Feedback Control of Hopf Bifurcations via Integral Averaging Method[1]

MingQing Xiao
Department of Mathematics
Southern Illinois University
Carbondale, IL 62901/USA
mxiao@math.siu.edu

Wei Kang
Department of Mathematics
Naval Postgraduate School
Monterey, CA 93943/USA
wkang@nps.navy.mil

Abstract

We study the local feedback stabilization of Hopf bifurcations for nonlinear systems of infinite-dimensions in the case where the linearized vector field has a pair of simple nonzero imaginary eigenvalues and all its other eigenvalues lie strictly in the left half-plane. Through discussing the normal form of nonlinear systems obtained by the integral averaging method, we discuss some conditions for controlling the stability of the systems, provided that the critical modes are uncontrollable. As an example, we apply the obtained results to the control of axial flow compressor model.

Key words: Nonlinear systems; Hopf bifurcation; Integral Averaging; Stabilization.

1 Introduction

Bifurcation phenomenon is one of the inherent behaviors of nonlinear systems and is of great physical interest ([11], [12], [20], [21]). Hopf bifurcation is an extremely important nonlinear behavior whose systems can be relevant in "describing" nature ([20]). It results from that the system has (linearized) modes with zero real part (called critical modes) and no unstable modes. In such a case, linear theory is inadequate for discussing the stability of the system. In the case of critical modes of the nonlinear system are uncontrollable, the exact controllability does not hold, and thus classical results of uniform feedback stabilization can no longer be applied.

The study of feedback stabilization of Hopf bifurcation control for finite-dimensional nonlinear systems was first seen in 1986, which was carried out by Abed and Fu ([1]) in which the linearizations of general nonlinear systems have a pair of simple, nonzero imaginary eigenvalues. Abed and Fu also discussed the stationary bifurcation, in which the critical linearized system possesses a simple zero eigenvalue ([2]). By applying the projection method (see [9]), Abed and Fu proposed a stabilizing feedback control by calculating the Lyapunov coefficient.Colonius and Kliemann considered the stabilization of one dimensional control systems [6]. Using normal form of nonlinear systems, Kang developed the analysis and control design algorithm for systems with one uncontrollable mode. [14]-[16]. Gu, Chen, Sparks, and Banda studied the bifurcation stabilization using output feedback where observability assumption is not necessary [13]. Wang and Murray studied the feedback stabilization of steady-state and Hopf bifurcations with multi-input case [25]. Zaslavsky investigated the feedback stabilization of connected nonlinear oscillators with uncontrollable linearization [27].

For *infinite-dimensional* nonlinear systems, the study of feedback stabilization of bifurcation control has not been seen yet, though infinite-dimensional models may represent the dynamics of physical systems more accurately than their finite-dimensional approximates in many practical systems. The major mathematical challenge is that the picture of the classification of all possible bifurcation diagrams is incomplete although there are several outstanding results concerning local bifurcations of codim 1 and 2 in infinite dimensions ([7], [18]). This is indeed the case even in higher finite-dimensions. For Hopf bifurcation only, the picture is relatively simple: there are totally three cases of Hopf bifurcation which are supercritical, subcritical, and degenerate ([20], [18]). All three cases are of great physical interest([8]). The supercritical case, for example, can describe the first stage of transition to turbulence in fluids as postulated by Landau [19], and the subcritical case occurs for flows that exhibit an immediate transition behavior [20].

2 Statement of the Problem

Let X, Y be Hilbert spaces with $Y \hookrightarrow X$, where "$\hookrightarrow$" denotes the continuous dense injection. We consider an

---

[1] Research supported in part by AFOSR under F49620-95-1-0409.

abstract evolution system

$$\frac{dx}{dt} = A(\mu)x + f_\mu(x, u), \tag{2.1}$$

where state $x \in X$, μ is a real parameter, $A(\mu)$ is the infinitesimal generator of an analytic semigroup on X for each μ in a neighborhood of the origin

$$f_\mu : Y \times U \to X$$

is a smooth function depending on the parameter μ with $f_\mu(0, 0) = 0$. Thus we may write f_μ as

$$f_\mu(x, u) = f_\mu^{(0)}(x) + f_\mu^{(1)}(x, u) \tag{2.2}$$

where $f_\mu^{(0)}(0) = 0$ and $f_\mu^{(1)}(x, 0) = 0$. A particular example is a parabolic control system of the form

$$\begin{aligned} &\frac{\partial \mathbf{x}}{\partial t} = D\Delta\mathbf{x} + M(\mu)\mathbf{x} + F_\mu\left(\xi, \mathbf{x}, u(\mathbf{x})\right), \\ &\xi \in \Omega = \text{smooth, open, bounded set in } \mathbf{R}^n, \\ &\mathbf{x} = 0 \quad \text{for} \quad \xi \in \partial\Omega. \end{aligned}$$

In this case $A(\mu) = D\Delta + M(\mu)$, where D is a positive diagonal matrix and $M(\mu)$ is a parameter-dependent matrix, and choices of X, Y would possibly be $H_0^2(\Omega)$, $L^2(\Omega)$ or $C_0^{2+\alpha}(\Omega)$, $C^\alpha(\Omega)$.

