

# A Generalized OGY Method for Controlling Higher Order Chaotic Systems

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## Abstract

In this paper, we discuss a generalization of the OGY chaos control method based on the invariant manifold theory. This control methodology can deal with higher order chaotic systems in the same spirit of the OGY method. The effectiveness of the methodology will be tested by controlling the third order Rossler chaos.

## 1 Introduction

Chaos control is of significant importance for solving many nontraditional real-world problems by means of nontraditional techniques [1]. Among the existing chaos control methodologies, the model free chaos control approach has attracted a great deal of attention due to the difficulty in obtaining a faithful model for a chaotic system in many real applications. The simple yet effective OGY method [5] has lately been extended and applied [2, 3, 4, 5, 6, 7]. Essentially, this kind of control techniques require an identification of the stable and unstable manifolds from time series and, based on that, a suitable control action is developed to bring the system orbit to the stable manifold. This type of control methods, although with features of classical control, exploits some particular properties of chaos.

It is now known that the OGY type of chaos control methods are effective only for the control of lower dimensional chaotic systems because it utilizes the prominent feature of saddle type of fixed points that have both stable and unstable manifolds. It is a common experience that controlling higher dimensional chaotic systems by this methodology is still quite difficult, often impossible [2]. This difficulty lies in situations where the system Jacobian at a fixed point has complex eigenvalues or multiple unstable eigenvalues. Even with distinct real eigenvalues, the construction of stable and unstable manifolds for higher dimensional chaotic systems is a technical challenge [2]. It is noted that the OGY method may be considered as a kind of “pole assignment” technique, which however does not completely characterize the essence of the method that drives the system state to the predetermined stable manifold.

In this paper, we propose a new chaos control method, which not only preserves the spirit of the OGY method to direct the system orbit to a designated stable manifold, but can also deal with higher order chaotic systems fairly easily. The novelty lies in the construction of suitable stable manifolds, which are selected according to some dynamic properties of the underlying chaotic system. It is noted that these stable manifolds are not necessarily the ones represented by the system stable

eigenvalues and their associated eigenvectors; therefore, they are independent stable manifolds. The chaos control task is then to force the system state to lie on any these selected stable manifolds, so that whenever the system state lies in the intersection of these manifolds, it will be guided towards the designated fixed point which corresponds to the originally targeted unstable periodic orbit.

## 2 The OGY Chaos Control Methodology

Consider a chaotic dynamic system

$$\dot{x} = f(x, p), \quad x \in R^n \quad (1)$$

Based on its time series, the phase flow of (1) can be constructed via the Poincaré section method, as

$$x(k+1) = F(x(k), p), \quad x(k) \in R^{n-1}, \quad p \in R^p \quad (2)$$

A fixed point of this map represents a periodic orbit [3]. Only saddle type of unstable periodic orbits (UPOs) are of interest here.

To demonstrate the principle of the OGY control method, consider a fixed point,  $x^*$ , satisfying

$$x^* = F(x^*, p) \quad (3)$$

Assume that the desired orbit to be stabilized is corresponding to a nominal parameter value,  $p_0$ . This control method suggests to first linearise the nonlinear chaotic system (1) about the desired fixed point  $x^*$  and the nominal parameter  $p_0$ , so as to obtain a locally linear model of the form

$$\Delta x(k+1) = A\Delta x(k) + B\Delta p(k) \quad (4)$$

where

$$A = \frac{\partial F}{\partial x} \quad \text{and} \quad B = \frac{\partial F}{\partial p}$$

with  $\Delta x(k) = x(k) - x^*$  and  $\Delta p = p(k) - p_0$ , which is used as a control input for stabilization. Typically, the OGY control method can be illustrated in a two dimensional phase plane [5, 2]. Assume, as mentioned above, the fixed point

$$\Delta x(k+1) = A\Delta x(k) \quad (5)$$

is a saddle. Then, there exist two eigenvalues,  $\lambda_u$  and  $\lambda_s$ , which satisfy  $|\lambda_s| < 1 < |\lambda_u|$ . Accordingly, there exist two associated left eigenvectors,  $v_u$  and  $v_s$ , such that

$$Av_u = \lambda_u v_u, \quad Av_s = \lambda_s v_s$$

On the other hand, the right eigenvectors,  $w_u$  and  $w_s$ , are defined as

$$w_u^\top A = \lambda_u w_u^\top, \quad w_s^\top A = \lambda_s w_s^\top$$

and one can easily verify that

$$[v_u \ v_s]^{-1} = [w_u \ w_s]^\top$$

in which

$$v_u^\top w_u = 1, \quad v_s^\top w_s = 1, \quad v_u^\top w_s = 0, \quad v_s^\top w_u = 0$$

