Comments on $\mathcal{L}_2$-gain Analysis of Systems with Persistent Outputs

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Abstract

In this paper, a generalization of the $\mathcal{L}_2$-gain inequality utilizing "point to set" distances [11, 10] is applied to cope with systems with persistent excitation. This generalization is compared with notions of *power gain* introduced in [4]. In particular it is found that the gain inequality associated with "point to set" distances freely admits a definition of available storage and allows the standard Hamilton-Jacobi-Bellman PDE to be generalized in a simple way. Issues concerning stability are also addressed. The nature of the generalized $\mathcal{L}_2$-gain inequality admits a simple understanding of stability, which is an improvement over the power gain generalization. However, casting optimal control problems is less satisfying in that it is more difficult to minimize the power generated by the system.

1 Introduction

Recent work [11, 10] has been directed towards generalizing finite gain analysis techniques to include nonlinear systems with persistent excitation such as limit cycles and chaos. Another less general technique considered in [4] accounts for persistent excitation by the addition of a linear term in time to the right hand side of the $\mathcal{L}_2$-gain inequality. This yields the following power gain inequality

$$\|z(s)\|^2_{\mathcal{L}_2[0,T]} \ \leq\ \gamma^2\|w(s)\|^2_{\mathcal{L}_2[0,T]} + \lambda T + \beta(x_0), \tag{1}$$

where w is the input or disturbance affecting the system, z is the output, and $\beta(x_0)$ is a measure of the energy stored by the system when initialized at state x_0. This definition is useful in the sense that the power of oscillations in the output can be directly gauged via the power bias λ. Furthermore, an optimal control problem can be defined to minimize this parameter [3]. However, one of the problems with the power gain inequality is that it is more difficult to quantify stability than is the case with the standard $\mathcal{L}_2$-gain property [13] or with properties such as "practical" ISS (see for example [11]). Furthermore, characterizing notions of stabilizing and antistabilizing solutions of the corresponding power gain PDEs [5] is more difficult due to multiple possible definitions of available storage for the power gain problem.

In this paper, we consider another simple generalization of the $\mathcal{L}_2$-gain inequality which uses "point to set" distances instead of euclidean norms (see for example [11]). In particular, we modify the $\mathcal{L}_2$-gain inequality by replacing the euclidean norm of z on the left hand side of the inequality by a distance measure from a set. That is, for some set Z in the output space, we analyse the case where the following "point to set" distance gain inequality holds:

$$\|z\|^2_{\mathcal{L}_{2,Z}[0,T]} \leq \gamma^2\|w\|^2_{\mathcal{L}_2[0,T]} + \beta(x_0) \tag{2}$$

where $\|z\|^2_{\mathcal{L}_{2,Z}[0,T]} := \int_0^T \|z(s)\|^2_Z\ ds$ and $\|z(s)\|_Z := \inf_{\zeta\in Z}|z(s)-\zeta|$.

Inequality (2) is useful in that it admits a simple unique definition of available storage, whilst the characterization of stabilizing and antistabilizing solutions of the corresponding PDEs is made much simpler. The downside is that the set Z could be difficult to determine and is not readily minimized in a control problem defined as in the power gain case.

The following class of systems is considered:

$$G : \begin{cases} \dot{x} &= A(x) + B(x)w, \\ z &= C(x) \end{cases} \tag{3}$$

where $x_0 = x(0) \in \mathbf{R}^n$ is the initial state, $x(t)$ is the state at time t, $w(t) \in \mathbf{R}^s$ is the disturbance, and $z(t) \in \mathbf{R}^r$ is the output.

Selected proofs of results are included in the Appendix.

2 Power Gain and "Point to Set" Distance $\mathcal{L}_2$-Gain

The power gain inequality (1) attempts to capture (via the λT term) the notion that systems without asymptotic stability can generate power.

Definition 2.1 (Power Gain) *A system has power gain* $\leq \gamma$ *if there exists a nonnegative power bias* $\lambda \in \mathbf{R}$ *and nonnegative energy bias function* $\beta : \mathbf{R}^n \to \mathbf{R}$ *such that the power gain inequality (1) holds for all* $w \in \mathcal{L}_2[0,T]$*, all* $T \geq 0$*, and all* $x_0 \in \mathbf{R}^n$.

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In view of inequality (1), systems with power gain must exhibit a nonnegative minimal power bias, called the *available power*. The available power, denoted by λ_a^γ is defined as per [4, 5] as follows:

$$\lambda_a^\gamma \quad = \limsup_{T\to\infty} \left\{ \frac{V^\gamma(x,T)}{T} \right\}, \tag{4}$$

where

$$V^\gamma(x,T) \ = \sup_{w\in\mathcal{L}_2[0,T]} \left\{ \|z\|^2_{\mathcal{L}_2[0,T]} - \gamma^2 \|w\|^2_{\mathcal{L}_2[0,T]} \right. \\ \left. : x(0) = x \right\} \tag{5}$$

Since the available power (4) characterizes the power generated by a nonlinear system, a simple definition of the *available storage* follows by subtracting the energy generated from the output energy. That is, the energy stored in the system on the interval $[0,T]$ is approximated by

$$E^\gamma(x,T) = V^\gamma(x,T) - \lambda_a^\gamma T$$

This yields two possible generalizations of the available storage: the finite horizon or *super* available storage V_a^γ, and the infinite horizon available storage V_b^γ, where

$$\begin{aligned} V_a^\gamma(x) &= \sup_{T\geq 0} \{E^\gamma(x,T)\}, \\ V_b^\gamma(x) &= \limsup_{T\to\infty} \{E^\gamma(x,T)\}. \end{aligned} \tag{6}$$

By inspection, $V_a^\gamma(x) \geq V_b^\gamma(x)$ for all $x \in \mathbf{R}^n$, and for $\lambda_a^\gamma > 0$ typically there exists $x \in \mathbf{R}^n$ such that $V_a^\gamma(x) > V_b^\gamma(x)$.

