbf{X} \in \mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$.

4 A pointwise gap metric

Before defining the pointwise gap metric, the following technical lemma is required.

Lemma 4.1 *When* $\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} \in \mathscr{L}(\mathcal{H}_3, \mathcal{H}_1 \oplus \mathcal{H}_2)$ *is an isometry,* $\boldsymbol{\mu}(\mathbf{B})^2 = 1 - \boldsymbol{\tau}(\mathbf{A})^2$, *and if* $\mathbf{A}$ *is also boundedly invertible,*

$$\boldsymbol{\mu}(\mathbf{B}\mathbf{A}^{-1}) = \boldsymbol{\mu}(\mathbf{B}) \Big/ \sqrt{1 - \boldsymbol{\mu}(\mathbf{B})^2} = \sqrt{1 - \boldsymbol{\tau}(\mathbf{A})^2} \Big/ \boldsymbol{\tau}(\mathbf{A}).$$

Similarly, if $(\tilde{\mathbf{A}} \quad \tilde{\mathbf{B}}) \in \mathscr{L}(\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{H}_3)$ *is a co-isometry, then* $\boldsymbol{\mu}(\tilde{\mathbf{B}})^2 = 1 - \boldsymbol{\tau}(\tilde{\mathbf{A}})^2$, *and if* $\tilde{\mathbf{A}}$ *is also boundedly invertible,*

$$\boldsymbol{\mu}(\tilde{\mathbf{A}}^{-1}\tilde{\mathbf{B}}) = \boldsymbol{\mu}(\tilde{\mathbf{B}}) \Big/ \sqrt{1 - \boldsymbol{\mu}(\tilde{\mathbf{B}})^2} = \sqrt{1 - \boldsymbol{\tau}(\tilde{\mathbf{A}})^2} \Big/ \boldsymbol{\tau}(\tilde{\mathbf{A}}).$$

Let $P, C, K, \tilde{K}, G$ and $\tilde{G}$ be as defined in the previous section. Then defining the scalar function $\boldsymbol{\rho}(P(\varphi), C(\varphi)) := \boldsymbol{\tau}(\tilde{G}K(\varphi))$, it is immediate that

$$\boldsymbol{b}(P, C) = \inf_{\varphi \in \mathbb{T}} \boldsymbol{\rho}(P(\varphi), C(\varphi)). \tag{5}$$

Moreover, since $\tilde{G}(K \quad \tilde{K}^*)(\varphi)$ is a co-isometry (cf. (2)) for all $\varphi \in \mathbb{T}$, it follows by Lemma 4.1 that

$$\boldsymbol{\mu}(\tilde{G}\tilde{K}^*(\varphi))^2 \quad = \quad 1 - \boldsymbol{\tau}(\tilde{G}K(\varphi))^2 \tag{6}$$

and

$$\boldsymbol{\mu}\left((\tilde{G}K)^{-1}\tilde{G}\tilde{K}^*(\varphi)\right) \quad = \quad \frac{\sqrt{1 - \boldsymbol{\rho}(P(\varphi), C(\varphi))^2}}{\boldsymbol{\rho}(P(\varphi), C(\varphi))} \tag{7}$$

for all $\varphi \in \mathbb{T}$.

Now, given a $C_1 \in \mathscr{D}$, with normalised right and left graph symbols K_1 and $\tilde{K}_1$, define

$$\boldsymbol{\kappa}(C(\varphi), C_1(\varphi)) := \boldsymbol{\mu}(\tilde{K}K_1(\varphi)) = \boldsymbol{\mu}(\tilde{K}_1K(\varphi)) =: \boldsymbol{\kappa}(C_1(\varphi), C(\varphi)),$$

where the second equality follows by Lemma 4.1, and the fact that $K^*(K_1 \quad \tilde{K}_1^*)(\varphi)$ is a co-isometry and $\begin{pmatrix} K^* \\ \tilde{K} \end{pmatrix} K_1(\varphi)$ is an isometry (cf. (1) and (2)) for all $\varphi \in \mathbb{T}$, which yields

$$\boldsymbol{\mu}(K^*\tilde{K}_1^*(\varphi)) = \boldsymbol{\mu}(\tilde{K}_1 K(\varphi)) = \sqrt{1 - \boldsymbol{\tau}(K^*K_1(\varphi))^2} = \boldsymbol{\mu}(\tilde{K}K_1(\varphi)).$$

It can also be shown that $\boldsymbol{\kappa}(C(\varphi), C_1(\varphi)) = 0$ if, and only if, $C(\varphi) = C_1(\varphi)$, and that $\boldsymbol{\kappa}(C(\varphi), C_2(\varphi)) \leq \boldsymbol{\kappa}(C(\varphi), C_1(\varphi)) + \boldsymbol{\kappa}(C_1(\varphi), C_2(\varphi))$ [6, 7]. As such, $\boldsymbol{\kappa}(\cdot,\cdot)$ is a metric pointwise in frequency. Furthermore, when $K^*K_1(\varphi)$ is invertible in $\mathscr{L}(\mathrm{L}^2_{\mathbb{H}}, \mathrm{L}^2_{\mathbb{H}})$, it follows by Lemma 4.1 that

$$\boldsymbol{\mu}\left(\tilde{K}K_1(K^*K_1)^{-1}(\varphi)\right) = \frac{\boldsymbol{\kappa}(P(\varphi), P_1(\varphi))}{\sqrt{1 - \boldsymbol{\kappa}(P(\varphi), P_1(\varphi))^2}}, \tag{8}$$

since $\begin{pmatrix} K^* \\ \tilde{K} \end{pmatrix} K_1(\varphi)$ is an isometry, for all $\varphi \in \mathbb{T}$.

