Numerical Computation of State Feedback Controllers for Systems with Persistent Outputs

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Abstract

In this paper, we consider the problem of computing a state feedback controller for a class of nonlinear plants without requiring asymptotic stability of the resulting closed loop system. A simple generalization of the $\mathcal{L}_2$-gain inequality which permits persistent outputs is used as the objective in the control design. By considering notions of dissipation and available storage, the controller can be computed by solving a Hamilton-Jacobi-Bellman-Isaac PDE. Although this PDE rarely admits analytical solutions, finite differences may be applied to compute an approximation to both the available storage and the desired state feedback controller. Due to problems with convergence of conventional Jacobi value space iterations, a mixed policy space / value space algorithm is applied.

1 Introduction

Recent work in the stability analysis of nonlinear systems has been focused on considering systems which do not exhibit asymptotically stable equilibria. Differing characterizations (with differing levels of generality) of this behaviour include inherently nonlinear notions of ISS [7] and $\mathcal{H}_\infty$ generalizations of power gain [3]. Of concern in these approaches is to understand the nature of stability of systems whose trajectories exhibit persistent excitation in the absence of disturbances. In characterizing the behaviour of systems such as limit cycle systems and chaotic systems, the input to state or input / output approaches mentioned above provide connections between gain inequalities and the "steady state" behaviour of the dynamics. In particular, the inequalities used make allowances for the fact that trajectories do not tend asymptotically to an equilibrium.

In this paper, we consider the problem of computing a state feedback controller K^* for a plant G which results in a closed loop system (G, K^*) which satisfies the gain inequality,

$$\|z\|^2_{\mathcal{L}_{2,Z}[0,T]} \leq \gamma^2 \|w\|^2_{\mathcal{L}_2[0,T]} + \beta(x) \tag{1}$$

where $\|z\|^2_{\mathcal{L}_{2,Z}[0,T]} := \int_0^T \|z(s)\|^2_Z \, ds$, $\|z(s)\|_Z := \inf_{\zeta \in Z} |z(s) - \zeta|$, and Z is a subset of the output space. As Z can be of nonzero measure, the gain inequality (1) permits finite gain for systems with persistent outputs. Exploiting connections between finite gain and dissipation, controllers can be computed by applying dynamic programming to the resulting definition of available storage [10, 9, 4]. The differential form of the dynamic programming equation (a HJBI PDE) can then be solved using a mixed policy space / value space finite difference scheme.

The following class of plants is considered in this paper:

$$G : \begin{cases} \dot{x} &= A(x) + B_1(x) w + B_2(x) u, \\ z &= C_1(x) + D_{12}(x) u, \end{cases} \tag{2}$$

where $x_0 = x(0) \in \mathbf{R}^n$ is the initial state, $x(t)$ is the state at time t, $w(t) \in \mathbf{R}^s$ is the disturbance, $u(t) \in \mathbf{R}^m$ is the control, and $z(t) \in \mathbf{R}^r$ is the output.

The class of controllers considered is limited to static state feedback controllers. That is, controllers of the form:

$$K : \{ \; u \; = K(x), \tag{3}$$

where $K : \mathbf{R}^n \to \mathbf{R}^m$ is independent of time. The plant G and controller K are interconnected as shown in Figure 1.

2 The Control Problem

Given a set $Z \subseteq \mathbf{R}^r$, the closed loop system (G, K) has $\mathcal{L}_{2,Z}$-gain $\leq \gamma$ if there exists a nonnegative function $\beta : \mathbf{R}^n \to \mathbf{R}$ such that the following "point to set" distance gain inequality holds:

$$\|z\|^2_{\mathcal{L}_{2,Z}[0,T]} \leq \gamma^2 \|w\|^2_{\mathcal{L}_2[0,T]} + \beta(x) \tag{1}$$

where $\|z\|^2_{\mathcal{L}_{2,Z}[0,T]} := \int_0^T \|z(s)\|^2_Z \, ds$ and $\|z(s)\|_Z := \inf_{\zeta \in Z} |z(s) - \zeta|$.

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The objective of the control problem studied in this paper is to design a state feedback controller K such that the closed loop system (G, K) as shown in Figure 1 has $\mathcal{L}_{2,Z}$-gain $\leq \gamma$ for a given gain γ and set of outputs $Z \in \mathbf{R}^r$.