As the available power (4) is computed over an infinite time horizon, V_b^γ is the more useful of the two definitions of available storage. However, due to the careful balancing defined by the limsup operation, it is difficult to prove bounds or even finiteness for the storage function V_b^γ. This means that proving stability results for worst case disturbances is difficult (see [5]).

An alternative to capturing the power generation of a system is to assume that the outputs of that system tend to a set (preferably compact). This can be achieved by considering the "point to set" distance of outputs from the given set.

Let Z be a subset of the output space $\mathbf{R}^r$. The "point to set" distance of output z from set Z is defined as

$$\|z\|_Z \ = \ \inf_{\zeta\in Z} |z - \zeta| \tag{7}$$

Clearly $\|\cdot\|_Z$ does not define a norm as $\|\zeta\|_Z = 0$ for all $\zeta \in Z$. Nor is $\|\cdot\|_Z$ a semi-norm, as the linearity property does not hold. $\|\cdot\|_Z$ is however continuous, so that $\lim_{k\to\infty} \|z_k\|_Z = 0$ implies that $\lim_{k\to\infty} z_k \in Z$.

Proposition 2.2 (Continuity of $\|\cdot\|_Z$) *Let Z be a bounded subset of $\mathbf{R}^r$. Then, the function $f(\zeta) := \|\zeta\|_Z$ is continuous in $\zeta \in \mathbf{R}^r$.*

Using (7) we can define a space of signals which are a finite distance from the set Z. That is,

$$\mathcal{L}_{2,Z}(\mathbf{R}^r, [0,T]) = \left\{ \begin{array}{l} z(s) \in \mathbf{R}^r \ \forall \ s \in [0,T], \\ \int_0^T \|z(s)\|_Z^2 \ ds < \infty \end{array} \right\}. \tag{8}$$

The size of the signal z is denoted by

$$\|z\|_{\mathcal{L}_{2,Z}} = \sqrt{\int_0^T \|z(s)\|_Z^2 \ ds} \tag{9}$$

Notions of gain using this "norm" of z can then be formulated. The most obvious one is to measure the gain from $\mathcal{L}_2$ inputs to $\mathcal{L}_{2,Z}$ outputs.

Definition 2.3 ($\mathcal{L}_{2,Z}$ gain) *Let Z be a subset of the output space $\mathbf{R}^r$. A system G has $\mathcal{L}_{2,Z}$-gain $\leq \gamma$ if there exists a nonnegative function $\beta : \mathbf{R}^n \to \mathbf{R}$ such that*

$$\|z\|^2_{\mathcal{L}_{2,Z}[0,T]} \leq \gamma^2 \|w\|^2_{\mathcal{L}_2[0,T]} + \beta(x) \tag{2}$$

for all $w \in \mathcal{L}_2[0,T]$, all $T \geq 0$, and all $x \in \mathbf{R}^n$.

To show that $\mathcal{L}_{2,Z}$-gain is useful in the power gain context, we show that $\mathcal{L}_{2,Z}$-gain implies power gain.

Theorem 2.4 *Let Z be a subset of the output space $\mathbf{R}^r$. Suppose that system G has $\mathcal{L}_{2,Z}$-gain $\leq \gamma$. Then, for any $\varepsilon \in (0,1)$, there exists gain $\bar{\gamma}_\varepsilon$ and power bias $\bar{\lambda}_\varepsilon$ such that system G has power gain $\leq \bar{\gamma}_\varepsilon$ with power bias $\bar{\lambda}_\varepsilon$, where*

$$\begin{aligned} \bar{\gamma}_\varepsilon &= \tfrac{\gamma}{\sqrt{1-\varepsilon}}, \\ \bar{\lambda}_\varepsilon &= \tfrac{1}{\varepsilon}\|Z\|^2, \end{aligned} \tag{10}$$

and $\|Z\| := \sup_{\zeta\in Z} |\zeta|$ is the sup norm of set Z.

This theorem shows that set gain is a stronger property than power gain. Perceivable problems with $\mathcal{L}_{2,Z}$-gain include how to define the set Z, what happens with systems that do not have outputs tending to a compact set, and what happens in the absence of detectability. To shed some light on the problem of defining Z, we provide a lower bound for the norm of Z in terms of the available power of a system.

Corollary 2.5 (Lower bound for $\|Z\|$) *Suppose that a system G has $\mathcal{L}_{2,Z}$-gain $\leq \gamma$ for some set Z. Then,*

$$\|Z\|^2 \quad \geq \lim_{\hat{\gamma}\to\infty} \{\lambda_a^{\hat{\gamma}}\}, \tag{11}$$

where λ_a^γ is the available power for system G.

