Robust Kalman Filter Design for Hybrid Systems with Norm-Bounded Unknown Nonlinearities[1]

Peng Shi[2], Mehmet Karan[3] and Yalcin Kaya[4]

Abstract

This paper considers the filtering problem for a class of linear hybrid systems with nonlinear uncertainties and Markovian jump parameters. The unknown nonlinearities in the system are time-varying and norm-bounded. First, we show the equivalence of the norm bounded linear and nonlinear uncertainty sets. Then, instead of the original hybrid linear system with nonlinear uncertainties, we consider the same system with linear uncertainties. By using a Riccati equation approach for this new system, a robust filter is designed using two sets of coupled Riccati-like equations such that the estimation error is guaranteed to have an upper bound.

1 Problem Formulation

Consider the following class of dynamical systems $\mathcal{S}_{NL}$

$$\mathcal{S}_{NL} \quad : \quad \left\{ \begin{array}{rcl} \dot{x}(t) & = & A(\eta_t)x(t) + f(t, x(t), \eta_t) + w(t, \eta_t), \\ y(t) & = & C(\eta_t)x(t) + g(t, x(t), \eta_t) + v(t) \end{array} \right. \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $y(t) \in \mathbb{R}^r$ is the measurement, $w(t, \eta_t) \in \mathbb{R}^n$ and $v(t) \in \mathbb{R}^r$ are the process and measurement noises which are assumed to be zero-mean Gaussian with covariance matrices $W(\eta_t)$ and $V(\eta_t)$ respectively that are independent from each other. The system mode $\{\eta_t, \ t \geq 0\}$ is a time homogeneous Markov process with right-continuous trajectories and taking values in a finite set $\Gamma = \{1, 2, \cdots, m\}$ with stationary transition probabilities $Pr(\eta_{t+h} = j | \eta_t = i)$ that are equal to $\lambda_{ij} h + o(h)$ for $i \neq j$ and $(1 + \lambda_{ii} h + o(h))$ for $i = j$ where $h > 0$, $\lim_{h \to 0} \frac{o(h)}{h} = 0$. $\lambda_{ij} \geq 0$ is the transition rate from mode i at time t to mode j at time $t + h$, and $\lambda_{ii} = -\sum_{j=1, j \neq i}^{m} \lambda_{ij}$. Initial conditions are given by $x(0) = x_0, \eta_0 = i$ for some x_0 and $i \in \Gamma$. The vector valued functions $f(t, x(t), \eta_t)$ and $g(t, x(t), \eta_t)$, for any $\eta_t \in \Gamma$, are unknown and they represent time-varying parametric uncertainties. These uncertainties satisfy $||f(t, x(t), \eta_t)|| \leq a(\eta_t)||x(t)||$ and $||g(t, x(t), \eta_t)|| \leq c(\eta_t)||x(t)||$ for all $x(t) \in \mathbb{R}^n$ where $a(\eta_t)$ and $c(\eta_t)$ are known constant non-negative numbers for each mode $\eta_t = i \in \Gamma$.

In this paper, we consider the problem of state estimation for linear systems with Markovian jump parameters subject to real time-varying nonlinear uncertainties. The next section shows the equivalence of the two sets of norm bounded linear and nonlinear uncertainties. Using this equivalence result, we design a guaranteed cost robust filter in Section 3 for the problem considered in this paper, via the solutions of two sets of coupled Riccati equations.

2 Equivalence of Linear and Nonlinear Uncertainty Sets

Define the following admissible nonlinear and linear uncertainty sets

$$\begin{array}{rcl} \Omega_x(t, x(t), \eta_t) & = & \{ \ f(t, x(t), \eta_t) \ : \\ & & \qquad ||f(t, x(t), \eta_t)|| \ \leq \ a(\eta_t)||x(t)|| \} \, , \\ \Omega_y(t, x(t), \eta_t) & = & \{ \ g(t, x(t), \eta_t) \ : \\ & & \qquad ||g(t, x(t), \eta_t)|| \ \leq \ c(\eta_t)||x(t)|| \} \, , \\ \overline{\Omega}_x(t, x(t), \eta_t) & \triangleq & \{ \ a(\eta_t) \, M(t, \eta_t) \, x(t) \ : \\ & & M(t, \eta_t) \in \mathbb{R}^{n \times n}, \rho(M(t, \eta_t)) \ \leq \ 1 \} \, , \\ \overline{\Omega}_y(t, x(t), \eta_t) & \triangleq & \{ \ c(\eta_t) \, N(t, \eta_t) \, x(t) \ : \\ & & N(t, \eta_t) \in \mathbb{R}^{n \times n}, \rho(N(t, \eta_t)) \ \leq \ 1 \} \, . \end{array}$$

Utilizing a similar technique used in [1], we establish the following relationships between these linear and nonlinear uncertainty sets.