The stable manifold is used as a vehicle to bring the system state to the fixed point: any drifting will be taken back to the stable manifold.

The OGY method suggests that we find a displacement of  $\Delta\rho$ , such that

$$w_u^\top \Delta x(k+1) = 0$$

which leads to a simple adjusting law for  $\Delta\rho$  as follows:

$$\Delta\rho = -\lambda_u \frac{w_u^\top \Delta x(k)}{w_u^\top B} \quad (6)$$

This is a simplified version of the original OGY method, which first performs the shifting and then the mapping. There are some other variations of the OGY method; nevertheless, the essence of this control method is to shift the orbit so that the movement along the direction perpendicular to the tangent of the stable manifold is zero, thereby keeping the orbit on the stable manifold. Thus, in the end, it will bring the orbit to the target.

For those second order Poincaré sections used for a third order chaotic system, if their corresponding  $A$  matrix contains complex eigenvalues and/or multiple unstable eigenvalues, then the OGY method cannot be applied directly. Partly, this is because (6) cannot be directly applied. Furthermore, for higher order chaotic dynamic systems, it is technically difficult to construct stable and unstable manifolds from time series, because there more likely exist some conjugate and/or multiple eigenvalues in the system Jacobian.

## 3 The Generalized OGY Chaos Control Methodology

As mentioned, the stable and unstable manifolds are constructed by using the system Jacobian eigenvalues and eigenvectors in the OGY method. Here, we propose to construct a set of stable manifolds within the framework of the chaotic system, independent of the system Jacobian eigenvalues and eigenvectors.

We first choose a set of desired stable manifolds, represented by

$$h(k) = h(\Delta x(k)) = C\Delta x(k) = 0 \in R^m \quad (7)$$

Similar to the idea of the OGY method, we will find a  $\Delta p(k)$  such that

$$h(k+1) = C\Delta x(k+1) = CA\Delta x(k) + CB\Delta p(k) = 0$$

This suggests that  $h(k) = 0$  will become the desired invariant manifolds, and the system orbit will lie on these manifolds.

Since these manifolds are to be determined, they will be designed to be asymptotically stable. This can be ensured by choosing a suitable  $C$  in the above. Moreover, they will be so designed that their intersection point is the desired fixed point which corresponds to the desired periodic orbit. Thus, once the trajectory reaches these stable manifolds, it will converge to the intersection point; subsequently, the controlled system orbit converges to the desired fixed point asymptotically. Therefore the desired periodic orbit is stabilized.

As indicated in [2], the one step ahead recursive method to reach a stable manifold is only effective in a vicinity of the desired fixed point, because the control magnitude is usually limited by the constraint  $|\Delta p(k)| \leq \bar{p}$  where  $\bar{p} > 0$ , which preserves the chaotic dynamics of the system. Outside the vicinity, it would be appropriate to keep the orbit to approach the stable manifolds, so that the convergence towards the stable manifolds becomes faster. An incremental control method can be used to realise this by limiting the stepsize of the control, so that the control force is always kept within the allowable range for chaos preservation.

The basic idea is, instead of reaching the desired fixed point within one step when the orbit is far away from it (which actually is impossible due the limitation of the control magnitude), we design the control such that the system orbit tends to (but may not reach) the stable manifolds. This leads to the condition

$$h_i(k) (h_i(k+1) - h_i(k)) < 0 \quad (8)$$

This condition shows that, if  $h_i(k) > 0$  then the control should be designed such that  $h_i(k+1) < h_i(k)$  (the orbit approaches  $h_i(k) = 0$  from one side where  $h_i(k) > 0$ ); if  $h_i(k) < 0$  then the control should be designed such that  $h_i(k) > h_i(k+1)$  (the orbit approaches  $h_i(k) = 0$  from the other side). To avoid overshoot on the manifold  $h(k) = 0$ , we define a boundary layer,  $\Omega$ , by

$$\Omega = \{ x(k) : \|h_i(k)\| \leq r, r > 0 \}$$

such that once the controlled orbit enters  $\Omega$ , the one step OGY chaos control formula is turned on; otherwise, the incremental control is activated.