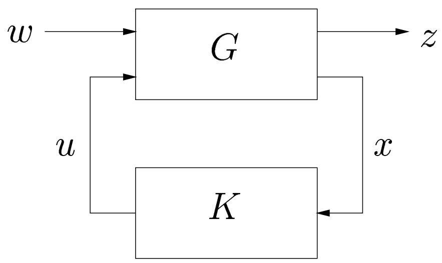

Figure 1: State Feedback Control Problem

One method for obtaining a controller which satisfies inequality (1) is to follow the conventional ideas of $\mathcal{H}_\infty$ control [8, 4]. From inequality (1), we can define a lower bound for all choices of the function $\beta(\cdot)$ for a given gain γ and controller K. This lower bound is the available storage for controller K and set Z, which we will denote by $V^\gamma_{a,K,Z}(\cdot)$:

$$V^\gamma_{a,K,Z}(x) = \sup_{T\geq 0, w\in\mathcal{L}_2[0,T]} \left\{ \|z\|^2_{\mathcal{L}_{2,Z}[0,T]} - \gamma^2 \|w\|^2_{\mathcal{L}_2[0,T]} \right\} \quad (4)$$

By applying dynamic programming arguments, it can be shown that $V^\gamma_{a,K,Z}(x)$ is a solution of the Hamilton-Jacobi-Bellman PDE

$$\sup_{w\in\mathbf{R}^p}\{\nabla_x V(x)[A(x) + B_2(x)K(x) + B_1(x)w] + \|C_1(x) + D_{12}(x)K(x)\|^2_Z - \gamma^2|w(s)|^2\} = 0. \quad (5)$$

Denote the LHS of (5) by $H_Z(x, K(x), \nabla_x V(x))$. Then,

$$\inf_{u\in\mathbf{R}^m} H_Z(x, u, \nabla_x V(x)) \leq H_Z(x, K(x), \nabla_x V(x)). \quad (6)$$

Define $K^*(x)$ as

$$K^*(x) = \operatorname{argmin}_{u\in\mathbf{R}^m} H_Z(x, u, \nabla_x V(x)) \quad (7)$$

Then, applying definition (7) in inequality (6),

$$H_Z(x, K^*(x), \nabla_x V(x)) \leq H_Z(x, K(x), \nabla_x V(x)) = 0.$$

So, $V(x)$ satisfies the above PDI for the closed loop system (G, K^*). From [5], this implies that the closed loop (G, K^*) is dissipative with supply rate $\gamma^2|w|^2 - \|z\|^2_Z$. That is, the closed loop (G, K^*) satisfies the "gain outside a set" inequality (1). This is summarized in the following theorem.

Theorem 2.1 *Suppose that the available storage $V_{a,K,Z}(\cdot)$ is locally bounded for a given set Z and controller K. Then, the closed loop system (G, K^*), with controller $K^*(x)$ defined by (7), has $\mathcal{L}_{2,Z}$-gain $\leq \gamma$.*

The controller K^* is also optimal in the sense that it minimizes the available storage $V^\gamma_{a,K,Z}(\cdot)$ over the class of static state feedback controllers. That is,

$$V^\gamma_{a,K^*,Z}(x) = \inf_K \sup_{T\geq 0, w\in\mathcal{L}_2[0,T]} \left\{ \|z\|^2_{\mathcal{L}_{2,Z}[0,T]} - \gamma^2 \|w\|^2_{\mathcal{L}_2[0,T]} \right\}.$$

Furthermore, $V^\gamma_{a,K^*,Z}(\cdot)$ is a solution of the Hamilton-Jacobi-Bellman-Isaacs PDE

$$\inf_{u\in\mathbf{R}^m} H_Z(x, u, \nabla_x V(x)) = 0. \quad (8)$$

Finally, define the set of states

$$S := \left\{ x \in \mathbf{R}^n : \nabla_x V^\gamma_{a,K^*,Z}(x) = 0, \ V^\gamma_{a,K^*,Z}(x) = 0 \right\}. \quad (9)$$

Suppose that set S is nonempty. By applying the dissipation inequality (see [1]) given $x \in S$,

$$V^\gamma_{a,K^*,Z}(x) + \int_0^T \gamma^2|w(s)|^2 - \|z(s)\|^2_Z ds \geq V^\gamma_{a,K^*,Z}(x(T)),$$

for any disturbance $w \in \mathcal{L}_2[0, T]$ and any $T \geq 0$, where $x(T)$ is the state of the closed loop system at time T. Noting that $V_{a,K^*,Z}(x) = 0$ by definition of $x \in S$, and choosing $w = 0$,