Proposition 4.2 *Given $P, C, C_1 \in \mathscr{D}$, suppose that $[P, C] \in \mathscr{D}\mathrm{L}^\infty_{\mathbb{T}}$ and that $(I - PC_1)$ is invertible in $\mathscr{D}$. If*

$$\sup_{\varphi \in \mathbb{T}} \boldsymbol{\kappa}(C(\varphi), C_1(\varphi)) < \inf_{\varphi \in \mathbb{T}} \boldsymbol{\rho}(P(\varphi), C(\varphi)) \ \ (\leq 1), \tag{9}$$

then $[P, C_1] \in \mathscr{D}\mathrm{L}^\infty_{\mathbb{T}}$.

Proof: The proof is similar to that of Lemma 3.6 in [7], and is based on the identity

$$\begin{aligned} \tilde{G}K_1 &= \tilde{G}\begin{pmatrix} K & \tilde{K}^* \end{pmatrix}\begin{pmatrix} K^* \\ \tilde{K} \end{pmatrix} K_1, \\ &= (\tilde{G}K)(I + (\tilde{G}K)^{-1}\tilde{G}\tilde{K}^*\tilde{K}K_1(K^*K_1)^{-1})(K^*K_1), \end{aligned} \tag{10}$$

which, with (7–9), leads to the result. For details see [14]. ∎

In addition to this appealing result, it also true that

$$\arcsin \boldsymbol{\rho}(P, C_1) \geq \arcsin \boldsymbol{\rho}(P, C) - \arcsin \boldsymbol{\kappa}(C, C_1), \tag{11}$$

for all $\varphi \in \mathbb{T}$. To see this, note from (10), that

$$\boldsymbol{\tau}(\tilde{G}K_1) \geq \boldsymbol{\tau}(\tilde{G}K)\boldsymbol{\tau}(K^*K_1) - \boldsymbol{\mu}(\tilde{G}\tilde{K}^*)\boldsymbol{\mu}(\tilde{K}K_1).$$

Now define $\phi := \arcsin \boldsymbol{\tau}(\tilde{G}K(\varphi))$ and $\theta := \arcsin \boldsymbol{\mu}(\tilde{K}K_1(\varphi))$ as angles in $[0, \pi/2]$, so that from (6) and (8), it follows that $\cos\phi = \boldsymbol{\mu}(\tilde{G}\tilde{K}^*(\varphi))$, $\cos\theta = \boldsymbol{\tau}(K^*K_1(\varphi))$, and

$$\begin{aligned} \boldsymbol{\tau}(\tilde{G}K_1) &\geq \sin\phi\cos\theta - \cos\phi\sin\theta = \sin(\phi - \theta) \\ &= \sin(\arcsin \boldsymbol{\tau}(\tilde{G}K) - \arcsin \boldsymbol{\mu}(\tilde{K}K_1)). \end{aligned}$$

Equation (11) then follows by taking the arcsin of both sides, since $\arcsin(\cdot)$ is monotonically increasing on $[0, \pi/2]$. Correspondingly, in view of (5),

$$\arcsin \boldsymbol{b}(P, C_1) \geq \arcsin \boldsymbol{b}(P, C) - \arcsin\left(\sup_{\varphi \in \mathbb{T}} \boldsymbol{\kappa}(C(\varphi), C_1(\varphi))\right).$$

The duality between $\boldsymbol{\kappa}(\cdot,\cdot)$ and $\boldsymbol{\rho}(\cdot,\cdot)$, suggested by Proposition 4.2 and (11), is strengthened further by the following result.