It would be interesting to see the inherent dynamic and geometric properties of the system when, the controlled orbit stays in the invariant manifolds  $h(k) = 0$ . On the invariant manifolds, an “equivalent” control can be considered [12], as

$$\Delta p_{av}(k) = -(CB)^{-1}CAx(k) \quad (9)$$

which is obtained by solving the manifold equation

$h(k+1) = 0$ . The resulting dynamics on the manifolds is described by

$$\begin{aligned} \Delta x(k+1) &= A\Delta x(k) + B\Delta p(k) \\ &= (A - B(CB)^{-1}CA)\Delta x(k) \\ &= (I - B(CB)^{-1}C)A\Delta x(k) \end{aligned} \quad (10)$$

It can be proved that this dynamics is invariant with respect to  $C$ .

It is known that the linear projector,  $P = I - B(CB)^{-1}C$  [8], actually decomposes the space  $R$  into subspaces  $R_1$  and  $R_2$ , so that for any  $x \in R$ ,

$$x = x_1 + x_2, \quad x_1 \in R_1, \quad x_2 \in R_2 \quad (11)$$

the linear operator  $P$  is a projector on  $R_1$  along  $R_2$ , if

$$Px = x_1, \quad Px_2 = 0$$

Also, define  $N(C)$  as the null space of  $C$  and  $R(C)$  the range of  $C$ . It can be shown that  $P$  is a projector projecting  $R^{n-1}$  on  $N(C)$  along  $R(B)$  [8]. Since  $\text{rank}(C) = m$  and  $\text{rank}(B) = m$ , we have  $\text{rank}(B(CB)^{-1}C) = m$ . Also,  $\text{rank}(N(C)) = n - 1 - m$ . Therefore, the projector  $P$  maps  $R^{n-1}$  to  $N(C)$ ; hence,  $P$  is at most of rank  $n - 1 - m$ . The geometrical interpretation is that the effect of the projector  $P$  in the order of the chaotic system is reduced, because the state vector is constrained to lie on  $N(C)$ , which is an  $(n - 1 - m)$  dimensional subspace. Consequently, we conclude that the chaotic systems, when confined on the invariant manifold so constructed, will have  $m$  zero eigenvalues and will have  $n - m$  eigenvalues being the transmission zeros of the “equivalent systems” [12, 8].

These  $n - 1 - m$  eigenvalues can be arbitrarily assigned by properly chosen  $C$ , for example, by choosing all stable eigenvalues for the purpose of constructing the desired invariant manifolds. For a single stable manifold, one can simply use the pole assignment technique instead; for multiple stable manifolds, one can use some well known algorithms [9, 13]. The strength of this new control method lies in that the stable invariant manifolds are constructed independently of the system stable and unstable Jacobian eigenvalues and their associated eigenvectors. As such, it relaxes the restriction that the unstable eigenvalues have to be real in order to construct the desired invariant manifolds for chaos control. Furthermore, it presents a unified framework for designing an OGY based chaos control strategy for chaotic systems of any order. It is also worth noting that the OGY method is a special case of our method: for example, for a third order chaotic system, if the invariant manifold is chosen as one that uses a stable eigenvalue as its exponent, then our method is identical to the OGY method.

It is noticed that this class of control methods, based on invariant manifolds, are robust against matched noise

and disturbance – one of the benefits of using the sliding mode control concept.