Under the further assumptions that $A(0)$ is the infinitesimal generator of analytic semigroup and $A(0)$ has only a pair of simple, nonzero imaginary eigenvalues, Crandall and Rabinowitz had shown an infinite-dimensional version of Hopf bifurcation theorem which guarantees the bifurcation of a family of time periodic solutions of an evolution equation from a family of equilibrium solutions by directly using the implicit function theorem ([7]) in the case of $u(x) \equiv 0$. Almost at the same time Chow and Mallet-Paret proved the similar result by using integral averaging method under the assumption that there exists a center manifold through the origin ([5]). The basic idea is to decompose the equation into three coupled equations which are equations for the amplitude r, the phase angle θ, and the stable part x_2. The series of coordinate changes, which is computable, decouples the r equation up to a certain order in r. One of the advantage to use integral averaging method is that system (2.1) can be transformed into a normal form (called averaged system) through a coordinate change, and the normal form not only can provide the information of the type of bifurcation but also can provide the structure information how the bifurcation forms, which is very useful for the controller design. We will elaborate it next.

3 Integral Average Method

Let us assume that $A(0)$ has only a pair of simple, nonzero imaginary eigenvalues (instead of finite many) and the spectrum of $A(0)$ is assumed to lie a positive distance from the imaginary axis except for the pair of nonzero imaginary eigenvalues. According to ([24]) we have the spectral decomposition

$$X = Z \oplus Q \text{ or } X = Z(\mu) \oplus Q(\mu) \text{ when } \mu \text{ is near } 0$$

where Z is the two-dimensional eigenspace of $A(0)$ corresponding to the simple, nonzero imaginary eigenvalues. Let $Q_1 = Q \cap Y$, then (2.1) admits a decomposition

$$\begin{aligned} x &= x_1 + x_2 \in Z \oplus Q_1 = Y, \\ \dot{x} &= x_1 + x_2 \in Z \oplus Q = X. \end{aligned}$$

Let $P : X \to Z$ be the projection defined by the direct sum, and let

$$\begin{aligned} f_{\mu 1}(x_1, x_2, \mu) &= Pf_\mu(x_1 + x_2, u(x_1 + x_2)) \\ f_{\mu 2}(x_1, x_2) &= (I - P)f_\mu(x_1 + x_2, u(x_1 + x_2)) \end{aligned}$$

System (2.1) can be written as

$$\begin{aligned} \dot{x}_1 &= A_Z(\mu)x_1 + f_{\mu 1}(x_1, x_2) \\ \dot{x}_2 &= A_Q x_2 + f_{\mu 2}(x_1, x_2) \end{aligned} \tag{3.3}$$

Without lose of generality, we can assume that when $\mu = 0$ system (2.1) has the form

$$\begin{aligned} \dot{x}_1 &= A_Z(0)x_1 \\ \dot{x}_2 &= A_Q x_2 \end{aligned} \tag{3.4}$$

Thus the linearized equation of (2.1) is given by

$$\begin{aligned} \dot{x}_1 &= A_Z(\mu)x_1 + \mu E(\mu)x_2 \\ \dot{x}_2 &= A_Q x_2 + \mu H(\mu)x_1 + \mu M(\mu)x_2 \end{aligned} \tag{3.5}$$

Where $E(\mu) \in \mathcal{L}(Q, Z)$, $H(\mu) \in \mathcal{L}(Z, Q)$, and $M(\mu) \in \mathcal{L}(Q, Q)$. Since f_μ is sufficiently smooth, we can expand the (2.1) as follows:

$$\begin{aligned} \dot{x}_1 &= B_0(x_2, \mu) + B_1(x_2, \mu)x_1 + B_2(x_2, \mu)x_1^2 + \ldots, \\ \dot{x}_2 &= \Gamma_0(x_1, \mu) + \Gamma_1(x_1, \mu)x_2 + \Gamma_2(x_1, \mu)x_2^2 + \ldots. \end{aligned}$$

where $B_i(x_2, \mu)$ and $\Gamma_i(x_1, \mu)$ are the **bounded symmetric *i*-linear operators** [17]. Then we can assume that

$$\begin{aligned} B_0(x_2, \mu) &= \mu E(\mu)x_2 + F(x_2, \mu)x_2^2, \\ B_1(x_2, \mu) &= A_Z(\mu) + G(x_2, \mu)x_2, \\ \Gamma_0(x_1, \mu) &= \mu H(\mu)x_1 + J(x_1, \mu)x_1^2, \\ \Gamma_1(x_1, \alpha) &= A_Q + \mu M(\mu) + N(x_1, \mu)x_1, \end{aligned}$$

where for fixed $x_2 \in Q$, $F(x_2, \mu) : Q \times Q \to Z$ is the bounded symmetric 2-linear operator, $G(x_2, \mu)x_2 \in \mathcal{L}(Q, Z)$, $J(x_1, \mu) : Z \times Z \to Q$ is the bounded symmetric 2-linear operator, and $N(x_1, \mu)x_1 \in \mathcal{L}(Z, Q)$.