Proposition 4.3 *Given $C \in \mathscr{D}$ such that $K^* \in \mathscr{D}\mathrm{L}^\infty_{\mathbb{T}}$, and $C_1 \in \mathscr{D}$, if $[P, C_1] \in \mathscr{D}\mathrm{L}^\infty_{\mathbb{T}}$ for all $P \in \mathscr{D}$ that satisfy $[P, C] \in \mathscr{D}\mathrm{L}^\infty_{\mathbb{T}}$ and $\boldsymbol{b}(P, C) > \beta$, then $\sup_{\varphi \in \mathbb{T}} \boldsymbol{\kappa}(C(\varphi), C_1(\varphi)) \leq \beta$.*

Proof: The proof is similar to that of Lemmas 3.12 and 3.13 in [7]. See [14] for details. ∎

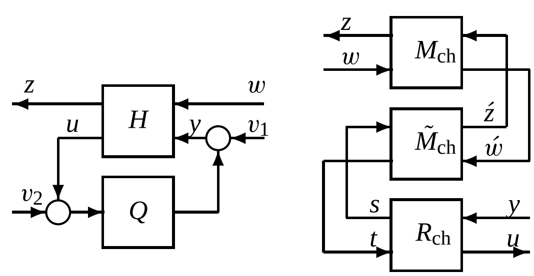
Corollary 4.4 *Given $\beta < 1$, $C \in \mathscr{D}$ such that $K^* \in \mathscr{D}\mathrm{L}^\infty_{\mathbb{T}}$, and $C_1 \in \mathscr{D}$, if*

$$\sup_{\varphi \in \mathbb{T}} \boldsymbol{\kappa}(C(\varphi), C_1(\varphi)) > \beta,$$

then there exist a $P \in \mathscr{D}$, such that $[P, C] \in \mathscr{D}\mathrm{L}^\infty_{\mathbb{T}}$ and $\boldsymbol{b}(P, C) > \beta$, and a frequency $\varphi_0 \in \mathbb{T}$, such that $g(\varphi_0) = k_1(\varphi_0)$ for some $k_1 \in \mathrm{gr}(\mathbf{L}_{C_1})$ and $g \in \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \mathrm{gr}(\mathbf{L}_P)$ (i.e. P is such that $\tilde{G}K_1(\varphi_0)$ has non trivial kernel.)

5 Capturing the sufficient condition in Proposition 4.3

In this section a "J-inner J-co-inner outer" factorisation is used to characterise all (plants) $P \in \mathscr{D}$ that satisfy $[P, \mathrm{C}] \in \mathscr{D}\mathrm{L}^\infty_{\mathbb{T}}$ and $\boldsymbol{b}(P, \mathrm{C}) > \beta$,[6] for a given (controller) $C \in \mathscr{R}$. The required factorisation is adapted from factorisations implicit in the DGKF solution to H^∞ synthesis problems [15]. First, some more notation, concerning linear fractional configurations. Given $H =: \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \in \mathscr{R}$ (resp. $\mathscr{D}$) and $Q \in \mathscr{R}$ (resp. $\mathscr{D}$), consider the general interconnection structure shown in Figure 2(a). The transfer function from (w, v_1, v_2) to (z, y, u) in Figure 2(a) is denoted by $[H, Q]$ (when it exists), and in particular, the transfer function from w to z is denoted by $\mathscr{F}_\ell(H, Q)$. When $(I - H_{22}Q)^{-1} \in \mathscr{R}$ (resp. $\mathscr{D}$) the interconnection is said to be well-posed and $\mathscr{F}_\ell(H, Q) = H_{11} + H_{12}P(I - H_{22}Q)^{-1}H_{21}$. The system $\mathscr{F}_\ell(H, Q)$ is commonly called a lower linear fractional transformation (LFT). Finally, given $\Theta =: \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix} \in \mathscr{R}$ (resp. $\mathscr{D}$),



(a) General Interconnection (b) Chain Scattering

Figure 2: Interconnection Structures

$$\Theta \star H := \begin{pmatrix} \mathscr{F}_\ell(\Theta, H_{11}) & \Theta_{12}(I - \Theta_{22}H_{11})^{-1}H_{12} \\ H_{21}(I - H_{11}\Theta_{22})^{-1}\Theta_{21} & \mathscr{F}_u(H, \Theta_{22}) \end{pmatrix},$$

whenever the inverses exist in $\mathscr{R}$ (resp. $\mathscr{D}$.), where $\mathscr{F}_u(H, Q) := H_{22} + H_{21}Q(I - H_{11}Q)^{-1}H_{12}$. This is known as the Redheffer star product.

Given a controller with transfer function $C \in \mathscr{R}$ define

$$H := \left(\begin{array}{cc|c} 0 & C & C \\ 0 & I & I \\ \hline -I & C & C \end{array}\right),$$

so that given a system with transfer function $P \in \mathscr{D}$,

$$\mathscr{F}_\ell(\mathrm{H}, P) = \begin{pmatrix} \mathrm{C} \\ I \end{pmatrix}(I - P\mathrm{C})^{-1}\begin{pmatrix} -P & I \end{pmatrix} \in \mathscr{D}$$

whenever $(I - P\mathrm{C})$ is invertible in $\mathscr{D}$. Correspondingly, $[P, \mathrm{C}] \in \mathscr{D}\mathrm{L}^\infty_{\mathbb{T}}$ and $\boldsymbol{b}(P, \mathrm{C}) > \beta \Leftrightarrow [\mathrm{H}, P] \in \mathscr{D}\mathrm{L}^\infty_{\mathbb{T}}$ and $\|\mathscr{F}_\ell(\mathrm{H}, P)\|_\infty <$

[6]Recall that the roman upright font is used to denote the corresponding "time-lifted" equivalent.