$$\int_0^T \|z(s)\|^2_Z ds \leq -V^\gamma_{a.K^*,Z}(x(T)) \leq 0$$

for all $T \geq 0$. That is, $z(t) \in Z$ for all $t \geq 0$. Hence,

$$x(0) \in S \implies z(t) \in Z$$

for all $t \geq 0$, where $z(t) = C_1(x(t)) + D_{12}(x(t))K^*(x(t))$. Recalling (8), the HJBI PDE can be rewritten (by completing squares) as

$$\nabla_x V(x)[A(x) + \tfrac{1}{4\gamma^2} B_1(x)B_1(x)'\nabla_x V(x)'] + \inf_{u\in\mathbf{R}^m}\{\nabla_x V(x)B_2(x)u + \|C_1(x) + D_{12}(x)u\|^2_Z\} = 0.$$

If $x \in S$, we immediately have that

$$\inf_{u\in\mathbf{R}^m}\{\|C_1(x) + D_{12}(x)u\|^2_Z\} = 0.$$

Hence, the infimum is not attained at a unique optimal control $u^* \in \mathbf{R}^m$ for $x \in S$.

Theorem 2.2 *Suppose that the set S given by (9) is nonempty. Then, for any $x \in S$, any control u satisfying the constraint $C_1(x) + D_{12}(x)u \in Z$ is optimal at x in the sense that it satisfies the the HJBI PDE (8).*

Remark 2.3 In the special case where $z = \begin{bmatrix} x \\ u \end{bmatrix}$, $0, z \in Z$ implies that $\begin{bmatrix} x \\ 0 \end{bmatrix} \in Z$. Hence, provided that $x \in S$ given by (9), zero control may be applied as the optimal control.

3 Computation of Controller K^*

Finite differences can be applied to the HJBI PDE (8) to compute both the controller K^* and the corresponding available storage. Unlike HJB PDEs such as (5), value space iterations [6, 2] derived from the finite difference approximation can be insufficient to guarantee convergence of the available storage and controller. An alternative is to apply mixed policy / value space iterations [6].

An iterative scheme for computing the available storage and controller using mixed policy / value space iterations is given in the remainder of this section.

For the purpose of notation, let $(\mathbf{R}^n)^{\delta_X}$ denote the state space grid, where δ_X is the state space grid spacing for the finite difference approximation. Similarly define $(\mathbf{R}^m)^{\delta_U}$ and $(\mathbf{R}^s)^{\delta_W}$ for the control and disturbance grids respectively. The grids must be restricted for finite computation time, so let

$$\begin{aligned} G_X &= \{x \in (\mathbf{R}^n)^{\delta_X} : |x| \le K_X\}, \\ G_U &= \{u \in (\mathbf{R}^m)^{\delta_U} : |u| \le K_U\}, \\ G_W &= \{v \in (\mathbf{R}^s)^{\delta_W} : |w| \le K_W\}, \end{aligned}$$

where K_X, K_U, and K_W are positive constants. Then, the maximum speed of the dynamics at each state is then given by

$$m(x) = \max_{u \in G_U, w \in G_W} |g(x,u,w)|$$

where $g(x,u,w) = A(x) + B_1(x)w + B_2(x)u$. This defines the interpolation time $\delta_T(x) = \frac{\delta_X}{m(x)}$ as the minimum time required to move from one state to another on the simplex

$$N^{\delta_X}(x) = \{x \pm \delta_X e_i : i = 1, \cdots, n\} \cup \{x\},$$

where e_i is the i^{th} unit vector in $\mathbf{R}^n$. The minimum interpolation time for any state on the grid G_X is

$$\begin{aligned} \delta_T &= \min_{x \in G_X}\{\delta_T(x)\} \\ &= \frac{\delta_X}{m}, \end{aligned}$$

where $m = \max_{x \in G_X}\{m(x)\}$. Define the transition probability [6]

$$p(x,\xi|u,w) = \begin{cases} 1 - \frac{|g(x,u,w)|_1}{m} & \xi = x, \\ \frac{g_i^{\pm}(x,u,w)}{m} & z = x \pm \delta_X e_i, \\ 0 & \text{elsewhere}, \end{cases}$$

where $g_i^{\pm} = \frac{|g_i| \pm g_i}{2}$, and g_i is the i^{th} component of $g(x,u,w) = A(x) + B_1(x)w + B_2(x)u$.

Let $V_k^{K_j}$ denote the k^{th} value space approximation to the available storage for the closed loop (G, K_j), where K_j is the j^{th} policy space approximation to the optimal controller K^*.