The most influential variation is the modelling error due to the linear regression based modeling, denoted as  $\rho(k)$ , such that

$$\Delta x(k+1) = A\Delta x(k) + B\Delta p(k) + \rho(k)$$

If  $\rho(k) \in R(C)$  (i.e., the matching condition holds), then the invariant manifold  $h(k) = 0$  is invariant with respect to  $\rho(k)$  [12]. However, since  $\rho(k)$  is the modelling error which may not belong to  $R(C)$ , we will have

$$h(k+1) = CA\Delta x(k) + CB\Delta p(k) + C\rho(k) = C\rho(k) \neq 0$$

meaning that the invariant manifold cannot be exactly constructed. To ensure that the orbit stay as close as possible to the intersection of the invariant manifolds, one can choose a control consisting of a time delayed term so as to compensate the mismatch:

$$\Delta p(k) = -(CB)^{-1}CA\Delta x(k)(k) - (CB)^{-1}C\rho(k-1) \quad (12)$$

This leads to

$$\|h(k+1)\| = \|C\rho(k) - C\rho(k-1)\|$$

The time delayed chaos control law (12) would be effective if  $\rho(k)$  is a slowly varying variation. In fact, the modelling error can be expressed as

$$\rho(k) = \frac{\partial^2 F}{\partial x^2} \Delta^2 x|_{x=x^*} + \frac{\partial^2 F}{\partial p^2} \Delta^2 p + O(\Delta^3 x(k))$$

Since both the terms  $\frac{\partial^2 F}{\partial x^2} \Delta^2 x|_{x=x^*}$  and  $\frac{\partial^2 F}{\partial p^2} \Delta^2 p$  have a fixed sign, the variation  $\rho(k)$  is slowly varying, hence,  $\|C\rho(k) - C\rho(k-1)\|$  would be usually smaller than that under a control without the time delayed compensation term.

The eigenvalues of the invariant manifolds can be arbitrarily assigned by properly chosen  $C$ , for example, by choosing all the eigenvalues of the invariant manifolds to be stable (but not necessarily determined by the system Jacobian eigenvalues). Some well known algorithm [13] can also be applied for this purpose.

#### 4 A Case Study

We now report simulation results of the application of the proposed method to the Rossler system.

The Rossler system is described by

$$\dot{x}_1 = -x_2 - x_1 \quad (13)$$

$$\dot{x}_2 = x_1 + ax_2 \quad (14)$$

$$\dot{x}_3 = b + x_3(x_1 - c) \quad (15)$$

where the constants  $a = 0.15$ ,  $b = 0.03$ ,  $c = 10$ . Figure 1 shows an example of chaotic motion. We first perturb the parameter  $c$ , that is

$$\dot{x}_1 = -x_2 - x_1 \quad (16)$$

$$\dot{x}_2 = x_1 + ax_2 \quad (17)$$

$$\dot{x}_3 = b + x_3(x_1 - c + \Delta c) \quad (18)$$

where the  $\Delta c$  is the control used in this paper. The Poincare section is chosen to be the one parallel to the  $x - z$  plane at  $x_2 = -5.1458$ . On this section, the unstable fixed point to be stabilized is located at  $(x_1^*, x_3^*) = (16.1785, 0.1385)$ . By using least-squares fitting on the sampled data around the unstable fixed point, a linear map was obtained as

$$\begin{bmatrix} \Delta x_1(k+1) \\ \Delta x_2(k+1) \end{bmatrix} = A \begin{bmatrix} \Delta x_1(k) \\ \Delta x_2(k) \end{bmatrix} + B \Delta p(k) \quad (19)$$

where

$$A = \begin{bmatrix} -1.2940 & -23.9196 \\ -0.0985 & -1.9425 \end{bmatrix}, \quad B = \begin{bmatrix} -3.4453 \\ -0.1818 \end{bmatrix}.$$

The eigenvalues of the second-order map (19) were found to be  $-3.1873$  and  $-0.0492$ .

We first transformed the second-order system (19) into the controllable canonical form, and then assigned the eigenvalue of the invariant manifold, which was of first order, to be 0.4. Transforming back to its original coordinates yields the controller using (13) as

$$\Delta p(k) = \begin{bmatrix} -0.4128 & -7.7799 \end{bmatrix} \begin{bmatrix} \Delta x_1(k) \\ \Delta x_2(k) \end{bmatrix} - C\rho(k-1)$$

where  $\rho(k-1) = x(k) - Ax(k-1) - Bu(k-1)$  and  $C = [-0.02253 \quad -1.2306]$ . The activating region of the OGY control is limited to  $\{\Delta x(k) : \Delta^2 x_1(k) + \Delta^2 x_2(k) < 1\}$ .