$\gamma := \frac{1}{\beta}$. Furthermore, setting $H_{\rm sc} := \begin{pmatrix} I & 0 \\ 0 & \gamma I \end{pmatrix} H \begin{pmatrix} \frac{1}{\gamma}I & 0 \\ 0 & \frac{1}{\gamma}I \end{pmatrix} \in \mathscr{R}$, it follows that $[P, \mathrm{C}] \in \mathscr{D}\mathrm{L}^\infty_{\mathbb{T}}$ and $\boldsymbol{b}(P, \mathrm{C}) > \beta \Leftrightarrow [\mathrm{H}_{\rm sc}, P] \in \mathscr{D}\mathrm{L}^\infty_{\mathbb{T}}$ and $\|\mathscr{F}_\ell(\mathrm{H}_{\rm sc}, P)\|_\infty < 1$.

Now suppose that the controller $C \in \mathscr{R}$ has stabilisable and detectable realisation $(\mathcal{A}, \mathcal{B}, \mathcal{C}, 0)$, and let $\mathcal{X} = \mathcal{X}^* \geq 0$ be the stabilising solution to the Generalised Control Algebraic Riccati Equation (GCARE)

$$\mathcal{A}^*\mathcal{X} + \mathcal{X}\mathcal{A} - \mathcal{X}\mathcal{B}\mathcal{B}^*\mathcal{X} + \mathcal{C}^*\mathcal{C} = 0, \tag{12}$$

and let $\mathcal{Z} = \mathcal{Z}^* \geq 0$ be the stabilising solution to the Generalised Filtering Algebraic Riccati Equation (GFARE)[7]

$$\mathcal{A}\mathcal{Z} + \mathcal{Z}\mathcal{A}^* - \mathcal{Z}\mathcal{C}^*\mathcal{C}\mathcal{Z} + \mathcal{B}\mathcal{B}^* = 0. \tag{13}$$

Then following steps similar to those used in [15, Sec. VI], it can be shown that when $\gamma > \sqrt{1 + \mathrm{rad}(\mathcal{X}\mathcal{Z})} =: 1/\boldsymbol{b}_{\rm opt}(C)$,

$$H_{\rm sc} = M \star \tilde{M} \star R, \tag{14}$$

where, $M \in \mathscr{R}\mathrm{H}^\infty_{\mathbb{C}_+}$ is inner with $M_{21}, M_{21}^{-1} \in \mathscr{R}\mathrm{H}^\infty_{\mathbb{C}_+}$, $\tilde{M} \in \mathscr{R}\mathrm{H}^\infty_{\mathbb{C}_+}$ is co-inner with $\tilde{M}_{12}, \tilde{M}_{12}^{-1} \in \mathscr{R}\mathrm{H}^\infty_{\mathbb{C}_+}$, and R is invertible in $\mathscr{R}$ with $R_{12}^{-1}, R_{21}^{-1} \in \mathscr{R}\mathrm{H}^\infty_{\mathbb{C}_+}$. Details of this factorisation are given in Appendix A. Exploiting the properties of M, $\tilde{M}$ and R mentioned above, it is possible to interpret the factorisation (14) within the chain-scattering formalism (see Figure 2(b).) To this end, define

$$\begin{aligned} M_{\rm ch} &:= \begin{pmatrix} M_{12} - M_{11}M_{21}^{-1}M_{22} & M_{11}M_{21}^{-1} \\ -M_{21}^{-1}M_{22} & M_{21}^{-1} \end{pmatrix} \in \mathscr{R}, \\ \tilde{M}_{\rm ch} &:= \begin{pmatrix} \tilde{M}_{12}^{-1} & -\tilde{M}_{12}^{-1}\tilde{M}_{11} \\ \tilde{M}_{22}\tilde{M}_{12}^{-1} & \tilde{M}_{21} - \tilde{M}_{22}\tilde{M}_{12}^{-1}\tilde{M}_{11} \end{pmatrix} \in \mathscr{R} \\ R_{\rm ch} &:= \begin{pmatrix} R_{12} - R_{11}R_{21}^{-1}R_{22} & R_{11}R_{21}^{-1} \\ -R_{21}^{-1}R_{22} & R_{21}^{-1} \end{pmatrix} \in \mathscr{R}, \end{aligned}$$

noting that

$$R_{\rm ch}^{-1} := \begin{pmatrix} R_{12}^{-1} & -R_{12}^{-1}R_{11} \\ R_{22}R_{12}^{-1} & R_{21} - R_{22}R_{12}^{-1}R_{11} \end{pmatrix} \in \mathscr{R}.$$