Let $T(t)$ denote the C_0 semigroup generated by $A(0)$ and $S(t)$ be the restriction of $T(t)$ on Q. Assume that Q

is $S(t)$-invariant, $e^{A_Q t} = S(t)$, and there exist positive constants M and α such that

$$\|S(t)\| \leq M e^{-\alpha t}, \qquad t \geq 0.$$

Then according to ([4]) or ([20]), there exists a (local) center manifold $\Pi : x_2 = h(x_1, \mu) \in Y$ through the origin, and tangent to the (x_1, μ) space. Let $x_1 = (r\cos\theta, r\sin\theta)$. Then the dynamics takes the form

$$\begin{aligned} \dot{r} &= [\mu E_1(\theta,\mu)x_2 + F_1(\theta,x_2,\mu)x_2^2] \\ &\quad + r[\mu + G_2(\theta,x_2,\mu)x_2] \\ &\quad + r^2 C_3(\theta,x_2,\mu) + r^3 C_4(\theta,x_2,\mu) + \ldots, \\ \dot{\theta} &= \frac{1}{r}[\mu \hat{E}_1(\theta,\mu)x_2 + \hat{F}_1(\theta,x_2,\mu)x_2^2] \\ &\quad + [\omega(\mu) + \hat{G}_2(\theta,x_2,\mu)x_2] \\ &\quad + rD_3(\theta,x_2,\mu) + r^2 D_4(\theta,x_2,\mu) + \ldots, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \dot{x}_2 &= \Gamma_0(r,\theta,\mu) + \Gamma_1(r,\theta,\mu)x_2 + \Gamma_2(r,\theta,\mu)x_2^2 \\ &\quad + \Gamma_3(r,\theta,\mu)x_2^3 + \ldots \end{aligned} \tag{3.7}$$

where $E_1, \hat{E}_1, F_1, \hat{F}_1, G_1, \hat{G}_1$ are operators which are computed from E, F, G on appropriate spaces and C_3, C_4, D_3, D_4 are real-value functions. Scale above equations by

$$r \to \epsilon r, \qquad x_2 \to \epsilon x_2, \qquad \mu \to \epsilon\mu$$

we have

$$\begin{aligned} \dot{r} &= \epsilon[\mu r + r^2 C_3(\theta,\epsilon x_2,\epsilon\mu) + \mu E_1(\theta,\epsilon\mu)x_2 \\ &\quad + F_1(\theta,\epsilon x_2,\epsilon\mu)x_2^2 + rG_2(\theta,\epsilon x_2,\epsilon\mu)x_2] \\ &\quad + \epsilon^2 r^3 C_4(\theta,\epsilon x_2,\epsilon\mu) + O(\epsilon^3), \end{aligned} \tag{3.8}$$

$$\begin{aligned} \dot{\theta} &= \omega_0 + \epsilon[\mu\omega'(0) + rD_3(\theta,\epsilon x_2,\epsilon\mu) \\ &\quad + \tfrac{\mu}{r}\hat{E}_1(\theta,\mu\epsilon)x_2 + \tfrac{1}{r}\hat{F}_1(\theta,\epsilon x_2,\epsilon\mu)x_2^2 \\ &\quad + \hat{G}_2(\theta,\epsilon x_2,\epsilon\mu)x_2] + O(\epsilon^2), \end{aligned} \tag{3.9}$$

$$\begin{aligned} \dot{x}_2 &= A_Q x_2 + \epsilon[\mu H(\epsilon\mu)x_1 + J(\epsilon x_1,\epsilon\mu)x_1^2 \\ &\quad + \mu M(\epsilon\mu)x_2 + N(\epsilon x_1,\epsilon\mu)x_1 x_2 \\ &\quad + \Gamma_2(\epsilon x_1,\epsilon\mu)x_2^2] + O(\epsilon^2). \end{aligned} \tag{3.10}$$

