Nonlinear Boundary Feedback Control of the One-Dimensional Wave Equation

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Abstract

In this paper, we analyze the dynamical behavior of the linear wave equation on an interval, where the right endpoint has a van der Pol type nonlinearity or boundary controller, while the left endpoint has a boundary condition involving displacement. The asymptotic behavior of the system can be classified into two basic types: classical unbounded instability, or spatial pointwise convergence to periodic points of a nonlinear map corresponding to the van der Pol condition.

1 Introduction

Acoustic chaos is a research topic that has seen a sharp surge of interest in engineering in recent years. A good number of papers [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] have been published. The research and applications in those papers include, e.g., the following:

(i) chaotic motions in acoustic cavitation [9, 8], which understandably is important for the detection of a submarine's underwater acoustic signature;

(ii) the development of musical instruments that play new types of sounds generated by chaotic acoustics, along with the corresponding musical notes for musicians to compose with [7, 17, 18];

(iii) study of sounds made by the vocal cords in the speech production process has revealed bifurcation and chaotic dynamics that could be used for personal (voice) identification and health diagnosis (for newborn infants from their cries [13]);

(iv) the design of *thermoacoustic* oscillators as possible musical instruments playing both ordinary and chaotic musical notes, and cooling/heating *thermoacoustic engines* whose control is achieved by boundary heating/cooling [17, 18].

Mathematically speaking, the basic model for sound wave propagation is the linear wave equation. In [1, 2, 3, 4], we have shown that various types of combinations of nonlinear and linear feedback control of the one-dimensional (1D) linear wave equation can cause chaotic vibration of the wave equation and, thus, acoustic chaos.

In this paper, we study the 1D wave equation

$$w_{xx}(x,t) - w_{tt}(x,t) = 0, \quad 0 < x < 1,\ t > 0, \tag{1}$$

with initial conditions

$$w(x,0) = w_0(x),\ w_t(x,0) = w_1(x), \quad 0 < x < 1, \tag{2}$$

and boundary feedback control containing a nonlinearity in the following form

$$w_t(0,t) + \gamma w(0,t) = 0, \quad \gamma \in \mathbb{R},\ t > 0; \tag{3}$$

$$w_x(1,t) = \alpha w_t(1,t) - \beta w_t^3(1,t), \tag{4}$$
$$0 < \alpha \le 1, \beta > 0,\ t > 0,$$

at, respectively, $x = 0$ and $x = 1$. Note that the left-end boundary condition (3) signifies either

$$w_t(0,t) = -\gamma w(0,t), \quad t > 0;$$

i.e., negative proportional (w) feedback to velocity (w_t), or

$$w(0,t) = -\frac{1}{\gamma} w_t(0,t), \quad t > 0;$$

i.e., negative velocity (w_t) feedback to displacement (w). And the right-end boundary condition (4) signifies a van der Pol-type nonlinear feedback of velocity (w_t) to force (w_x) which is self-regulating ([1, 2]). We want to study the long-term behavior of the solution w under the control effect. In (1) we have assumed both the wave speed and the interval to be 1 just for convenience.

The major technical difficulty in this study is due to the presence of the displacement $w(0,t)$ in (3). This is the first time we are able to successfully treat this situation involving $w(0,t)$, which constitutes the main thrust of this paper.

2 Nonlinear Iterative Maps and the Statements of Main Results

Define

$$u(x,t) = \frac{1}{2}[w_x(x,t) + w_t(x,t)], \tag{5}$$

$$v(x,t) = \frac{1}{2}[w_x(x,t) - w_t(x,t)]. \tag{6}$$

Then (u,v) satisfies the diagonalized first-order hyperbolic system

$$\begin{aligned}&\frac{\partial}{\partial t}\begin{bmatrix}u(x,t)\\ v(x,t)\end{bmatrix} \\ &= \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix}u(x,t)\\ v(x,t)\end{bmatrix}, \quad 0<x<1,\ t>0,\end{aligned} \tag{7}$$

and the method of characteristics can be applied. The boundary conditions (3) and (4) lead to, respectively, the following reflection relations

$$\begin{aligned}&v(0,t) + \gamma\int_0^t v(0,s)ds = u(0,t) \\ &\quad + \gamma\int_0^t u(0,s)ds + \gamma w(0,0), \quad t>0;\end{aligned} \tag{8}$$

$$\begin{aligned}&\beta[u(1,t)-v(1,t)]^3 + (1-\alpha)[u(1,t)-v(1,t)] \\ &\quad + 2v(1,t) = 0, \quad t>0.\end{aligned} \tag{9}$$