Lemma 1

$$\begin{array}{rcl} \Omega_x(t, x(t), \eta_t) & = & \overline{\Omega}_x(t, x(t), \eta_t) \\ \Omega_y(t, x(t), \eta_t) & = & \overline{\Omega}_y(t, x(t), \eta_t). \end{array}$$

Now, let us define the system $\mathcal{S}_L$ below

$$\mathcal{S}_L : \left\{ \begin{array}{rcl} \dot{x}(t) & = & (A(\eta_t) + a(\eta_t)M(t, \eta_t)) \, x(t) + w(t, \eta_t), \\ y(t) & = & (C(\eta_t) + c(\eta_t)N(t, \eta_t)) \, x(t) + v(t) \end{array} \right. \tag{2}$$

The unknown matrices $M(t, \eta_t)$ and $N(t, \eta_t)$ represent time-varying parametric uncertainties and satisfy

[1] This work is supported by Divisional Research Management Fund, University of South Australia.

[2] School of Mathematics, The University of South Australia. Present address: Land Operations Division, Defence Science and Technology Organisation, PO Box 1500, Salisbury, SA 5108, Australia. Email: peng.shi@dsto.defence.gov.au

[3] Cooperative Research Center for Sensor Signal and Information Processing (CSSIP), 1 Warrendi Rd, Mawson Lakes, SA 5095, Australia. Email: mehmet@cssip.edu.au

[4] School of Mathematics, The University of South Australia, The Levels Campus, Mawson Lakes, SA 5095, Australia. Email: yalcin.kaya@unisa.edu.au

$M^T(t,\eta_t)M(t,\eta_t) \leq I$ and $N^T(t,\eta_t)N(t,\eta_t) \leq I$ for all t, $x(t)$ and $\eta_t \in \Gamma$. We also assume that there exist known matrices $\bar{M}_i, \bar{N}_i, E_i$ such that $\begin{bmatrix} M_i(t) \\ N_i(t) \end{bmatrix} = \begin{bmatrix} \bar{M}_i(t) \\ \bar{N}_i(t) \end{bmatrix} F_i(t)E_i(t)$ where $F_i(t)$ is an unknown matrix satisfying $F_i^T(t)F_i(t) \leq I$ for all t and $i \in \Gamma$.

For the robust filtering problem investigated in this paper, Lemma 1 suggests that instead of the system $\mathcal{S}_{NL}$ in (1) with the nonlinear uncertainties from the sets Ω_x and Ω_y, one can consider the system $\mathcal{S}_L$ with only linear uncertainties from sets $\overline{\Omega}_x$ and $\overline{\Omega}_y$. Lemma 1 also suggests that the robust filter designed for the linear system $\mathcal{S}_L$ subject to linear uncertainties would work for $\mathcal{S}_{NL}$. Therefore, our attention will be focused on designing a robust filter for the system $\mathcal{S}_L$.

3 Robust Filtering

Let $x(t,x_0,\eta_0)$ denote the trajectory of the state $x(t)$ from the initial state x_0 with an initial system mode η_0. In order to solve the robust state estimation problem, we assume that the nominal system $\mathcal{S}_{NL}$ in (1) (setting $f(t,x(t),\eta_t) \equiv \mathbf{0}$) is *stochastically stable*, i.e. $\int_0^\infty \mathsf{E}\left\{||x(t,x_0,\eta_0)||^2\right\}\ dt < \infty$ when $w(t) \equiv 0$. From [2], it is known that $\mathcal{S}_{NL}$ is stochastically stable if and only if there exists a set of positive definite matrices P_i satisfying $A_i^T P_i + P_i A_i + \sum_{j=1}^m \lambda_{ij} P_j + V_i = 0$ for any given set of positive definite matrices V_i for $i \in \Gamma$.

Given a hybrid system $\mathcal{S}$ with a state vector $x(t)$ and a system mode $\eta_t = i \in \Gamma$, an estimator $\mathcal{F}$ with the state equations $\mathcal{F} : \dot{\hat{x}}(t) = G_i \hat{x}(t) + K_i y(t), i \in \Gamma, \hat{x}(0) = x_0$, is said to be a guaranteed cost state estimator for this system $\mathcal{S}$ if there exists a constant symmetric matrix $P \geq 0$ such that $\mathsf{E}\left\{(x-\hat{x})(x-\hat{x})^T\right\} \leq P$. The following theorem gives the parameters of a stochastically stable estimator such that the error covariance of state $x(t)$ and its estimate $\hat{x}(t)$ is bounded for all admissible uncertainties.