Using the following coordinate transformation

$$\overline{r} = r + \epsilon\phi_1(r,\theta,\mu,\epsilon) + \epsilon w(r,\theta,\mu,\epsilon)x_2 + \epsilon^2\phi_2(r,\theta,\mu,\epsilon)$$

where ϕ_1, ϕ_2 are scalar functions which have been chosen so that the ϵ-term and ϵ^2-term can be killed, and $w(r,\theta,\mu,\epsilon) = r\hat{w}(\theta) \in Q_1^*$ (Q_1^* is the dual space of Q_1) satisfies

$$G_2(\theta,0,0) + d\hat{w}(\theta)/d\theta\omega_0 + \hat{w}(\theta)A_Q = 0, \tag{3.11}$$

system (2.1) can then be transformed into the averaged system of the form

$$\dot{\overline{r}} = \epsilon\mu\overline{r} + \epsilon^2\kappa\overline{r}^3 + O(\epsilon^3), \tag{3.12}$$

$$\dot{\theta} = \omega_0 + O(\epsilon), \tag{3.13}$$

$$\dot{x}_2 = A_Q x_2 + O(\epsilon), \tag{3.14}$$

where $\kappa := \kappa_1 + \kappa_2$ and

$$\begin{aligned} \kappa_1 &= \tfrac{1}{2\pi}\int_0^{2\pi} \left[C_4(\theta,0,0) - \tfrac{1}{\omega_0}C_3(\theta,0,0)D_3(\theta,0,0)\right] d\theta \\ \kappa_2 &= \tfrac{1}{2\pi}\int_0^{2\pi} \hat{w}(\theta)J(0,0)(\cos\theta,\sin\theta)^2 d\theta. \end{aligned}$$

All periodic solutions of (2.1) bifurcating from the origin and $\mu = 0$ can be obtained from (3.12)-(3.14).

4 Feedback Control of Hopf Bifurcations

Theorem 1. *Suppose that when $u(x) \equiv 0$, $\kappa \neq 0$. Then the necessary condition for controlling the stability of system (2.1) through the term $(I-P)f_\mu^{(1)}(x,u)u$ is $G_2(\cdot,0,0) \neq 0$, where G_1 appears in (3.6).*

Theorem 2. *Suppose that*

$$Pf_\mu^{(1)}(x_1 + x_2, u) = \beta_4(x_2,u)x_1^4 + \beta_5(x_2,u)x_1^5 + \cdots \tag{4.15}$$

and $G_2(\cdot,0,0) \neq 0$. If there exists a function $v \in U$ such that

1. *$f_\mu^{(1)}(x_1,u) = \mathcal{F}_\mu(x_1)x_1^2$, where $\mathcal{F}_\mu : Z \times Z \to Q$ is a bounded symmetric linear operator,*
2. *$\int_0^{2\pi} \hat{w}(\theta)\mathcal{F}_0(0)(\cos\theta,\sin\theta)^2 d\theta \neq 0$.*

then there exists a control function $u = cv$ which can determine the stability of the system (2.1) by choosing approbate scalar c. Furthermore if we assume $Re\lambda'(0) = \alpha$ and let

$$\kappa_3 = \frac{1}{2\pi}\int_0^{2\pi} \hat{w}(\theta)\mathcal{F}_0(0)(\cos\theta,\sin\theta)^2 d\theta \tag{4.16}$$

and $\kappa = \kappa_1 + \kappa_2 + \kappa_3$, then the radius r_0 of the periodic solution is about

$$r_0 \sim \sqrt{\left|\frac{\alpha}{\kappa}\right|}. \tag{4.17}$$

Let us consider the following simple example: let $x_1 = (x_{11}, x_{12})$ and

$$\begin{aligned} \dot{x}_{11} &= \mu x_{11} - x_{12} + x_{11}x_2 + x_{11}(x_{11}^2 + x_{12}^2) + O(x_1,x_2)^4 \\ \dot{x}_{12} &= x_{11} + \mu x_{12} + x_{12}x_2 + x_{12}(x_{11}^2 + x_{12}^2) + O(x_1,x_2)^4 \\ \dot{x}_2 &= -x_2 + (x_{11} + x_{12})^2 + u + O(x_1,x_2,u)^3 \end{aligned}$$

where $O(x)^k$ stands for homogeneous polynomials of degree k in x. In this case $G_2 = \omega_0 = \hat{w} = 1$ and $J(x_1) = (1,1,1,1)$. When $u \equiv 0$, a direct calculation yields $\kappa_1 = \kappa_2 = 1$, and thus the above system has the averaged normal form:

$$\begin{aligned} \dot{\overline{r}} &= \epsilon\mu\overline{r} + 2\epsilon^2\overline{r}^3 + O(\epsilon^3) \\ \dot{\theta} &= 1 + O(\epsilon) \\ \dot{x}_3 &= -x_3 + O(\epsilon) \end{aligned}$$

which indicates that the system undergoes the subcritical Hopf bifurcation at $\mu = 0$. For small μ, the radius of the bifurcating periodic solution is about $\sqrt{\frac{|\mu|}{2}}$. If we let $u = c(x_1 + x_2)^2$, then we have $\kappa_3 = c$, hence $\kappa = c + 2$. If we choose $c < -2$ then the bifurcation is supercritical and the radius of the bifurcating periodic solution is near $\sqrt{|\frac{\mu}{c+2}|}$.