Example 2.6 Consider the limit cycle system

$$\left\{ \begin{array}{ll} \dot{x} & = \left\{ \begin{array}{ll} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} x - |x|^2 x + \frac{x}{|x|} v, & |x| \neq 0, \\ 0 & |x| = 0, \end{array} \right. \\ z & = x, \end{array} \right.$$

where $x \in \mathbf{R}^2$. This system has available power given by

$$\lambda_a^\gamma \quad = \tfrac{2}{3} - \tfrac{2}{27}\gamma^2 + \tfrac{2}{27}\left(\gamma^2 + 3\right)\sqrt{\tfrac{\gamma^2+3}{\gamma^2}}$$

Hence, a necessary condition for the system to exhibit $\mathcal{L}_{2,Z}$-gain $\leq \gamma$ for any gain γ and set Z is that (11) hold. That is, $\|Z\|^2 \geq 1$. Note however that in the absence of disturbances, $|z(t)| \to 1$ as $t \to \infty$. Furthermore, $x = 0$ is an unstable equilibrium, which implies that $0 \in Z$. Hence, one choice for Z would be $B[0;1]$, which agrees with the bound $\|Z\| \geq 1$ obtained above.

3 The Available Storage

The available storage for systems with gain defined using the "point to set" distance is defined in an analogous way to $\mathcal{H}_\infty$ theory [12, 7]. Since the definition depends on the set Z for which inequality (2) holds, we denote the available storage by $V_{a,Z}^{\gamma}$ for gain γ. Then,

$$V_{a,Z}^{\gamma}(x) \quad = \sup_{T \geq 0} \{V_Z^{\gamma}(x,T)\} \tag{12}$$

where $V_Z(x,T)$ is the finite horizon value function defined by

$$V_Z^{\gamma}(x,T) = \sup_{w \in \mathcal{L}_2[0,T]} \left\{ \int_0^T \left[\|z(s)\|_Z^2 - \gamma^2 |w(s)|^2 \right] \, ds \right\} \tag{13}$$

It can be shown that $V_Z^{\gamma}(x,T)$ is nondecreasing in T. Hence, the sup in (12) can be replaced by a limit,

$$V_{a,Z}^{\gamma}(x) \quad = \lim_{T \to \infty} \{V_Z^{\gamma}(x,T)\}\,. \tag{14}$$

Furthermore, if a system has $\mathcal{L}_{2,Z}$-gain $\leq \gamma$ then it is immediate from (2) and (13) that the limit (12) exists and is finite, since $V_{a,Z}^{\gamma}(x) \leq \beta(x) < \infty$ for all $x \in \mathbf{R}^n$.

With regard to interpretation, the available storage (14) for systems with gain using "point to set" distances is much more satisfying that the existing pair of power gain definitions (6). The presence of the limsup in the definition of V_b^{γ} means that it is more difficult to show that V_b^{γ} is bounded below. Proving dynamic programming properties is also more difficult. Using the definition V_a^{γ} on the other hand is unsatisfying since it does not appear to deal explicitly with behaviour on an infinite time horizon. However, the definition (12) uses a sup whilst dealing with an infinite time horizon, as shown by the equivalent expression (14). This fundamental simplification in the definitions allows an easier development of theory analogous to $\mathcal{H}_\infty$ than is possible in the power gain case.

Theorem 3.1 *For any $\varepsilon \in (0,1)$ and any $x \in \mathbf{R}^n$,*

$$V_a^{\gamma}(x) \quad \leq \left(\tfrac{1}{1-\varepsilon}\right) V_{a,Z_\varepsilon}^{\gamma_\varepsilon}(x) \tag{15}$$

where $Z_\varepsilon = B[0; \sqrt{\varepsilon \lambda_a^{\gamma}}]$ and $\gamma_\varepsilon = \gamma\sqrt{1-\varepsilon}$.

A weaker result can be obtained by considering the λ-storage function V_λ^{γ}, which is defined as

$$\begin{aligned} V_\lambda^{\gamma}(x) \quad &= \sup_{T \geq 0} \{E_\lambda^{\gamma}(x,T)\} \\ E_\lambda^{\gamma}(x,T) \quad &= V^{\gamma}(x,T) - \lambda T \end{aligned} \tag{16}$$

where $V^{\gamma}(x,T)$ is given by (5). Note that for $\lambda = \lambda_a^{\gamma}$, $V_\lambda^{\gamma} \equiv V_a^{\gamma}$. Then, the weaker statement of the result is as follows:

Theorem 3.2 *For any $\varepsilon \in (0,1)$, $x \in \mathbf{R}^n$, $\lambda \geq \lambda_a^{\gamma_\varepsilon}$,*

$$V_\lambda^{\gamma}(x) \leq \left(\tfrac{1}{1-\varepsilon}\right) V_{a,Z_\varepsilon}^{\gamma_\varepsilon}(x) \tag{17}$$

where $Z_\varepsilon = B[0; \sqrt{\varepsilon\lambda}]$ and $\gamma_\varepsilon = \gamma\sqrt{1-\varepsilon}$.