To study eq. (9), we define a nonlinear map

$$F\colon\ v \mapsto u = F(v)$$

where for each given $v \in \mathbb{R}$, $u \in \mathbb{R}$ is the unique real solution of the cubic equation

$$\beta(u-v)^3 + (1-\alpha)(u-v) + 2v = 0. \tag{10}$$

Using F, we now define a map

$$\begin{aligned}\mathcal{F}\colon\ X \equiv\ & C([0,1]) \times C([0,1]) \times \mathbb{R} \to X;\\ &\mathcal{F}(f_0, g_0, a_0) = (f_1, g_1, a_1);\\ &\quad f_0, f_1, g_0, g_1 \in C([0,1]); \quad a_0, a_1 \in \mathbb{R},\end{aligned}$$

where

$$\begin{aligned}f_1(x) &= F(g_0(1-x)),\\ g_1(x) &= f_0(1-x) + \gamma e^{-\gamma(1-x)} \cdot a_0,\\ a_1 &= e^{-\gamma}a_0.\end{aligned}$$

We also define the projection operators $P_1, P_2\colon\ X \to C([0,1])$ by

$$\begin{aligned}&P_1(f_0,g_0,a_0) = f_0, \quad P_2(f_0,g_0,a_0) = g_0,\\ &\quad \text{for} \quad (f_0,g_0,a_0) \in X.\end{aligned} \tag{11}$$

Upon using the method of characteristics, the solution (u,v) of (7) can be expressed explicitly as follows: for given $(x,t), x \in [0,1], t>0$, write $t = n+\tau$, where $n \in \mathbb{N} = \{0,1,2,\dots\}$, $0 \le \tau < 1$. Then

$$v(x,t) = \begin{cases} P_1 \circ \mathcal{F}^{n+1}(u_0(1+x-\tau), \\ v_0(1+x-\tau), v(0,0)), \\ \hfill \text{if } x < \tau, \\ P_1 \circ \mathcal{F}^{n}(u_0(x-\tau), \\ v_0(x-\tau), w(0,0)), \\ \hfill \text{if } x \ge \tau; \end{cases} \tag{12}$$

$$u(x,t) = \begin{cases} P_2 \circ \mathcal{F}^{n}(u_0(x+\tau), \\ v_0(x+\tau), w(0,0)), \\ \hfill \text{if } x+\tau < 1, \\ P_2 \circ \mathcal{F}^{n+1}(u_0(x+\tau-1), \\ v_0(x+\tau-1), w(0,0)), \\ \hfill \text{if } x+\tau \ge 1, \end{cases} \tag{13}$$

where in (12) and (13), $\mathcal{F}^k$ denotes the k-times iterative composition $\mathcal{F} \circ \mathcal{F} \circ \cdots \circ \mathcal{F}$ of the map $\mathcal{F}$ (rather than the kth power of $\mathcal{F}$) for $k = n$ and $n+1$. Therefore, we see that the long-term behavior of (u,v) is completely determined by that of the iterates $\mathcal{F}^n$ as $n \to \infty$. However, we can further reduce the order of $\mathcal{F}$ as follows. Let

$$(u_n, v_n, a_n) = \mathcal{F}(u_{n-1}, v_{n-1}, a_{n-1}), \text{ for } n = 1, 2, \dots .$$