The scheme is initialized as follows:

$$\begin{aligned} K_0(x) &= \begin{cases} \text{any stabilizing state feedback} \\ \text{for plant } G, \end{cases} \\ V_0^{K_0}(x) &= 0, \end{aligned} \tag{10}$$

for all $x \in G_X$. Given controller approximation K_j and available storage value space approximation $V_{N_{value}}^{K_{j-1}}$ corresponding to the controller approximation K_{j-1}, N_{value} value space iterations are then performed to compute an approximation for the available storage $V_{N_{value}}^{K_j}$ for controller K_j.

Value space initialization:

$$V_0^{K_j}(x) = V_{N_{value}}^{K_{j-1}}. \tag{11}$$

Value space iteration:

$$\begin{aligned} &\tilde{V}_{k+1}^{K_j}(x) = \\ &\quad \max_{w \in G_W}\Big\{ \textstyle\sum_{\xi \in N^{\delta_X}(x)} [V_k^{K_{j-1}}(\xi) p(x,\xi; K_j(x), w)] \\ &\qquad + \delta_T[\|C_1(x) + D_{12}(x)K_j(x)\|_Z^2 - \gamma^2 |w|^2 \Big\} \\ &V_{k+1}^{K_j}(x) = \tilde{V}_{k+1}^{K_j}(x) - \tilde{V}_{k+1}^{K_j}(x_0) \end{aligned} \tag{12}$$

where x_0 is a reference state used for normalization purposes. The worst case disturbance $W_k^{K_j}(\cdot)$ is also updated at each step (for later use in the policy space iteration):

$$\begin{aligned} &W_k^{K_j}(x) = \\ &\text{argmax}_{w \in G_W}\Big\{ \textstyle\sum_{\xi \in N^{\delta_X}(x)} [V_k^{K_{j-1}}(\xi) p(x,\xi; K_j(x), w)] \\ &\qquad + \delta_T[\|C_1(x) + D_{12}(x)K_j(x)\|_Z^2 - \gamma^2 |w|^2 \Big\} \end{aligned} \tag{13}$$

N_{value} value space iterations (12), (13) yields the approximations $V_{N_{value}}^{K_j}$ and $W_{N_{value}}^{K_j}$ to the available storage and corresponding worst case disturbance respectively for the controller K_j. These approximations are then used to update the controller using a policy space iteration.

Policy space iteration:

$$\begin{aligned} &K_{j+1}(x) = \text{argmin}_{u \in G_U}\Big\{ \\ &\quad \textstyle\sum_{\xi \in N^{\delta_X}(x)} [V_{N_{value}}^{K_j}(\xi) p(x,\xi; u, W_{N_{value}}^{K_j}(x))] \\ &\quad + \delta_T[\|C_1(x) + D_{12}(x)K_j(x)\|_Z^2 - \gamma^2 |W_{N_{value}}^{K_j}(x)|^2 \Big\} \end{aligned} \tag{14}$$

By repeatedly computing new approximations to the controller and the corresponding available storage, the controller tends to the optimal controller. A summary of the algorithm is as follows:

Initialize control and storage function as per (10);
for $j = 0$ to N_{policy} do
  for $k = 0$ to $N_{value} - 1$ do
    Compute $(k+1)^{th}$ approximation $V_{k+1}^{K_j}$
      to the available storage for (G, K_j) using (12);
    Update the worst case disturbance $W_{k+1}^{K_j}$
      using (13);
  end;
  Compute $(j+1)^{th}$ approximation to the optimal
    controller K^* using (14);
end;
(15)

4 Examples

Example 4.1 Consider the unstable linear plant

$$G : \begin{cases} \dot{x} &= \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \\ z &= \begin{bmatrix} x \\ u \end{bmatrix}. \end{cases}$$

The aim is to compute the controller K^* which yields a closed loop system with $\mathcal{L}_{2,Z}$-gain ≤ 4, where $Z = B[0;3] \subseteq \mathbf{R}^3$. Define the state space, disturbance, and control grids by setting $\delta_X = 0.05$, $K_X = 3$, $\delta_W = 0.025$, $K_W = 0.4$, $\delta_U = 0.1$, $K_U = 12$. Using 25 policy space iterations with 400 value space iterations per policy space iteration, an approximation to the controller K^* can be computed using the scheme (15). The resulting approximations for the available storage, controller K^*, and worst case disturbance are illustrated in Figures 2, 3, and 4. Figure 5 illustrates that the controller yields a closed loop limit cycle system. Figure 6 shows that $|z(t)|^2 \leq 9$ (ie $z(t) \in Z$) holds for sufficiently large time t. Figure 7 illustrates the evolution of the absolute error for the numerical scheme versus policy / value space iteration. Figure 8 demonstrates that the Hamiltonian defined by the left hand side of (8) is near zero after 25 policy space iterations.