5 Application to the analysis of axial flow compressor model

Problems of compressor instability have been of concern to aircraft engine designers for two decades because the engine performance is effectively reduced by rotating stall and surge. Rotating stall, which corresponds to a traveling wave of gas around the annulus of the compressor, occurs when a nonaxisymmetric flow disturbance develops (around the annulus of the rotor) and causes drastic reduction in the performance of the compressor. As rotating stall develops, axisymmetric oscillations across the compression system, known as surge, occur. Surge is a low-frequency, large-amplitude oscillation of the mean flow rate in the compressor which induces high blade, causing stress levels and possible reverse flow which affects flow conditions throughout the entire compression system.

The following are pressure-balance equations of a compression system, called the Moore-Greitzer model, which describes the behavior of the axial flow compression system and whose derivation can be found in [22]:

$$l_c \frac{d\Phi}{dt} = -\Psi(t) + \frac{1}{2\pi} \int_0^{2\pi} \psi_c(\Phi - \phi'_\eta \big|_{\eta=0}) d\theta \quad (5.18)$$

$$l_c \frac{d\Psi}{dt} = \frac{1}{4B^2}(\Phi(t) - \Phi_T) \quad (5.19)$$

Here, the subscripts t, θ, η denote partial differentiation; ϕ' denotes the upstream disturbance velocity; Φ is the annulus-averaged axial flow coefficient; Ψ is the annulus-averaged total-to-static pressure rise coefficient; t is the dimensionless time variable; ψ_c is the characteristic function of the compressor; B is the plenum/compressor volume ratio, which is called Greitzer B-parameter; m is the duct parameter; a is the internal compressor lag; θ is the circumferential angle around the compressor annulus; $\nu > 0$ is the viscous coefficient; and Φ_T is the throttle characteristic, which controls the dimensionless mass flow through the throttle.

Under the assumption that the velocity field is unperturbed at the entrance, the upstream disturbance velocity ϕ' satisfies Laplace's equation:

$$\phi'_{\theta\theta} + \phi'_{\eta\eta} = 0, \quad (\theta, \eta) \in [0, 2\pi] \times (-\infty, 0). \quad (5.20)$$

The boundary conditions are periodic in θ, and at $\eta = 0$

$$\begin{aligned} &\tfrac{\partial}{\partial t}(m\phi' + \tfrac{1}{a}\phi'_\eta) - \Big(\psi_c(\Phi + \phi'_\eta) \\ &- \tfrac{1}{2\pi}\textstyle\int_0^{2\pi} \psi_c(\Phi + \phi'_\eta)d\theta - \tfrac{1}{2a}\phi'_{\theta\eta} + \tfrac{\nu}{2}\phi'_{\eta\theta\theta}\Big) = 0 \end{aligned} \quad (5.21)$$

and when $\eta = -\infty$

$$\phi' = 0 \quad (5.22)$$

which provides the expression for the pressure rise between the upstream reservoir and the exit duct discharge. Equation (5.18) is obtained by computing the circumferential mean of boundary condition (5.21) and describes the change of the mass flow through the compressor. Equation (5.19) accounts for dynamic pressure changes downstream the compressor exit, in the plenum and across the throttle. The compressor characteristic has the cubic form:

$$\psi_c(\Phi) = a_0 + a_1\Phi + a_2\Phi^2 + a_3\Phi^3. \quad (5.23)$$

The throttle characteristic is given by $\Phi_T(\Psi) = \mu\sqrt{\Psi}$.

Let $\dot{L}^2(0, 2\pi)$ be the space of all square integrable 2π-periodic functions with zero average. If we define the flow disturbance at the compressor face to be $g := \phi'_\eta|_{\eta=0}$, then it can be written as

$$g(t, \theta) = \sum_{n \in \mathbf{Z}\backslash\{0\}} \overline{\phi}_n(t) e^{|n|\eta + in\theta}\Big|_{\eta=0}$$

where $\overline{\phi}_n = n\tilde{\phi}_n$. Next we introduce a linear operator $K : \dot{L}^2(0, 2\pi) \to \dot{L}^2(0, 2\pi)$ by

$$K(\phi) = \sum_{n \in \mathbf{Z}\backslash\{0\}} \{1 + \frac{am}{|n|}\} \tilde{\phi}_n e^{in\theta}$$

for any $\phi = \sum_{n \in \mathbf{Z}\backslash\{0\}} \tilde{\phi}_n e^{in\cdot} \in \dot{L}^2(0, 2\pi)$. Clearly, K is a positive definite, self-adjoint linear operator on $\dot{L}^2(0, 2\pi)$. Thus