Example 3.3 Consider a simple scalar linear system with affine output,

$$\left\{ \begin{array}{rcl} \dot{x} &=& -x + w, \\ z &=& x + 1. \end{array} \right. \tag{18}$$

This system has power gain $\leq \gamma$ for all $\gamma > 1$ and has available power $\lambda_a^{\gamma} = \frac{\gamma^2}{\gamma^2 - 1}$. Set $\gamma = 2$. Since the dynamics are stable, $\lim_{t \to \infty}\{z(t)\} = 1$ in the absence of disturbances. Hence, set $Z_\varepsilon \supseteq B[0;1]$. In particular, choose $\lambda = 1.4$ and $\varepsilon = 0.75$. This yields $Z_\varepsilon \approx B[0; 1.025]$, $\gamma_\varepsilon = 1$. The two functions in (17) are illustrated in Figure 1.

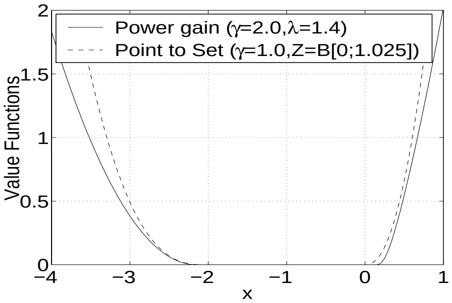

Figure 1: $V_\lambda^{\gamma}(x)$ and $\left(\frac{1}{1-\varepsilon}\right) V_{a,Z_\varepsilon}^{\gamma_\varepsilon}(x)$ versus x for scalar affine system (18).

4 Dissipation

We now define analogous notions of dissipation [14] with the supply rate $\gamma^2|w|^2 - \|z\|_Z^2$.

Definition 4.1 *System (3) is dissipative with gain γ relative to the set of outputs $Z \subseteq \mathbf{R}^r$ if there exists a nonnegative storage function $V : \mathbf{R}^n \to \mathbf{R}$ which satisfies the dissipation inequality*

$$V(x) + \int_0^T [\gamma^2 |w(s)|^2 - \|z(s)\|_Z^2] \, ds \geq V(x(T)) \tag{19}$$

for all $w \in \mathcal{L}_2[0,T]$, all $T \geq 0$, and all $x \in \mathbf{R}^n$.

Connections between dissipation and finite gain follow immediately.

Theorem 4.2 *Let $Z \subseteq \mathbf{R}^r$.*

- *(i). System (3) has $\mathcal{L}_{2,Z}$-gain $\leq \gamma$ iff system (3) is dissipative with the same gain.*
- *(ii). System (3) is dissipative with gain γ relative to the set of outputs $Z \subseteq \mathbf{R}^r$ iff there exists a viscosity solution of the PDI*

$$\begin{aligned} \sup_{w \in \mathbf{R}^s} \{ &\nabla_x V(x)[A(x) + B(x)w] \\ &+ \|C(x)\|_Z^2 - \gamma^2 |w|^2 \} \leq 0 \end{aligned} \tag{20}$$

The existence of a zero set of the available storage is useful for proving PDE and stability results.

Lemma 4.3 *Given set* $Z \subseteq \mathbf{R}^r$ *and gain* $\gamma < \infty$, *suppose that system* G *has* $\mathcal{L}_{2,Z}$*-gain* $\leq \gamma$. *Then,*

$$\inf_{x \in \mathbf{R}^n} \{V_{a,Z}^{\gamma}(x)\} = 0.$$

Theorem 4.4 *Given set* $Z \subseteq \mathbf{R}^r$, *suppose that the available storage is locally bounded, proper (ie the sets* $\{\xi \in \mathbf{R}^n : 0 \leq V_{a,Z}^{\gamma}(\xi) \leq c\}$ *are compact for every* $c > 0$*) and is lower semicontinuous. Then, there exists a state* $\bar{x} \in \mathbf{R}^n$ *such that* $V_{a,Z}^{\gamma}(\bar{x}) = 0$.

The available storage can then be shown to satisfy a simple generalization of the Hamilton-Jacobi-Bellman PDE found in $\mathcal{L}_2$-gain analysis.

Theorem 4.5 *Given a set* $Z \subseteq \mathbf{R}^r$, *suppose that the available storage* $V_{a,Z}^{\gamma}(x)$ *is locally bounded. Then,* $V_{a,Z}^{\gamma}(x)$ *is a viscosity solution of the PDE*

$$\begin{aligned}\sup_{w \in \mathbf{R}^s}\{\nabla_x V(x)[A(x) + B(x)w] \\ + \|C(x)\|_Z^2 - \gamma^2 |w(s)|^2\} = 0.\end{aligned} \tag{21}$$

5 Stability

Via dissipation, it can now be shown that the finite gain inequality (2) implies that the output and the state of the system tend to a set (assuming detectability).

Continuity of the operator $\|\cdot\|_Z$ can be used to show that in the absence of disturbances, the output z of a system with finite $\mathcal{L}_{2,Z}$-gain for a given set Z tends to the set Z.

Proposition 5.1 (Outputs without disturbances) *Suppose a system* G *has* $\mathcal{L}_{2,Z}$*-gain* $\leq \gamma$ *for a given subset* $Z \subseteq \mathbf{R}^r$. *Then, in the absence of disturbances, the outputs of* G *tend to* Z.

Theorem 5.2 *Suppose a system* G *has* $\mathcal{L}_{2,Z}$*-gain* $\leq \gamma$ *for a given subset* $Z \subseteq \mathbf{R}^r$. *Then,* $V_{a,Z}^{\gamma}(x_0) = 0$ *implies that* $C(x_0) \in Z$. *Furthermore, with* $x(0) = x_0$, $C(x(s)) \in Z$ *for all* $s \geq 0$.