Since $M_{21}^{-1} \in \mathscr{R}\mathrm{H}^\infty_{\mathbb{C}_+}$, $\tilde{M}_{12} \in \mathscr{R}\mathrm{H}^\infty_{\mathbb{C}_+}$, $R_{21}^{-1} \in \mathscr{R}\mathrm{H}^\infty_{\mathbb{C}_+}$ and $R_{12}^{-1} \in \mathscr{R}\mathrm{H}^\infty_{\mathbb{C}_+}$, it follows that $M_{\rm ch} \in \mathscr{R}\mathrm{H}^\infty_{\mathbb{C}_+}$, $\tilde{M}_{\rm ch} \in \mathscr{R}\mathrm{H}^\infty_{\mathbb{C}_+}$ and $R_{\rm ch}, R_{\rm ch}^{-1} \in \mathscr{R}\mathrm{H}^\infty_{\mathbb{C}_+}$. Furthermore, using the fact that M is inner and that $\tilde{M}$ is co-inner, it follows that

$$M_{\rm ch}^*(\varphi)J_1M_{\rm ch}(\varphi) = J_2 \quad \text{and} \quad \tilde{M}_{\rm ch}(\varphi)J_2\tilde{M}_{\rm ch}^*(\varphi) = J_3 \tag{15}$$

for all $\varphi \in j\mathbb{R}$, where J_1, J_2 and J_3 are signature matrices of the form $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, with appropriate partitioning conformal to that of $M_{\rm ch}$ and $\tilde{M}_{\rm ch}$. Note that $J_nJ_n = I$, for each $n = 1, 2, 3$. In summary, $M_{\rm ch}$ is J-inner, $\tilde{M}_{\rm ch}$ is J-co-inner and $R_{\rm ch}$ is outer.

Now, given any signals $y, u \in \mathrm{L}^2_{j\mathbb{R}}$, define $\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right) := R_{\rm ch}\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)$. Since multiplication by $\tilde{M}_{\rm ch}$ is surjective (cf. (15)), there exist signals $\acute{z}, \acute{w} \in \mathrm{L}^2_{j\mathbb{R}}$ such that $\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right) = \tilde{M}_{\rm ch}\left(\begin{smallmatrix} \acute{z} \\ \acute{w} \end{smallmatrix}\right)$. Furthermore, the corresponding signals $\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right) := M_{\rm ch}\left(\begin{smallmatrix} \acute{z} \\ \acute{w} \end{smallmatrix}\right) \in \mathrm{L}^2_{j\mathbb{R}}$ satisfy

$$\left(\begin{smallmatrix} z \\ u \end{smallmatrix}\right) = H_{\rm sc}\left(\begin{smallmatrix} w \\ y \end{smallmatrix}\right). \tag{16}$$

[7]The required stabilising solutions $\mathcal{X}$ and $\mathcal{Z}$ exist since $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is stabilisable and detectable [10, Corol. 13.8]

This chain-scattering characterisation of the signals in (16), in conjunction with the properties of $M_{\rm ch}$, $\tilde{M}_{\rm ch}$ and $R_{\rm ch}$, gives rise to the following result, which characterises (in terms of a necessary condition) all transfer functions $P \in \mathscr{D}$ that satisfy $[P, \mathrm{C}] \in \mathscr{D}\mathrm{L}^\infty_{\mathbb{T}}$ and $\boldsymbol{b}(P, \mathrm{C}) := \inf_{\varphi \in \mathbb{T}} \boldsymbol{\rho}(P(\varphi), \mathrm{C}(\varphi)) > \beta$

Theorem 5.1 *Given a controller $C \in \mathscr{R}$ and a number $\beta < \boldsymbol{b}_{\rm opt}(C)$, let $R \in \mathscr{R}$ be defined as in (19) with $\gamma = \frac{1}{\beta}$. Moreover, let $P \in \mathscr{D}$ be such that $[P, \mathrm{C}] \in \mathscr{D}\mathrm{L}^\infty_{\mathbb{T}}$ with $\boldsymbol{b}(P, \mathrm{C}) > \beta$. Then for any $\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \in \mathrm{gr}(\mathbf{L}_P) \subset \mathrm{L}^2_{\mathbb{T}}$,*

$$\langle \mathrm{R}_{\rm ch}\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)(\varphi), J_3\mathrm{R}_{\rm ch}\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)(\varphi)\rangle < 0 \ \ \textit{for all } \varphi \in \mathbb{T}, \tag{17}$$

where J_3 is a signature matrix of the form $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ partitioned conformably with $\mathrm{R}_{\rm ch}$.[8]

Proof: For $\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \in \mathrm{gr}(\mathbf{L}_P)$, let $\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right) = \mathrm{R}_{\rm ch}\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \acute{z} \\ \acute{w} \end{smallmatrix}\right) = J_2\tilde{\mathrm{M}}_{\rm ch}^*J_3\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right)$ so that

$$\begin{aligned} \langle \left(\begin{smallmatrix} \acute{z} \\ \acute{w} \end{smallmatrix}\right)(\varphi), J_2\left(\begin{smallmatrix} \acute{z} \\ \acute{w} \end{smallmatrix}\right)(\varphi)\rangle &= \langle J_2\tilde{\mathrm{M}}_{\rm ch}^*J_3\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right)(\varphi), \tilde{\mathrm{M}}_{\rm ch}^*J_3\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right)(\varphi)\rangle \\ &= \langle \left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right)(\varphi), J_3\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right)(\varphi)\rangle \quad \text{for all } \varphi \in \mathbb{T}. \end{aligned}$$