$$< \phi, \psi >_K := < \phi, K\psi >_{L^2(0,2\pi)}$$

defines an equivalent inner product on $\dot{L}^2(0, 2\pi)$, and we denote by $\dot{L}^2_K(0, 2\pi)$ the space which consists of elements of $\dot{L}^2(0, 2\pi)$ with inner product $< \cdot, \cdot >_K$. Let $X = L^2_K(0, 2\pi) \times \mathbf{R} \times \mathbf{R}$, with inner product

$$\begin{aligned} < x_1, x_2 > = a^{-1} < g_1, g_2 >_{\dot{L}^2_K(0,2\pi)} \\ + l_c\Phi_1\Phi_2 + (4l_cB^2)\Psi_1\Psi_2 \end{aligned} \quad (5.24)$$

where $x_i = (g_i, \Phi_i, \Psi_i) \in X, i = 1, 2$, and let the norm on this space be defined by $\|x\| := \sqrt{< x, x >}$ for $x \in X$. Let $x_e = \{g_e, \phi_e, \psi_e\}$ be an equilibrium point of (5.25) with $g_e = 0, \psi_e = \psi_c(\phi_e)$ and

$$f_\mu(g, \Phi, \Psi) = \begin{bmatrix} aK^{-1}(\psi_c(\Psi + g) - \overline{\psi}_c) \\ \frac{1}{l_c}(\overline{\psi}_c - \Psi) \\ \frac{1}{4l_cB^2}(\Phi - \Phi_T) \end{bmatrix},$$

where

$$\overline{\psi}_c = \frac{1}{2\pi}\int_0^{2\pi} \psi_c(\Phi + g)d\theta$$

If we introduce the linear operator A:

$$\begin{aligned} A(\mu) &= \begin{bmatrix} K^{-1}\left(\frac{\nu}{2}\frac{\partial^2}{\partial\theta^2} - \frac{1}{2}\frac{\partial}{\partial\theta}\right) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ D(A) &= \Big\{x \in X \mid \frac{\partial g}{\partial\theta}, \frac{\partial^2 g}{\partial\theta^2} \in \dot{L}_K^2(0, 2\pi) \text{ and} \\ &\quad g(0) = g(2\pi) \text{ where } x = (g, \Phi, \Psi) \in X\Big\} \end{aligned}$$

and then the compressor model can be written as an evolution equation on X as follows:

$$\frac{dx}{dt} = A(\mu)x + f_\mu(x). \qquad (5.25)$$

The $A(\mu) + df_{x_e}(\mu)$ has a pair of complex conjugate eigenvalues $\lambda(\mu)$ and $\overline{\lambda}(\mu)$ such that

$$\text{Re } \lambda'(\mu_c) < 0, \qquad \text{Re } \lambda(\mu_c) = 0, \qquad \text{Im } \lambda(\mu_c) \neq 0,$$

By using the integral averaging method, system (5.25) undergoes subcritical Hopf bifurcation as μ is reduced to a critical value $\mu = \mu_c$ (see [26]), so there is a hysteresis effect in recovering from rotating stall. This analytic result matches the observed behavior obtained through experiments. The averaged system of (5.25) has been found with $\kappa = \kappa_1 + \kappa_2$, where

$$\kappa_1 = -\frac{3\pi a_3}{4}, \qquad \kappa_2 = -\frac{3\nu(2 + am)\psi_c''(\phi_e(\mu_c))}{4[(2 + am)^2 + 9\nu^2]}$$

and $a_3 < 0$. The radius r_0 of the (unstable) periodic solution is

$$r_0 \sim \sqrt{\left|\frac{\alpha}{\kappa}\right|} = \sqrt{\left|\frac{\alpha}{\kappa_1 + \kappa_2}\right|}$$

where

$$\alpha := \frac{a}{1 + am}\psi_c''(\phi_e(\gamma_c))\phi_e'(\gamma_c).$$

Next we consider the throttle feedback control:

$$\Phi_T(\Psi, g) := \left(\mu + \frac{c}{\pi}\int_0^{2\pi} g^2(\theta)d\theta\right)\sqrt{\Psi}. \qquad (5.26)$$

In this case

$$f_\mu^{(1)}(x, u) = \begin{bmatrix} 0 \\ 0 \\ \frac{c}{\pi}\int_0^{2\pi} g^2(\theta)d\theta\sqrt{\Psi} \end{bmatrix} \qquad (5.27)$$

and

$$\begin{aligned} f_\mu^{(1)}(x_1, u) &= \begin{bmatrix} 0 \\ 0 \\ \frac{c}{\pi}\int_0^{2\pi}(s_1\cos\theta + s_2\sin\theta)^2 d\theta\sqrt{\psi_e} \end{bmatrix} \\ &= c\sqrt{\psi_e}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}^2 \end{aligned}$$