The following definitions are useful in determining the stability of system G.

Definition 5.3 *Let* $X \subset \mathbf{R}^n$ *and* $Z \subset \mathbf{R}^r$ *be sets of states and outputs respectively. A system* G *is*

(i). X*-state* Z*-observable if* $w(t) = 0$, $z(t) \in Z$ *for all* $t \geq 0$ *implies that* $x(t) \in X$ *for all* $t \geq 0$,

(ii). X*-state* Z*-detectable if* $w(t) = 0$, $z(t) \in Z$ *for all* $t \geq 0$ *implies that* $\lim_{t \to \infty} \|x(t)\|_X = 0$.

Theorem 5.4 *Assume that system* G *has* $\mathcal{L}_{2,Z}$*-gain* $\leq \gamma$ *for a given set of outputs* $Z \subseteq \mathbf{R}^r$ *and gain* $\gamma < \infty$. *Then,*

(i). G *is* X*-state* Z*-observable* $\Rightarrow V_{a,Z}^{\gamma}(x) > 0$ *for all* $x \notin X$.

(ii). $V_{a,Z}^{\gamma}(x) > 0$ *for all* $x \notin X$, $V_{a,Z}^{\gamma}(\cdot)$ *is proper,* G *is* X*-state* Z*-detectable* $\Rightarrow x(t) \to X$ *as* $t \to \infty$.

6 Strict Bounded Real Lemma

The following results are developed in parallel with those of [6].

Theorem 6.1 *Assume there exists sets* $Z \subseteq \mathbf{R}^r$ *and* $X \subseteq \mathbf{R}^n$ *such that system (3) has* $\mathcal{L}_{2,Z}$*-gain* $< \gamma$ *and system*

$$\begin{cases} \dot{x} &= A(x) + B(x)w, \\ z &= x \end{cases} \tag{22}$$

has $\mathcal{L}_{2,X}$*-gain* $\leq \hat{\gamma}$. *Then, there exists a locally bounded lower semicontinuous function* $\hat{V}(x) > 0$ *with* $\hat{V}(x) = 0$ *for all* $x \in X$ *satisfying the PDE*

$$\begin{aligned}\sup_{w \in \mathbf{R}^s}\{\nabla_x \hat{V}(x)[A(x) + B(x)w] \\ + \|C(x)\|_Z^2 - \hat{\gamma}^2 |w(s)|^2 + \delta^2 \|x\|_X^2\} \leq 0\end{aligned} \tag{23}$$

in the viscosity sense.

Theorem 6.2 *Given sets* $Z \subseteq \mathbf{R}^r$ *and* $X \subseteq \mathbf{R}^n$, *suppose there exists a locally bounded lower semicontinuous nonnegative viscosity solution* $\hat{V}(x)$ *of the PDI (23). Then, there exists a locally bounded nonegative viscosity solution* $V(x)$ *of the PDE*

$$\begin{aligned}\sup_{w \in \mathbf{R}^s}\{\nabla_x \hat{V}(x)[A(x) + B(x)w] \\ + \|C(x)\|_Z^2 - \hat{\gamma}^2 |w(s)|^2\} = 0.\end{aligned} \tag{21}$$

Furthermore, if $V(x)$ *is smooth then the vector field*

$$A^*(x) = A(x) + B(x)B(x)'\nabla_x V(x)' \tag{24}$$

is stable in the sense that any trajectory defined by $\dot{x}^* = A^*(x^*)$ *tends to* X. *That is,* $x^*(t) \to X$ *as* $t \to \infty$.

7 Computation of the Available Storage

Given a set $Z \subseteq \mathbf{R}^r$, the available storage $V_{a,Z}^{\gamma}$ defined by (12) can be computed via Theorem 4.5. By applying finite differences to the nonstationary form of PDE (21) (ie, the PDE for $V_Z^{\gamma}(x,T)$), an iterative scheme [2, 1, 9] can be obtained for the available storage $V_{a,Z}^{\gamma}$.

Example 7.1 Consider the limit cycle system of Example 2.6. Choose a state space grid $x_1, x_2 \in \{x \in [-3,3] : x = 0.2k, k \in \mathcal{Z}\}$ and a disturbance grid $v \in \{x \in [-0.1, 0.3] : x = 0.005k, k \in \mathcal{Z}\}$. The available storage is shown in Figures 2. Observe that the available storage is zero inside $B[0;1]$.

Appendix - Selected Proofs

Proof: [Theorem 2.4] For any $\varepsilon \in (0,1)$,

$$|z - y|^2 \quad \geq (1-\varepsilon)|z|^2 - (\tfrac{1}{\varepsilon} - 1)|y|^2.$$

Applying the definition of the "point to set distance",

$$\begin{aligned}\|z(s)\|_Z^2 &= \inf_{y \in Z} |z(s) - y|^2 \\ &\geq (1-\varepsilon)|z(s)|^2 - (\tfrac{1}{\varepsilon} - 1)\|Z\|^2,\end{aligned}$$