Then

$$\begin{aligned}u_n(x) &= F(v_{n-1}(1-x)),\\ v_n(x) &= u_{n-1}(1-x) + \gamma e^{-\gamma(1-x)},\\ a_n &= e^{-\gamma}a_{n-1}.\end{aligned} \tag{14}$$

From (14) we obtain

$$\begin{aligned}u_n(x) &= F(u_{n-2}(x) + \gamma e^{-\gamma x}a_{n-2}),\\ a_n &= e^{-2\gamma}a_{n-2}, \text{ for } n \ge 2.\end{aligned} \tag{15}$$

From (15), we obtain also

$$v_n(x) = u_{n-1}(x) + \gamma e^{-\gamma(1-x)}a_{n-1}, \text{ for } n \ge 1.$$

Therefore, it is sufficient to study the iterates of the following reduced-order map

$$\widetilde{\mathcal{F}}\colon\ \widetilde{X} \equiv C([0,1]) \times \mathbb{R} \to \widetilde{X}, \tag{16}$$

$$\widetilde{\mathcal{F}}(f_0, a_0) = (f_1, a_1), \tag{17}$$

where

$$f_1(x) = F(f_0(x) + \gamma e^{-\gamma x} a_0), \quad a_1 = e^{-2\gamma} a_0. \qquad (18)$$

Our major theorems can now be stated below.

Theorem 1. *Assume that* $\gamma < 0$. *Let* $(y_n, a_n) = \mathcal{F}^n(y_0, a_0) \in \widetilde{X}$. *Then*

(i) if $a_0 < 0$, *then* $a_n \to -\infty$, $\sup_{[0,1]} y_n \to +\infty$, *as* $n \to \infty$.

(ii) if $a_0 > 0$, *then* $a_n \to +\infty$, $\inf_{[0,1]} y_n \to -\infty$, *as* $n \to \infty$,

(iii) if $a_0 = 0$, *then* $a_n = 0$ *for all* $n = 1, 2, \ldots$, *and for each* $x \in (0,1)$,

one of the following three cases will hold:

(a) $\lim_{n\to\infty} y_n(x) = 0$; *(b)* $\lim_{n\to\infty} y_{2n+1}(x) = v_p$, $\lim_{n\to\infty} y_{2n}(x) = -v_p$;

(c) $\lim_{n\to\infty} y_{2n+1}(x) = -v_p$, $\lim_{n\to\infty} y_{2n}(x) = v_p$,

where $v_p > 0$ *is the unique (period-2) point of* F *such that* $F(v_p) = -v_p$, $F(-v_p) = v_p$. ☐

Theorem 2. *Assume that* $\gamma > 0$. *Let* $(y_n, a_n) = \mathcal{F}^n(y_0, a_0) \in \widetilde{X}$. *Then*

(i) if $a_0 = 0$, *then the conclusion of case (iii) in Theorem 1 remains valid.*

(ii) if $a_0 \neq 0$, *then* $\lim_{n\to\infty} a_n = 0$ *and there is a constant* $B > 0$ *depending only on* α *and* β *in (4) such that*

$$\|y_n\|_{C([0,1])} \le B, \qquad \forall\, n \ge N,$$

for some N *sufficiently large.* ☐

3 Proof of Theorem 1

First, we recall the profile and certain properties of the nonlinear map F. The graph of F is given in Fig. 1.

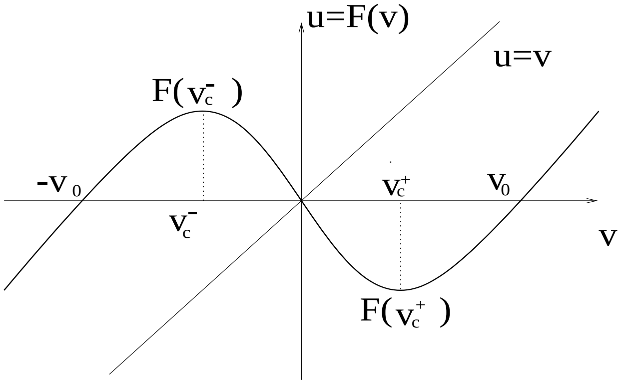

Fig. 1 The graph of the nonlinear map F defined from (10). F is an odd function with 3 intercepts: $-v_0, 0$ and v_0. There are two local extremal points v_c^- and v_c^+. Here we have used $\alpha = 0.5$ and $\beta = 1$.