Now we need to solve the following equation:

$$G_2(\zeta, 0, \gamma_c) + \frac{1}{2}(w^*)'(\zeta) + w^*(\zeta)A_Q(0) = 0. \qquad (5.28)$$

Assume

$$G_2(\zeta, 0, \gamma_c) = \sum_{n=-\infty}^{\infty} g_n e^{in\zeta}, \quad g_n \in (V^\perp)^*,$$

where $g_n \in (V^\perp)^*$, $V = span\{\cos\theta, \sin\theta\}$. By expanding $w^*(\zeta)$ as a Fourier series

$$w^*(\zeta) = \sum_{n=-\infty}^{\infty} w_n e^{in\zeta},$$

inserting this into Eq.(5.28) and equating coefficients, we arrive at

$$w_n = -g_n(A_Q(\mu_c) + \frac{in}{2})^{-1}.$$

We then have

$$\begin{aligned} \kappa_3 &= \frac{c\sqrt{\psi_e}}{2\pi}\int_0^{2\pi} w^*(\zeta)\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\zeta \\ \sin\zeta \end{bmatrix}^2 d\zeta \\ &= \frac{c\sqrt{\psi_e}}{2\pi}\int_0^{2\pi} w_0(\zeta)\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\zeta \\ \sin\zeta \end{bmatrix}^2 d\zeta. \end{aligned}$$

Since $w_0 = g_0 A_Q^{-1}(\mu_c)$, we next determine the linear functional g_0. Note that for any $v = [v_1, v_2, v_3]^T \in V^\perp$,

$$\begin{aligned} G_2(\zeta, 0, \gamma_c)v = \psi_c''(\phi_e(\gamma_c))\Big[&\int_0^{2\pi}(v_1 + v_2)\cos^2\theta d\theta\cos^2\zeta \\ &+ 2\int_0^{2\pi}(v_1 + v_2)\cos\theta\sin\theta d\theta\sin\zeta\cos\zeta \\ &+ \int_0^{2\pi}(v_1 + v_2)\sin^2\theta d\theta\sin^2\zeta\Big] \\ = \psi_c''(\phi_e(\gamma_c))&\int_0^{2\pi}(v_1 + v_2)\cos^2(\theta - \zeta)d\theta. \end{aligned}$$

Clearly, for all $[v_1, v_2, v_3]^T \in V^\perp$

$$g_0\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \frac{1}{2\pi}\int_0^{2\pi} G_2(\zeta, 0, \gamma_c)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} d\zeta$$

$$= \frac{\psi_c''(\phi_e(\gamma_c))}{2\pi}\int_0^{2\pi}\int_0^{2\pi}(v_1 + v_2)\cos^2(\theta - \zeta)d\theta d\zeta$$

and

$$A_Q^{-1}(\mu_c)\begin{bmatrix} 0 \\ 0 \\ \sqrt{\psi_e} \end{bmatrix}$$

$$= 4l_c^2 B^2 \left(1 - \frac{\mu_c}{2\sqrt{\psi_c}}\right)^{-1} \begin{bmatrix} 0 \\ \frac{\sqrt{\psi_c}}{l_c} \\ \frac{\psi_c'(\phi_e)\sqrt{\psi_e}}{l_c} \end{bmatrix}.$$

Therefore we have

$$g_0 A_Q^{-1}(\mu_c) \begin{bmatrix} 0 \\ 0 \\ \sqrt{\psi_e} \end{bmatrix} = 4\pi l_c B^2 \left(1 - \frac{\mu_c}{2\sqrt{\psi_c}}\right)^{-1} \sqrt{\psi_e}$$

and

$$\kappa_3 = -4c\psi_c''(\phi_e(\gamma_c))\pi l_c B^2 \left(1 - \frac{\mu_c}{2\sqrt{\psi_c}}\right)^{-1} \psi_e \quad (5.29)$$

Choose c such that $\kappa = \kappa_1 + \kappa_2 + \kappa_3 < 0$, then the feedback throttle control (5.30) can stabilize the system (2.1).

Theorem 3. *The feedback controller*

$$\Phi_T(\Psi, g) := \left(\mu + \frac{c}{\pi}\int_0^{2\pi} g^2(\theta)d\theta\right)\sqrt{\Psi}. \quad (5.30)$$

stabilizes the compression system (5.25) locally by choosing appropriate c. That is, it can change the Hopf bifurcation from subcritical to supercritical. The radius of the periodic solution can be controlled by c, given by

$$r_0 \sim \sqrt{\left|\frac{\alpha}{\kappa}\right|} = \sqrt{\left|\frac{\alpha}{\kappa_1 + \kappa_2 + \kappa_3}\right|}$$

where

$$\alpha := \frac{a}{1 + am}\psi_c''(\phi_e(\gamma_c))\phi_e'(\gamma_c).$$

References

[1] E. Abed and J. Fu. Local feedback stabilization and bifurcation control, I. Hopf bifurcation. *Systems & Control Letters*, 7(1986), pages 11-17.