Then note that $\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right) = \mathrm{M}_{\rm ch}\left(\begin{smallmatrix} \acute{z} \\ \acute{w} \end{smallmatrix}\right)$ satisfies $\left(\begin{smallmatrix} z \\ u \end{smallmatrix}\right) = \mathrm{H}_{\rm sc}\left(\begin{smallmatrix} w \\ y \end{smallmatrix}\right)$ (cf. (16)), and hence, using (15), that

$$\begin{aligned} &\langle \mathrm{R}_{\rm ch}\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)(\varphi), J_3\mathrm{R}_{\rm ch}\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)(\varphi)\rangle = \langle \left(\begin{smallmatrix} \acute{z} \\ \acute{w} \end{smallmatrix}\right)(\varphi), \mathrm{M}_{\rm ch}^*J_1\mathrm{M}_{\rm ch}\left(\begin{smallmatrix} \acute{z} \\ \acute{w} \end{smallmatrix}\right)(\varphi)\rangle \\ &= \langle \mathrm{M}_{\rm ch}\left(\begin{smallmatrix} \acute{z} \\ \acute{w} \end{smallmatrix}\right)(\varphi), J_1\mathrm{M}_{\rm ch}\left(\begin{smallmatrix} \acute{z} \\ \acute{w} \end{smallmatrix}\right)(\varphi)\rangle = \langle \left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right)(\varphi), J_1\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right)(\varphi)\rangle < 0, \end{aligned}$$

since $\boldsymbol{b}(P, \mathrm{C}) > \beta \Rightarrow \|\mathscr{F}_\ell(\mathrm{H}_{\rm sc}, P)\|_\infty < 1 \Rightarrow \|z\| < \|w\|$. ■

Remark 5.2 *If $P \in \mathscr{R}$ is such that $\mathscr{F}_\ell(H_{\rm sc}, P) \in \mathscr{R}\mathrm{H}^\infty_{\mathbb{C}_+}$ and $\boldsymbol{b}(P, C) > \beta$, then $Q := \mathscr{F}_\ell(R, P) \in \mathscr{R}\mathrm{H}^\infty_{\mathbb{C}_+}$ and $\|Q\|_\infty < 1$ [14, 15]. The LFT here is well-posed since the $\mathcal{D}_{22}$-term in any realisation of R is zero. However, for arbitrary $P \in \mathscr{D}$, it is not always possible to guarantee well-posedness of $\mathscr{F}_\ell(\mathrm{R}, P)$ (in $\mathscr{D}$.)*

6 SD approximation in $\boldsymbol{\kappa}(\cdot, \cdot)$

The purpose of this section is to present the main result of the paper, which follows directly from Corollary 4.4 and Theorem 5.1. This leads to a procedure for SD approximation (in $\boldsymbol{\kappa}$) of a given $C \in \mathscr{R}$.

Theorem 6.1 *Given $C \in \mathscr{R}$, with stabilisable and detectable realisation $(\mathcal{A}, \mathcal{B}, \mathcal{C}, 0)$, and $\beta < \boldsymbol{b}_{\rm opt}(C)$, define $R \in \mathscr{R}$ as in (19) with $\gamma = \frac{1}{\beta}$. If $C_1 \in \mathscr{D}$ satisfies*

$$\langle \mathrm{R}_{\rm ch}\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)(\varphi), J_3\mathrm{R}_{\rm ch}\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)(\varphi)\rangle \geq 0 \ \ \textit{for all } \varphi \in \mathbb{T} \tag{18}$$

and all $\left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right) \in \mathrm{gr}(\mathbf{L}_{C_1})$, then $\sup_{\varphi \in \mathbb{T}} \boldsymbol{\kappa}(\mathrm{C}(\varphi), C_1(\varphi)) \leq \beta$.

Proof: Suppose that $\sup_{\varphi \in \mathbb{T}} \boldsymbol{\kappa}(\mathrm{C}(\varphi), C_1(\varphi)) > \beta$, so that by Corollary 4.4,[9] there exist a $P \in \mathscr{D}$ such that $\boldsymbol{b}(P, \mathrm{C}) > \beta$, and a frequency $\varphi_0 \in \mathbb{T}$, such that $k_1(\varphi_0) = g(\varphi_0)$ for some $k_1 \in \mathrm{gr}(\mathbf{L}_{C_1})$ and $g \in \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)\mathrm{gr}(\mathbf{L}_P)$. Now note, by Theorem 5.1, that

$$\langle \mathrm{R}_{\rm ch}\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)g(\varphi_0), J_3\mathrm{R}_{\rm ch}\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)g(\varphi_0)\rangle_{\mathrm{L}^2_{\mathbb{H}}} < 0,$$

[8]Recall that upright roman font denotes "time-lifted" equivalent in $\mathscr{D}$.