Let us list the key properties of F that will be used in this paper:

(P$_1$) F is an odd function: $F(-x) = -F(x)$, for all $x \in \mathbb{R}$;

(P$_2$) F has 3 x-axis intercepts: $-v_0, 0$ and v_0;

(P$_3$) F has a positive local maximum at v_c^- and a negative local minimum at v_c^+; $v_c^+ = -v_c^-$;

(P$_4$) $F(v_c) > -v_0$;

(P$_5$) $F(x) > x$ for $x \in (-\infty, 0)$ and $x > F(x)$ for $x \in (0, \infty)$;

(P$_6$) $0 < F'(x) < 1$ for all $x \in (-\infty, v_c^-) \cup (v_c^+, \infty)$;

(P$_7$) For any $u > v_c^+$,

$$0 < m_u^+ \equiv \min_{x\in[u,\infty)} F'(x) = F'(u) < 1.$$

Similarly, for any $u < v_c^-$,

$$0 < m_u^- \equiv \min_{x\in(-\infty,u)} F'(x) = F'(u) < 1.$$

The verification of (P$_1$)–(P$_7$) above may be found in [1].

From the reduced order map $\widetilde{\mathcal{F}}$ in (16), let us fix our attention at a given point $x_0 \in [0,1]$ and consider a 2×2 iterative map $\widetilde{\mathcal{F}}_{x_0}$:

$$(y_{n+1}, a_{n+1}) = \widetilde{\mathcal{F}}_{x_0}(y_n, a_n) = (y_n + \gamma e^{-\gamma x_0} a_n, e^{-2\gamma} a_n)$$
$$\text{for any } (y_n, a_n) \in \mathbb{R}^2, \quad n = 0, 1, 2, \ldots .$$

This map $\widetilde{\mathcal{F}}_{x_0}$ is obviously related to $\widetilde{\mathcal{F}}$ by being the pointwise evaluation of $\widetilde{\mathcal{F}}$ at x_0. By abuse of notation, we still write $\widetilde{\mathcal{F}}_{x_0}$ as $\widetilde{\mathcal{F}}$.

Let $a_0 < 0$, and $y_0 \in \mathbb{R}$. Note that $\gamma < 0$. If we can show that

$$\lim_{n\to\infty} y_n = \infty,$$

then (i) will be established. We proceed in two steps in the following.

(i) We show that there exists an integer $N_0 \geq 0$ such that $y_{N_0} \in (-v_0, \infty)$.

We write $b = \gamma e^{-\gamma x_0} < 0$. If $y_0 \in (-v_0, \infty)$, then just take $N_0 = 0$. Otherwise $y_0 \in (-\infty, -v_0]$. If $y_n \notin (-\infty, -v_0]$ for all $n = 1, 2, \ldots$, then we will show a contradiction. Define

$$N_0 = \left[\frac{-v_0 - y_0}{m^-_{-v_0} b a_0}\right] + 1; \text{ cf. } (\mathrm{P}_7) \text{ for } m^-_{-v_0}; \quad [r]$$

denotes the integral part of a given real number r.

Then because $-v_0 - y_0 \geq 0$, $m^-_{-v_0} b a_0 > 0$, we have $N_0 \geq 1$. It follows that

$$\begin{aligned} y_{N_0} &= F(y_{N_0-1} + b a_{N_0-1}) \\ &= F(y_{N_0-1}) + F'(\xi_{N_0-1}) b a_{N_0-1} \\ &\quad (\ \xi_{N_0-1} \in (y_{N_0-1}, y_{N_0-1} + b a_{N_0-1}) \subseteq (y_0, -v_0)\ ) \\ &\geq F(y_{N_0-1}) + m^-_{-v_0} b a_{N_0-1} \\ &\geq F(y_{N_0-2}) + m^-_{-v_0} b a_{N_0-2} + m^-_{-v_0} b a_{N_0-1} \\ &\geq F(y_{N_0-2}) + 2 m^-_{-v_0} b a_{N_0-2} \\ &\ \vdots \\ &\geq F(y_0) + N_0 m^-_{-v_0} b a_0 \\ &> y_0 + N_0 m^-_{-v_0} b a_0 > -v_0, \end{aligned} \tag{19}$$

which is a contradiction.