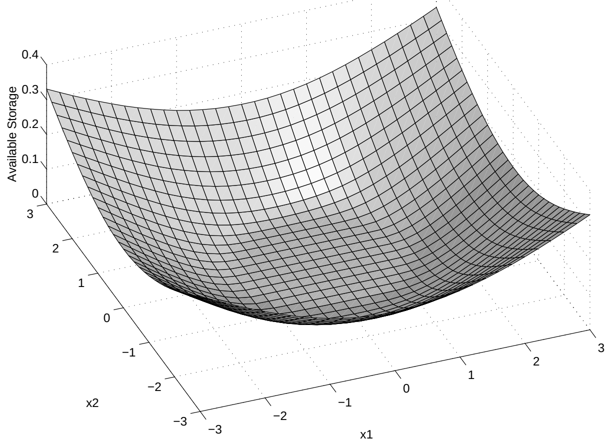

Figure 2: Available storage $V_{a,Z}^{\gamma}(x)$

where $\|Z\| := \sup_{y\in Z} |y|$ is defined to be the sup norm of the set Z. Hence,

$$\int_0^T |z(s)|^2\ ds \quad \le \tfrac{1}{1-\varepsilon}\int_0^T \|z(s)\|_Z^2\ ds + \tfrac{1}{\varepsilon}\|Z\|^2 T \tag{25}$$

Combining (25) with (2),

$$\begin{aligned}\int_0^T |z(s)|^2\ ds \quad &\le \left(\tfrac{\gamma}{\sqrt{1-\varepsilon}}\right)^2 \int_0^T |w(s)|^2\ ds + \tfrac{1}{\varepsilon}\|Z\|^2 T \\ &\qquad + \tfrac{1}{1-\varepsilon}\beta(x),\end{aligned}$$

which is the power gain inequality (1). ■

Proof: [Corollary 2.5] Rewrite (10) as $\|Z\|^2 \ge \varepsilon\bar{\lambda}_\varepsilon \ge \varepsilon\lambda_a^{\bar{\gamma}_\varepsilon}$ and send $\varepsilon \uparrow 1$. ■

Proof: [Theorem 3.1] Write V_a^γ given by (6) as

$$\begin{aligned}V_a^\gamma(x) \quad &= \sup_{T\ge 0,\ w\in\mathcal{L}_2[0,T]} \Big\{ \int_0^T [(|z(s)|^2 - \lambda_a^\gamma) \\ &\qquad -\gamma^2 |w(s)|^2]\ ds\Big\}.\end{aligned} \tag{26}$$

Define the indicator function

$$\chi_Z(\zeta) = \begin{cases} 1 & x \in Z, \\ 0 & x \notin Z.\end{cases}$$

Then, in (26),

$$\begin{aligned}|z(s)|^2 - \lambda_a^\gamma \quad &= [|z(s)|^2 - \lambda_a^\gamma]\chi_{Z_\varepsilon}(z(s)) + \\ &\qquad [|z(s)|^2 - \lambda_a^\gamma]\chi_{\mathbf{R}^r\setminus Z_\varepsilon}(z(s)),\end{aligned}$$

where $Z_\varepsilon := B[0; \sqrt{\varepsilon\lambda_a^\gamma}]$. But, following an argument similar to the proof of Theorem 2.4, for any $\varepsilon \in (0,1)$,

$$|z(s)|^2 - \lambda_a^\gamma \quad \le \tfrac{1}{1-\varepsilon}[|z(s)| - \sqrt{\varepsilon\lambda_a^\gamma}]^2.$$

Combining the above two equations,

$$\begin{aligned}|z(s)|^2 - \lambda_a^\gamma \quad &\le [|z(s)|^2 - \lambda_a^\gamma]\chi_{Z_\varepsilon}(z(s)) + \\ &\qquad \tfrac{1}{1-\varepsilon}[|z(s)|^2 - \sqrt{\varepsilon\lambda_a^\gamma}]^2 \chi_{\mathbf{R}^r\setminus Z_\varepsilon}(z(s))\end{aligned}$$

But, by definition of Z_ε,

$$\begin{aligned}0 \quad &\ge [|z(s)|^2 - \lambda_a^\gamma]\chi_{Z_\varepsilon}(z(s)), \\ \|z(s)\|_{Z_\varepsilon}^2 \quad &= [|z(s)|^2 - \lambda_a^\gamma]\chi_{\mathbf{R}^r\setminus Z_\varepsilon}(z(s)).\end{aligned}$$

Hence,

$$|z(s)|^2 - \lambda_a^\gamma \quad \le \tfrac{1}{1-\varepsilon}\|z(s)\|_{Z_\varepsilon}^2.$$

Substituting in (26),

$$\begin{aligned}V_a^\gamma(x) \quad &\le \sup_{T\ge 0,\ w\in\mathcal{L}_2[0,T]} \Big\{ \int_0^T \tfrac{1}{1-\varepsilon}\|z(s)\|_{Z_\varepsilon}^2 \\ &\qquad -\gamma^2|w(s)|^2]\ ds\Big\} \\ &= \tfrac{1}{1-\varepsilon}\sup_{T\ge 0,\ w\in\mathcal{L}_2[0,T]} \Big\{ \int_0^T \|z(s)\|_{Z_\varepsilon}^2 \\ &\qquad -(\gamma\sqrt{1-\varepsilon})^2|w(s)|^2]\ ds\Big\} \\ &= \tfrac{1}{1-\varepsilon} V_{a,Z_\varepsilon}^{\gamma_\varepsilon}(x)\end{aligned}$$

where $\gamma_\varepsilon = \gamma\sqrt{1-\varepsilon}$ and $Z_\varepsilon := B[0; \sqrt{\varepsilon\lambda_a^\gamma}]$. ■