Example 4.2 Consider the limit cycle system

$$G : \begin{cases} \dot{x} &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} x - |x|^2 x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \\ z &= \begin{bmatrix} x \\ u \end{bmatrix}. \end{cases}$$

The aim is to compute the controller K^* which yields a closed loop system with $\mathcal{L}_{2,Z}$-gain ≤ 0.5, where $Z = B[0;1] \subseteq \mathbf{R}^3$. Define the state space, disturbance, and control grids by setting $\delta_X = 0.1$, $K_X = 2$, $\delta_W = 0.02$, $K_W = 0.6$, $\delta_U = 0.05$, $K_U = 2$. Using 200 policy space iterations with 100 value space iterations per policy space iteration, an approximation to

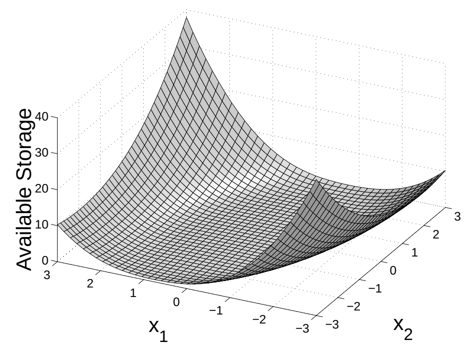

Figure 2: Available Storage for Example 4.1

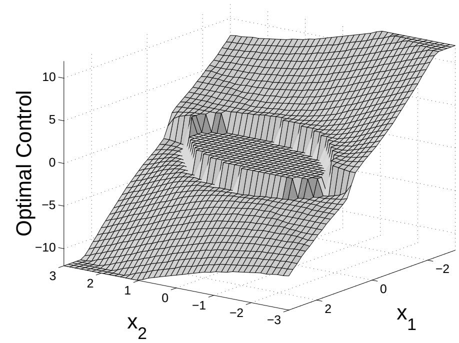

Figure 3: Controller K^* for Example 4.1

the controller K^* can be computed using the scheme (15). Convergence was obtained to within 10^{-8} absolute error in the available storage. The resulting approximations for the available storage, controller K^*, and worst case disturbance are illustrated in Figures 9, 10, and 11. Figure 12 illustrates that the controller yields a closed loop limit cycle with amplitude < 1. Figure 13 shows that $|z(t)|^2 \leq 1$ (ie $z(t) \in Z$) holds for sufficiently large time t.

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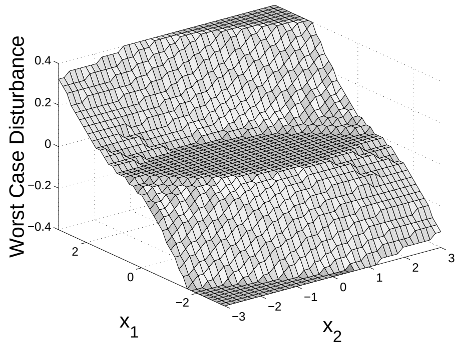

Figure 4: Worst case disturbance for Example 4.1

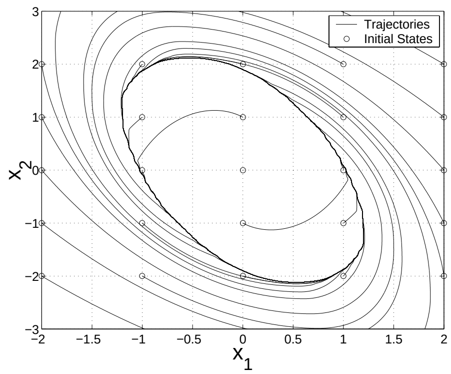

Figure 5: Sample closed loop trajectories ($w = 0$) for Example 4.1

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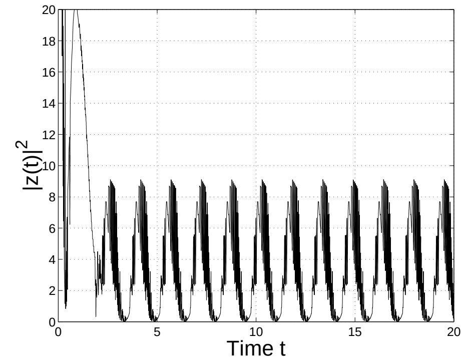