(ii) Now, we prove that if $y_0 \in (-v_0, \infty)$, $a_0 < 0$, then $y_n \to \infty$ as $n \to \infty$.

Since F is continuous, $F(v_0) = 0$ and $F(\infty) = \infty$, there is a $v_1 \in (v_0, \infty)$ such that $F(v_1) = v_0$. It is clear that if $y_0 \in (-v_0, \infty)$, then $y_n \in [F(v_0), \infty]$ for $n \geq 1$, by (P_4). Because a_n increases exponentially in magnitude with n, there is an integer $N > 1$ such that

$$F(v_c^+) + b a_N > v_1.$$

We claim that $y_{N+i} \geq 0$ for $i \geq 1$. This will be done by induction on i. When $i = 1$,

$$\begin{aligned} y_{N+1} &= F(y_N + b a_N) \\ &\geq F(F(v_c^+) + b a_N) \\ &\quad \text{since } F \text{ is increasing on } [v_c^+, \infty) \\ &\quad \text{and } y_N \in (F(v_c^+), \infty)) \\ &> F(v_1) = v_0 > 0. \end{aligned}$$

Assume that when $i \leq k$, the statement that $y_{N+k} \geq 0$ is valid. Then

$$\begin{aligned} y_{N+k+1} &= F(y_{N+k} + b a_{N+k}) \\ &\geq F(F(v_c^+) + b a_N) \\ &\geq F(v_1) = v_0 > 0. \end{aligned}$$

Therefore $y_{N+i} \geq 0$ for all $i = 1, 2, \ldots$.

Now, we have

$$\begin{aligned} y_{N+i+1} &= F(y_{N+i} + b a_{N+i+1}) \\ &\geq F(b a_{N+i+1}). \end{aligned}$$

Since $b a_{N+i+1} \to \infty$ as $i \to \infty$ and due to the fact that F is increasing on (v_c, ∞) and $F(x) \to \infty$ as $x \to \infty$, we have $y_{N+i+1} \to \infty$ as $i \to \infty$. Therefore (i) has been proved. The proof of (ii) is the same as (i). We now consider (iii), where $a_0 = 0$. This implies $a_n = 0$ for all $n = 1, 2, \ldots$. Therefore the iteration becomes $y_{n+1} = F(y_n)$, the same as in that in [1]. Since F has only one fixed point $x = 0$ (which is repelling) and two periodic points, v_p and $-v_p$, of period two (which are attracting), and no other periodic points, we see that (iii) follows immediately. ☐

4 Proof of Theorem 2

Throughout this section, $\gamma > 0$. The case $a_0 = 0$ is easy. So let us consider only the case $a_0 \neq 0$. We may assume that $a_0 > 0$ because the case $a_0 < 0$ may be treated by using the oddness of the map F. We now show that we may choose $B = v_0$ in the statement of Theorem 2 and make it valid.

By (P_6), we have $1 - F'(x) > 0$ for $x \in [v_0, \infty)$. Therefore

$$\begin{aligned} &G(x) \equiv x - F(x) \text{ is strictly increasing on} \\ &\quad [v_0, \infty); \text{ implying} \\ &x - F(x) > v_0 - F(v_0) = v_0, \text{ for all } x \geq v_0. \end{aligned} \tag{20}$$

Because a_n decreases exponentially with n, we can find an integer $N_1 > 0$ such that

$$0 < b a_n < \min\left(\frac{v_0}{2}, v_0 - F(v_c^-)\right), \text{ for all } n > N_1. \tag{21}$$

Using (P_3) and applying the mean value theorem just as in (19), we can establish that

$$\text{for } y_0 \in [-v_0, v_0], \text{ we have } y_{N_1} \in [-M, M] \tag{22}$$

$$\text{where } M \equiv \max\{F(v_c^-), v_0 + N_1 b a_0\}. \tag{23}$$

Because N_1 in (23) is an arbitrary positive integer, any $M > 0$ sufficiently large will make (22) work.