[2] E. Abed and J. Fu. Local feedback stabilization and bifurcation control, II. Stationary bifurcation. *Systems & Control Letters*, 8(1987), pages 467-473.

[3] D. G. Aronson. The asymptotic speed of propagation of a simple epidemic. *NSF-CBMS Regional Conf. Nonlinear Diffusion Equations, Univ. Houston, Houston, Tex., 1976*, pp. 1–23. Res. Notes Math., No. 14, Pitman, London, 1977

[4] J. Carr. Applications of Centre Manifold Theory. *Springer-Verlag*, 1981.

[5] S. Chow and J. Mallet-Paret. Integral averaging and bifurcation. *Journal of Differential Equations*, 26(1977), pages 112-159.

[6] F. Colonius and W. Kliemann. Controllability and stabilization of one dimensional systems near bifurcation points. *Systems and Control Letters*, **24**, 87-95, 1995.

[7] M. Crandall and P. Rabinowitz. The Hopf bifurcation theorem in infinite dimensions. *Arch. Rat. Mech. Anal.*, 67(1977), pages 53-72.

[8] M. Crandall and P. Rabinowitz. Mathematical theory of bifurcation. *In: Bifurcation Phenomena in Mathematical Physics and Related Topics*, edited by C. Bardos and D. Bessis. Reidel: Dodrecht, 1980.

[9] G. Iooss and D. D. Joseph. Elementary Stability and Bifurcation Theory. *Springer*, New York, 1980.

[10] I.J. Day, E.M. Greitzer and N.A. Cumpsty. Prediction of compressor performance in rotating stall. Journal of Engineering for Power, Vol. 100, 1-14, 1978.

[11] P. C. Fife. Asymptotic states for equations of reaction and diffusion. *Bull. Am. Math. Soc.*, 84, 1978, pages 693-726.

[12] P. C. Fife. Mathematical Aspects of Reaction-Diffusion Systems. *Lecture Notes in Biomath.*, 28, Springer, Berlin, 1979.

[13] G. Gu, X. Chen, A. G. Sparks and S. S. Banda. Bifurcation stabilization with local output feedback linearization of nonlinear systems. *SIAM J. Control and Optimization*, **30** (1999), 934-956.

[14] W. Kang. Bifurcation and normal form of nonlinear control systems, Part I. *SIAM J. Control and Optimization*, 1(36), 1998, pages 193-212.

[15] W. Kang. Bifurcation and normal form of nonlinear control systems, Part II. *SIAM J. Control and Optimization*, 1(36), 1998, pages 213-232.

[16] W. Kang. Bifurcation control via state feedback for systems with a single uncontrollable mode. *SIAM J. Control and Optimization*, to appear 1999-2000.

[17] T. Kato. Perturbation Theory for Linear Operators. *Berlin : Springer*, 1995.

[18] Y. A. Kuznetsov. Elements of Applied Bifurcation Theory. *Springer*, New York, 1995.

[19] L. Landau and E. Lifshitz. Fluid Mechanics. *Pergamon*, Oxford, 1959.

[20] J. E. Marsden and M. F. McCracken. The Hopf Bifurcation and Its Applications. *Springer-Verlag*, 1976.

[21] F. McCaughan. Bifurcation analysis of axial flow compressor stability. *SIAM J. of Appl. Math. 50*, 1232-1253, 1990.

[22] F. K. Moore and E. M. Greitzer. A theory of post-stall transients in axial compression systems: Part I–Development of Equations *ASME Journal Engineering for Gas Turbines and Power,* 108:68-76, 1986.

[23] D.C. Liaw and E.H. Abed. Active control of compressor stall inception: a bifurcation theoretic approach. *Automatica*, Vol. 32, 109-115, 1996.

[24] B. Sz-Nagy and C. Foias. Harmonic Analysis of Operators on Hilbert Space. *American Elsevier*, New York, 1970.

[25] Y. Wang and R. M. Murray. Feedback stabilization of steady-state feedback and Hopf bifurcations. *Proc. of the 38th IEEE CDC*, pp. 2431-2437, 1999.

[26] M. Xiao and T. Başar. Center manifold of the viscous Moore-Greitzer PDE model. *SIAM Journal of Applied Mathematics,* to appear 2000.

[27] B. Zaslavsky. Feedback stabilization of connected nonlinear oscillators with uncontrollable linearization. *Systems Control Lett.*, **27**, no. 3, 181–185, 1996.