The simulation was carried out with system initial state at  $(14.1785, -4.1458, 1.1385)$ . The simulation results are shown in Figure 2. Figure 2 (a) depicts the orbit of the controlled Rossler chaos in the three-dimensional space. From Figure 2 (b), one can see that the control action gradually decreases in time, and eventually the system orbit converges to the desired unstable periodic orbit. We also tested different initial conditions and our chaos control strategy was found to be robust. In fact, without the time delayed term, the control is effective as well.

#### 5 Conclusions

A generalized OGY chaos control method has been proposed in this paper. The advantages of the new method include: first, it preserves the spirit of the OGY control method in the sense that the controller derives the

chaotic orbit to a stable manifold for target tracking; second, the construction of the desired stable invariant manifolds are independent of the system Jacobian eigenvalues and eigenvectors, which is advantageous in case where the system Jacobian is not available from time series. This method is expected to be useful particularly for the control of hyperchaos pertaining to higher order systems, for such applications as secure communication and fluid mixing.

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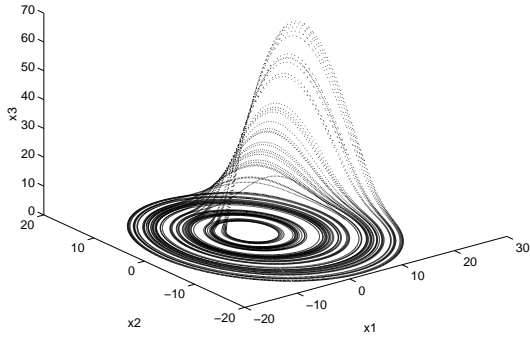


Figure 1(a): Rossler chaos in 3D space

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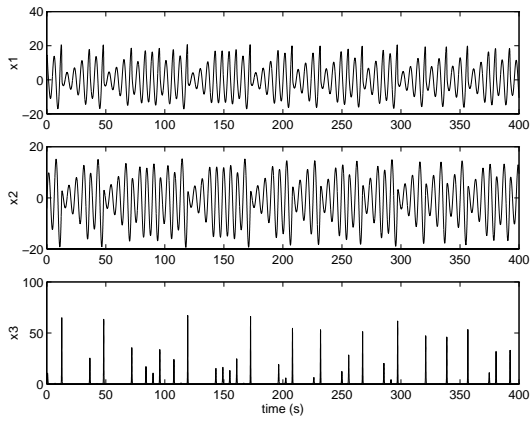


Figure 1(b): Time responses of Rossler chaos.

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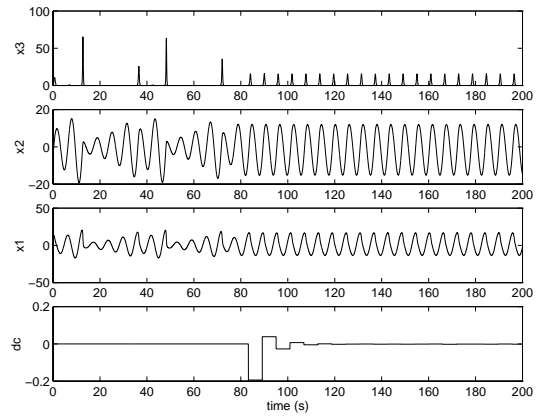


Figure 2(b): Time responses of the controlled Rossler chaos.

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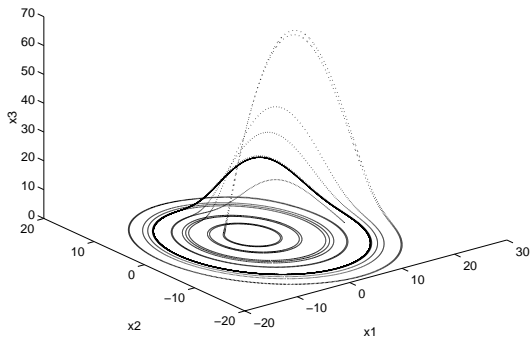


Figure 2(a): The controlled Rossler chaos in 3D space