We can now show the following:

Claim: Let $M > 0$ be given and $y_0 \in [-M, M]$. If $0 < ba_0 < \min\left(\frac{v_0}{2}, v_0 - F(v_c^-)\right)$, then $y_n \in [-v_0, v_0]$ for all $n > N$.

Proof of Claim: It is clear that once $y_{n_0} \in [-v_0, v_0]$ for some positive integer $n_0 > N_1$, then $y_n \in [-v_0, v_0]$ for $n > n_0$ because

$$\begin{aligned} &F(v_c^-) = \max_{x \in [-v_0, v_0]} F(x) \text{ and} \\ &F(v_c^-) + ba_n < v_0, \text{ by (21).} \end{aligned}$$

Set $N = \left[\frac{2M}{v_0}\right] + 1$. We want to prove that if $y_0 = M$, then $y_n \in [-v_0, v_0]$ for $n > N$. Assume the contrary. Then $y_k \notin [-v_0, v_0]$ for $k = 0, \ldots, N$ due to the fact established in the preceding paragraph. Thus $y_k > v_0$ for $k = 0, \ldots, N$, since it is clear that $y_k \notin (-\infty, -v_0]$ for $k = 0, \ldots, N$, if a_0 is sufficiently small (without loss of generality). Now,

$$\begin{aligned} y_N &= F(y_{N-1} + ba_{N-1}) \\ &= F(y_{N-1}) + F'(\xi_{N-1})ba_{N-1} \\ &\quad (\xi_{N-1} \in (y_{N-1}, y_{N-1} + ba_{N-1})) \\ &< F(y_{N-1}) + ba_{N-1} \\ &\quad (\ F'(\xi_{N-1}) < 1 \text{ by } (\mathrm{P}_6)\ ) \\ &< F(y_{N-1}) + ba_0 < F(y_{N-1}) + \frac{v_0}{2} \\ &\quad \text{(by assumption of the Claim)} \\ &< y_{N-1} - v_0 + \frac{v_0}{2} \\ &\quad (\text{because } y_{N-1} \in (v_0, \infty) \text{ and (21)}) \\ &= y_{N-1} - \frac{v_0}{2} \\ &< y_{N-2} - \frac{v_0}{2} \cdot 2 < \cdots \\ &< y_0 - \frac{v_0}{2} \cdot N \\ &= M - \frac{v_0}{2}\left(\left[\frac{2M}{v_0}\right] + 1\right) < 0, \end{aligned}$$

contradicting $y_N > v_0$. So the Claim has been verified.

Similarly, if $y_0 = -M$, then $y_n \in [-v_0, v_0]$ for $n > N$.

Therefore, we have justified the claim.

The rest of the proof is easy. For any $y_0 \in \mathbb{R}$, through N_1 iterations for N_1 sufficiently large satisfying (22), there are two possibilities:

(i) $y_{N_1} \in [-v_0, v_0]$; (ii) $y_{N_2} \in (-\infty, -v_0) \cup (v_0, \infty)$.

In case (i), we may apply the Claim directly. In case (ii), we set $M = |y_{N_1}|$. Then by the Claim, there is an $n_0 > 0$ such that $y_n \in [-v_0, v_0]$ for $n > n_0$. Thus $|y_n| \leq v_0 = B_0$ for $n > n_0$. □

Acknowledgement G. Chen was partially supported by Texas A&M University Interdisciplinary Research Initiative IRI 99-22. S.B. Hsu was partially supported by a grant from the National Research Council of Republic of China.

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