[9]Note that the conditions of the corollary hold since a normalised right graph symbol for $\mathrm{C}(\in \mathscr{D})$ satisfying $\mathrm{K}^* \in \mathscr{D}\mathrm{L}^\infty_{\mathbb{T}}$ exists. To see this note that a realisation for K with invertible $\mathcal{A}$-term can always be obtained by "time-lifting" a realisation of a normalised right graph symbol for $C(\in \mathscr{R})$.

since $\boldsymbol{b}(P, C) > \beta$. But from (18)

$$\begin{aligned}&\langle \mathrm{R}_{\mathrm{ch}} \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right) g(\varphi_0),\, J_3 \mathrm{R}_{\mathrm{ch}} \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right) g(\varphi_0)\rangle \\ &\quad = \langle \mathrm{R}_{\mathrm{ch}} \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right) k_1(\varphi_0),\, J_3 \mathrm{R}_{\mathrm{ch}} \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right) k_1(\varphi_0)\rangle \geq 0,\end{aligned}$$

which contradicts the previous inequality. ∎

Now, given $C \in \mathscr{R}$ and a pre-filtered, periodic SD controller, with corresponding transfer function $C_{\mathrm{sd}} \in \mathscr{D}$,[10] suppose that the LFT configuration $[\mathrm{R}^{-1}, C_{\mathrm{sd}}] \in \mathscr{D}\mathrm{H}^{\infty}_{\mathbb{D}}$ with $\|\Phi\|_\infty \leq 1$, where $\Phi := \mathscr{F}_\ell(\mathrm{R}^{-1}, C_{\mathrm{sd}})$ and R is defined as in Theorem 6.1. Then for every $\left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right) \in \mathrm{gr}(\mathbf{L}_{C_{\mathrm{sd}}})$,

$$\langle \mathrm{R}_{\mathrm{ch}}\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)(\varphi),\, J_3 \mathrm{R}_{\mathrm{ch}}\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)(\varphi)\rangle_{\mathrm{L}^2_{\mathbb{H}}} \geq 0 \ \text{ for all } \varphi \in \mathbb{T},$$

and thus $\sup_{\varphi \in \mathbb{T}} \boldsymbol{\kappa}(\mathrm{C}(\varphi), C_{\mathrm{sd}}(\varphi)) \leq \beta$, by Theorem 6.1. Also, given any $P \in \mathscr{R}$ which satisfies $[\mathrm{P}, \mathrm{C}] \in \mathscr{D}\mathrm{H}^{\infty}_{\mathbb{D}}$ and $\boldsymbol{b}(\mathrm{P}, \mathrm{C}) > \beta$, $\mathrm{Q} := \mathscr{F}_\ell(\mathrm{R}, \mathrm{P}) \in \mathscr{D}\mathrm{H}^{\infty}_{\mathbb{D}}$ and $\|\mathrm{Q}\|_\infty < 1$ (cf. Remark 5.2.) As such it follows by standard small-gain arguments that $[\mathrm{Q}, \Phi] \in \mathscr{D}\mathrm{H}^{\infty}_{\mathbb{D}}$ (see Figure 3.) Correspondingly, since R_{ch} is a unit in $\mathscr{D}\mathrm{H}^{\infty}_{\mathbb{D}}$,

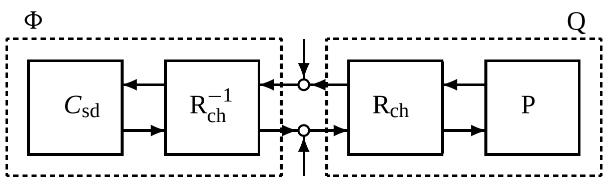


Figure 3: Chain-scattered $[\Phi, \mathrm{Q}]$

$[\mathrm{P}, C_{\mathrm{sd}}] \in \mathscr{D}\mathrm{H}^{\infty}_{\mathbb{D}}$ (i.e. can just move R_{ch} across the disturbances in Figure 3.) In conjunction with the bounds (11), this suggests the following SD approximation procedure for a given controller $C \in \mathscr{R}$:

Procedure 1 Find the smallest $\beta < \boldsymbol{b}_{\mathrm{opt}}(C)$, such that (for a fixed sampling period and pre-filter) there exists a pre-filtered SD controller satisfying $[\mathrm{R}^{-1}, C_{\mathrm{sd}}] \in \mathscr{D}\mathrm{H}^{\infty}_{\mathbb{D}}$ and $\|\mathscr{F}_\ell(\mathrm{R}^{-1}, C_{\mathrm{sd}})\|_\infty \leq 1$, where R (depending on C and β only) is defined in (19) with $\gamma := \frac{1}{\beta}$, and $C_{\mathrm{sd}} \in \mathscr{D}$ denotes the transfer function of the SD approximation to be synthesised. This can be achieved via standard H^∞ SD synthesis methods [8, 16, 17].