Proof: [Theorem 4.2] Continuity of the operator $\|\cdot\|_Z$ (Proposition 2.2) ensures that the standard argument of [8] holds. ■

Proof: [Lemma 4.3] By Theorem 4.2, system G is dissipative with gain γ. Furthemore, $V_{a,Z}^\gamma(\cdot)$ is a candidate storage function. Hence,

$$V_{a,Z}^\gamma(x) + \int_0^T [\gamma^2|w(s)|^2 - \|z(s)\|_Z^2]\ ds \ge V_{a,Z}^\gamma(x(T)).$$

Taking the inf over $w \in \mathcal{L}_2[0,T]$ and $T \ge 0$ yields that

$$\begin{aligned}V_{a,Z}^\gamma(x) + \inf_{\substack{T\ge 0 \\ w\in\mathcal{L}_2[0,T]}} &\left\{\gamma^2\|w\|_{\mathcal{L}_2[0,T]}^2 - \|z\|_{\mathcal{L}_{2,Z}[0,T]}^2\right\} \\ &\ge \inf_{\substack{T\ge 0 \\ w\in\mathcal{L}_2[0,T]}} \left\{V_{a,Z}^\gamma(x(T))\right\} \\ &\ge \inf_{x\in\mathbf{R}^n} \left\{V_{a,Z}^\gamma(x)\right\}.\end{aligned}$$

But, the second term on the LHS of this inequality is $-V_{a,Z}^\gamma(x)$. Hence, $0 \ge \inf_{x\in\mathbf{R}^n}\left\{V_{a,Z}^\gamma(x)\right\}$. ■

Proof: [Theorem 4.4] Since $V_{a,Z}^\gamma(\cdot)$ is locally bounded, Lemma 4.3 implies that $\inf_{x\in\mathbf{R}^n}\{V_{a,Z}^\gamma(x)\} = 0$. Hence, we can define a sequence x_k such that $\lim_{k\to\infty}\{V_{a,Z}^\gamma(x_k)\} = 0$. Hence, as $V_{a,Z}^\gamma(\cdot)$ is nonnegative, given $\varepsilon > 0$, there exists an $n \ge 0$ such that

$$k \ge n \ \Rightarrow\ 0 \le V_{a,Z}^\gamma(x_k) < \varepsilon.$$

Now, the set $L_\varepsilon = \{\xi \in \mathbf{R}^n : 0 \le V_{a,Z}^\gamma(\xi) < \varepsilon\}$ is assumed compact. Hence, there exists a convergent subsequence x_{k_j} such that $\lim_{j\to\infty}\{x_{k_j}\} = \bar{x} \in L_\varepsilon$ and $\lim_{j\to\infty}\{V_{a,Z}^\gamma(x_{k_j})\} = 0$. Since $V_{a,Z}^\gamma(\cdot)$ is lower semicontinuous, $V_{a,Z}^\gamma(\bar{x}) = 0$. ■

Proof: [Theorem 4.5] The proof is very similar to that given in the Appendix of [6] and is omitted. ■

Proof: [Theorem 6.1] Since system (3) has $\mathcal{L}_{2,Z}$-gain $< \gamma$, there exists an $\varepsilon > 0$ such that

$$\int_0^T \|z(s)\|_Z^2\ ds \quad \le (\gamma^2 - \varepsilon^2)\int_0^T |w(s)|^2\ ds + \beta(x)$$

for all $w \in \mathcal{L}_2[0,T]$, $T \ge 0$, $x \in \mathbf{R}^n$. Rearranging,

$$\begin{aligned}\int_0^T \|z(s)\|_Z^2\ ds &+ \varepsilon^2\int_0^T |w(s)|^2\ ds \\ &\le \gamma^2\int_0^T |w(s)|^2\ ds + \beta(x)\end{aligned} \tag{27}$$

Since system (22) has $\mathcal{L}_{2,X}$-gain $\le \hat{\gamma}$,

$$\int_0^T \|x(s)\|_X^2\ ds \le \hat{\gamma}^2\int_0^T |w(s)|^2\ ds + \hat{\beta}(x).$$

Rearranging,

$$\int_0^T |w(s)|^2\ ds \ge \tfrac{1}{\hat{\gamma}^2}\int_0^T \|x(s)\|_X^2\ ds - \tfrac{\hat{\beta}(x)}{\hat{\gamma}^2}. \tag{28}$$

Combining inequalities (27) and (28),

$$\begin{aligned}\int_0^T &\left[\|z(s)\|_Z^2 + \tfrac{\varepsilon^2}{\hat{\gamma}^2}\|x(s)\|_X^2\right]\ ds \\ &\le \gamma^2\int_0^T |w(s)|^2\ ds + \left[\tfrac{\varepsilon^2}{\hat{\gamma}^2}\hat{\beta}(x) + \beta(x)\right]\end{aligned} \tag{29}$$