Figure 6: $|z(t)|^2$ versus t for Example 4.1

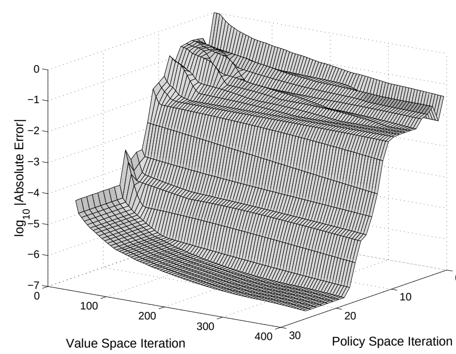

Figure 7: Numerical Absolute Error for Example 4.1

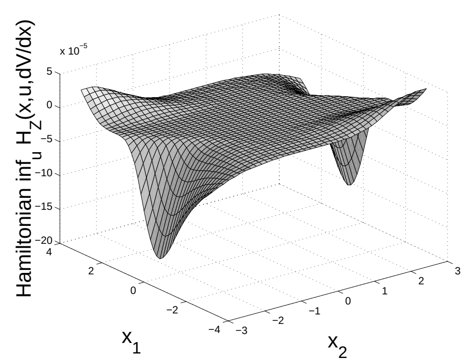

Figure 8: Hamiltonian for Example 4.1

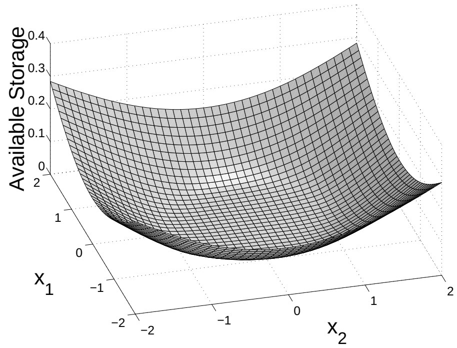

Figure 9: Available Storage for Example 4.2

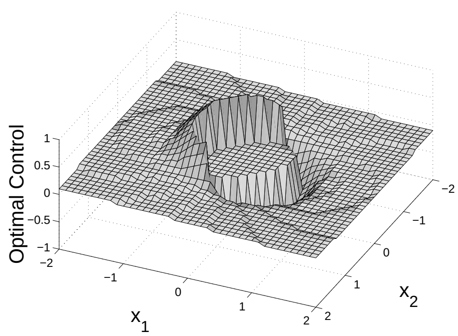

Figure 10: Controller K^* for Example 4.2

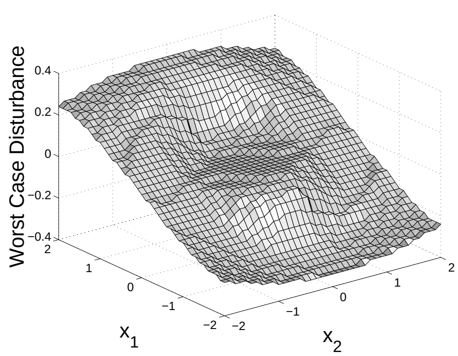

Figure 11: Worst case disturbance for Example 4.2

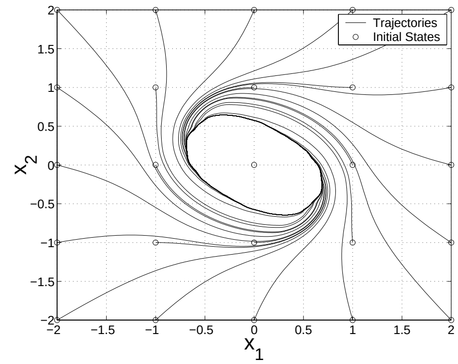

Figure 12: Sample closed loop trajectories ($w = 0$) for Example 4.2

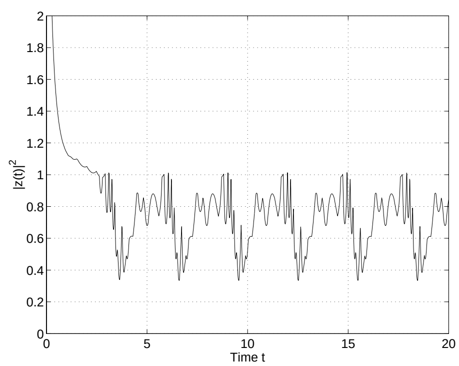

Figure 13: $|z(t)|^2$ versus t for Example 4.2