Remark 6.2 *Given a plant $P \in \mathscr{R}$ and a controller $C \in \mathscr{R}$ suppose that $[P, C] \in \mathscr{R}\mathrm{H}^{\infty}_{\mathbb{C}_+}$. An SD approximation obtained via the procedure above would only be useful if the corresponding smallest β achieved is much less than $\boldsymbol{b}(P, C)$ (cf. (11).) If this is not the case, then the pre-filter or sampling period chosen at the beginning of the procedure must be re-designed.*

A Factorisation of H_{sc}

In this section details of the factorisation (14) are presented. Define:

$$M := \left(\begin{array}{c|cc} \mathcal{A} + \mathcal{B}\mathcal{F}_\infty & \begin{pmatrix} 0 & \frac{1}{\gamma}\mathcal{B} \end{pmatrix} & \frac{\sqrt{\gamma^2-1}}{\gamma}\mathcal{B} \\ \hline \begin{pmatrix} \mathcal{C} \\ \mathcal{F}_\infty \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\gamma} I \end{pmatrix} & \begin{pmatrix} 0 \\ \frac{\sqrt{\gamma^2-1}}{\gamma} I \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} I & 0 \\ 0 & \frac{\sqrt{\gamma^2-1}}{\gamma} I \end{pmatrix} & \begin{pmatrix} 0 \\ -\frac{1}{\gamma} I \end{pmatrix} \end{array}\right),$$

with $\mathcal{F}_\infty := -\mathcal{B}^*\mathcal{X}$;

$$\tilde{M} := \left(\begin{array}{c|cc} \mathcal{A}_{\mathrm{tmp}} + \gamma \mathcal{L}_{\mathrm{tmp}}\mathcal{C} & \begin{pmatrix} -\mathcal{L}_{\mathrm{tmp}} & \frac{1}{\sqrt{\gamma^2-1}}\mathcal{B} \end{pmatrix} & \frac{\gamma}{\sqrt{\gamma^2-1}}\mathcal{Y}_{\mathrm{tmp}}\mathcal{F}_\infty^* \\ \hline \frac{-\gamma}{\sqrt{\gamma^2-1}}\mathcal{F}_\infty & \begin{pmatrix} 0 & 0 \end{pmatrix} & I \\ \gamma\mathcal{C} & \begin{pmatrix} -I & 0 \end{pmatrix} & 0 \end{array}\right),$$

with

$$\mathcal{A}_{\mathrm{tmp}} := \mathcal{A} - \frac{1}{\gamma^2 - 1}\mathcal{B}\mathcal{F}_\infty, \quad \mathcal{L}_{\mathrm{tmp}} := -\gamma\mathcal{Y}_{\mathrm{tmp}}\mathcal{C}^*,$$

$$\mathcal{Y}_{\mathrm{tmp}} = \mathcal{Y}_\infty(I - \mathcal{X}\mathcal{Y}_\infty)^{-1} \geq 0, \quad \mathcal{Y}_\infty := \frac{1}{\gamma^2 - 1}\mathcal{Z};$$

and

$$R := \left(\begin{array}{c|cc} \mathcal{A}_R & -\mathcal{L}_{\mathrm{tmp}} & \mathcal{B}_{2R} \\ \hline \frac{-\gamma}{\sqrt{\gamma^2-1}}\mathcal{F}_\infty & 0 & \frac{1}{\sqrt{\gamma^2-1}} I \\ \gamma\mathcal{C} & I & 0 \end{array}\right), \tag{19}$$

with

$$\mathcal{A}_R := \mathcal{A}_{\mathrm{tmp}} + \frac{\gamma^2}{\gamma^2 - 1}\mathcal{Y}_{\mathrm{tmp}}\mathcal{F}_\infty^*\mathcal{F}_\infty, \ \mathcal{B}_{2R} := \frac{\gamma}{\gamma^2-1}\mathcal{B} - \frac{\gamma}{\gamma^2-1}\mathcal{Y}_{\mathrm{tmp}}\mathcal{F}_\infty^*.$$

It follows readily that, $H_{\mathrm{sc}} = M \star \tilde{M} \star R$, $M \in \mathscr{R}\mathrm{H}^{\infty}_{\mathbb{C}_+}$ is inner with $M_{21}, M_{21}^{-1} \in \mathscr{R}\mathrm{H}^{\infty}_{\mathbb{C}_+}$, $\tilde{M} \in \mathscr{R}\mathrm{H}^{\infty}_{\mathbb{C}_+}$ is co-inner with $\tilde{M}_{12}, \tilde{M}_{12}^{-1} \in \mathscr{R}\mathrm{H}^{\infty}_{\mathbb{C}_+}$, and R is invertible in $\mathscr{R}$ with $R_{12}^{-1}, R_{21}^{-1} \in \mathscr{R}\mathrm{H}^{\infty}_{\mathbb{C}_+}$. See [14] for details.

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[10] See end of Section 2 for details of how to construct this for a time-domain representation.