The integrand in the right hand side of inequality (29) may be rewritten as

$$
\begin{aligned}
&\|z(s)\|_Z^2 + \|x(s)\|_X^2 \\
&= \inf_{\zeta \in Z} |z(s) - \zeta|^2 + \tfrac{\varepsilon^2}{\hat{\gamma}^2} \inf_{\xi \in X} |x(s) - \xi|^2 \\
&= \inf_{(\zeta,\xi) \in Z \times X} \left\{ |z(s) - \zeta|^2 + \tfrac{\varepsilon^2}{\hat{\gamma}^2} |x(s) - \xi|^2 \right\} \\
&= \inf_{(\zeta,\xi) \in Z \times X} \left| \left[\begin{array}{c} z(s) - \zeta \\ \frac{\varepsilon}{\hat{\gamma}}[x(s) - \xi] \end{array} \right] \right|^2 \\
&= \inf_{(\zeta,\eta) \in Z \times \hat{X}} \left| \left[\begin{array}{c} z(s) - \zeta \\ \frac{\varepsilon}{\hat{\gamma}} x(s) - \eta \end{array} \right] \right|^2 \\
&= \left\| \left[\begin{array}{c} z(s) \\ \frac{\varepsilon}{\hat{\gamma}} x(s) \end{array} \right] \right\|_{Z \times \hat{X}}^2 ,
\end{aligned}
$$

where $\hat{X} := \left\{ x \in \mathbf{R}^n : \frac{\varepsilon}{\hat{\gamma}} x \in X \right\}$. Hence, the system

$$
\left\{ \begin{array}{ll} \dot{x} & = A(x) + B(x) w, \\ z & = \left[\begin{array}{c} C(x) \\ \frac{\varepsilon}{\hat{\gamma}} x \end{array} \right] \end{array} \right. \tag{30}
$$

has $\mathcal{L}_{2,Z \times \hat{X}}$-gain $\leq \gamma$.

Theorem 4.2 then implies inequality (23). The integrated form of (23) is

$$
\begin{aligned}
\hat{V}(x(T)) \leq V(x) + \textstyle\int_0^T [\gamma^2 |w(s)|^2 - [\|C(x(s))\|_Z^2 + \\
\tfrac{\varepsilon^2}{\hat{\gamma}^2} \|x(s)\|_X^2]]\, ds
\end{aligned}
$$

Setting $w = 0$ and choosing $x \in \mathbf{R}^n$ such that $V(x) = 0$,

$$
\hat{V}(x(T)) \leq \tfrac{\varepsilon^2}{\hat{\gamma}^2} \textstyle\int_0^T \|x(s)\|_X^2 \, ds
$$

for all $T \geq 0$. Hence, $V(x(T))$ is strictly decreasing for all $x \notin X$. To avoid a contradiction, it follows that $x \in X$. ■

Proof: [Theorem 6.2] Since $\hat{V}(x)$ is a solution of the PDI (23), Theorem 4.2 implies that $\hat{V}(x)$ satisfies the dissipation inequality (19). That is,

$$
\begin{aligned}
\hat{V}(x) &\geq \sup_{T \geq 0} \sup_{w \in \mathcal{L}_2[0,T]} \{ \hat{V}(x(T)) + \\
&\qquad \textstyle\int_0^T [\|z(s)\|_Z^2 + \tfrac{\varepsilon^2}{\hat{\gamma}^2} \|x(s)\|_X^2 - \gamma^2 |w(s)|^2]\, ds \} \\
&\geq \sup_{T \geq 0} \sup_{w \in \mathcal{L}_2[0,T]} \{ \textstyle\int_0^T [\|z(s)\|_Z^2 - \gamma^2 |w(s)|^2]\, ds \} \\
&= V_{a,Z}^{\gamma}(x).
\end{aligned}
$$

But, $V_{a,Z}^{\gamma}(x)$ is a viscosity solution of the PDE (21). Hence, as $\hat{V}(x)$ is locally bounded, so is $V_{a,Z}^{\gamma}(x)$. Given $T \geq 0$, choose $w^* \in \mathcal{L}_2[0,T]$ to be optimal in the definition of $V_{a,Z}^{\gamma}(x)$. Then, by dynamic programming,

$$
V_{a,Z}^{\gamma}(x) = \textstyle\int_0^T [\|z^*(s)\|_Z^2 - \gamma^2 |w^*(s)|]\, ds + V_{a,Z}^{\gamma}(x^*(T)).
$$

Substituting $w^* \in \mathcal{L}_2[0,T]$ in the dissipation inequality (19) for $\hat{V}(x)$,

$$
\begin{aligned}
\hat{V}(x) \geq \textstyle\int_0^T [\|z^*(s)\|_Z^2 + \tfrac{\varepsilon^2}{\hat{\gamma}^2} \|x^*(s)\|_X^2 - \gamma^2 |w^*(s)|^2]\, ds \\
+ \hat{V}(x^*(T)).
\end{aligned}
$$

Subtracting the preceding equation,

$$
\begin{aligned}
\hat{V}(x^*(T)) - V_{a,Z}^{\gamma}(x^*(T)) \leq [\hat{V}(x) - V_{a,Z}^{\gamma}(x)] \\
- \tfrac{\varepsilon^2}{\hat{\gamma}^2} \textstyle\int_0^T \|x^*(s)\|_X^2 \, ds
\end{aligned}
$$

But, $\hat{V}(x) - V_{a,Z}^{\gamma}(x) \geq 0$ for all $x \in \mathbf{R}^n$. Choosing $x = x^*(0) \notin X$ implies that $x^*(t) \to X$ as $t \to \infty$. Note that if $V_{a,Z}^{\gamma}(x)$ is smooth, then $w^* = B(x)' \nabla_x V_{a,Z}^{\gamma}(x)$.

■

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