Theorem 1 *Let the system $\mathcal{S}_L$ in (2) be stochastically quadratically stable. Then, the estimator $\mathcal{F}$ given by $\mathcal{F} : \dot{\hat{x}}(t) = G_i\hat{x}(t) + K_i y(t)$ is a stochastic stable quadratic state estimator with guaranteed cost $\mathsf{E}\left\{(x-\hat{x})^T(x-\hat{x})\right\} \leq tr(Q)$ where $G_i = \bar{A}_i - K_i\bar{C}_i$, and $K_i = \left(Q_i\bar{C}_i^T + \frac{1}{\varepsilon_i}a_i c_i \left(I + Z_i P_i^{-1}\right)\bar{M}_i\bar{N}_i^T\right)\bar{R}_i^{-1}$. Here Q is defined by $Q = Q_j$ where $j = \arg\max_{i\in\Gamma}\ tr(Q_i)$. The matrices P_i, Q_i and Z_i are found from the following equations such that P_i and $\begin{bmatrix} P_i & Z_i^T \\ Z_i & Q_i \end{bmatrix}$ are positive definite.*

$$
\begin{aligned}
&A_i P_i + P_i A_i^T P_i + \varepsilon_i P_i E_i^T E_i P_i \\
&+(W_i + \frac{a_i^2}{\varepsilon_i}\bar{M}_i\bar{M}_i^T) + \sum_{j=1}^m \lambda_{ij} P_j \;=\; 0, \qquad (3) \\
&\bar{A}_i Q_i + Q_i \bar{A}_i^T + \bar{W}_i P_i^{-1}(Q_i + Z_i^T) \\
&+(Q_i + Z_i)P_i^{-1}\bar{W}_i + \varepsilon_i(Z_i E_i^T E_i Q_i + Q_i E_i^T E_i Z_i^T) \\
&+\sum_j \lambda_{ij}(Z_j P_i^{-1} Q_i + Q_i P_i^{-1} Z_j^T) + W_i \\
&+2\varepsilon_i Z_i E_i^T E_i Z_i^T + \sum_{j=1}^m \lambda_{ij}(Z_j P_i^{-1} Z_i^T + Z_i P_i^{-1} Z_j^T) \\
&-\left(Q_i\bar{C}_i^T + \frac{1}{\varepsilon_i}a_i c_i (I + Z_i P_i^{-1})\bar{M}_i\bar{N}_i^T\right)\bar{R}_i^{-1} \\
&\times\left(Q_i\bar{C}_i^T + \frac{1}{\varepsilon_i}a_i c_i (I + Z_i P_i^{-1})\bar{M}_i\bar{N}_i^T\right)^T = 0, \qquad (4) \\
&\bar{A}_i \;=\; A_i + \left(W_i + \frac{1}{\varepsilon_i}a_i^2\bar{M}_i\bar{M}_i^T\right)P_i^{-1} + \varepsilon_i Z_i E_i^T E_i \\
&+\sum_j \lambda_{ij} Z_j P_i^{-1}, \quad \bar{C}_i \;=\; C_i + \frac{1}{\varepsilon_i}a_i c_i \bar{N}_i \bar{M}_i^T P_i, \\
&\bar{R}_i \;=\; R + \frac{1}{\varepsilon_i}c_i^2\bar{N}_i\bar{N}_i^T, \quad \bar{W}_i \;=\; W_i + \frac{1}{\varepsilon_i}a_i c_i \bar{M}_i \bar{M}_i^T.
\end{aligned}
$$

The first equation (3) can be solved using linear matrix inequality (LMI) techniques detailed in [3]. Several approaches can be applied to solve the equation in (4) since it is more complicated. For example, Z_i can be chosen to be the zero matrix to simplify this equation, but special care has to be taken to make sure that the matrix $\bar{A}_i$ is stable for all $i \in \Gamma$. If this is not the case, different values of ε_i can be chosen so that $\bar{A}_i$ becomes stable. Also, higher dimensional LMIs may be used to solve (4).

4 Conclusion

This paper investigated stability and filtering problems for a class of linear continuous-time systems with nonlinear uncertainties and Markovian jumping parameters. It has been shown that the sets of linear and nonlinear uncertainties are equivalent, hence these problems have been tackled via equivalent linear hybrid systems with linear uncertainties. It has been shown that the robust filtering problem is solvable if two sets of coupled algebraic Riccati-like equations have symmetric positive definite solutions. We derived a sufficient condition to solve these coupled Riccati-like equations such that the underlying system is stochastically stable. Finally, we obtained the parameters of a stochastic quadratic estimator which guarantees both the stability and boundedness of the estimation error dynamics.

References

[1] P. Shi and C. Y. Kaya, “Robust Kalman filtering for continuous-time systems with norm-bounded nonlinear uncertainties,” *IMA J. Mathe. Contr. Inform*, 2000. to appear.

[2] X. Feng, K. A. Loparo, Y. Ji, and H. J. Chizeck, “Stochastic stability properties of jump linear systems,” *IEEE Trans. Automat. Control*, vol. 37, no. 1, pp. 38– 53, 1992.

[3] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, vol. 15 of *Studies in Applied Mathematics*. Philadelphia, PA: SIAM, June 1994.