Optimal control of backward stochastic differential equations: The linear-quadratic case

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Abstract

This paper is concerned with optimal control of linear backward stochastic differential equations (BSDEs) with a quadratic cost criteria, or backward linear–quadratic (BLQ) control. The solution of this problem is obtained completely and explicitly by using an approach which is based primarily on the completion-of-squares technique. Two alternative, though equivalent, expressions for the optimal control are obtained. The first of these involves a pair of Riccati type equations, an uncontrolled BSDE and an uncontrolled forward stochastic differential equation (SDE), while the second is in terms of a Hamiltonian system. A key step in our derivation is a proof of global solvability of the aforementioned Riccati equations. Although of independent interest, this issue has particular relevance to the BLQ problem since these Riccati equations play a central role in our solution. Last but not least, it is demonstrated that the optimal control obtained coincides with the solution of a certain *forward* linear–quadratic (LQ) problem. This, in turn, reveals the origin of the Riccati equations introduced.

Key words: Backward stochastic differential equations (BSDEs), linear–quadratic (LQ) optimal control, Riccati equations, completion of squares.

1 Introduction

A backward stochastic differential equation (BSDE) is an Ito stochastic differential equation for which a *random terminal condition* on the state has been specified. The linear version of this type of equations was first introduced by Bismut [4] as the adjoint equation in the stochastic maximum principle (see also [3, 18, 21]). General nonlinear BSDEs, introduced independently by Pardoux and Peng [17] and Duffie and Epstein [9], have received considerable research attention in recent years due to their nice structure and wide applicability in a number of different areas, especially in mathematical finance (see, e.g., [7, 10, 11, 14, 16, 20]).

Unlike a (forward) stochastic differential equation (SDE), the solution of a BSDE is a *pair* of adapted processes $(x(\cdot),\, z(\cdot))$. The additional term $z(\cdot)$ may be interpreted as a risk-adjustment factor and is required for the equation to have *adapted* solutions. This restriction of solutions to the class of *adapted processes* is necessary if the insights gained from the study of BSDEs are to be useful in applications. For recent accounts on BSDE theory and applications, the reader is referred to the books [16, 20].

Since a BSDE is a well-defined dynamic system, it is very natural and appealing, first at the theoretical level, to consider the optimal control of the BSDE. As for applications, optimally controlled BSDEs promise to have a great potential. For example, an optimal control problem of a linear BSDE comes out in the process of solving a *forward* stochastic linear–quadratic (LQ) control problem in [6]. Moreover, controlled BSDEs are expected to have important applications in mathematical finance. For instance, a situation in which funds may be injected or withdrawn from the replication process of a contingent claim so as to achieve some other goal may be viewed quite naturally as an optimal BSDE control problem. However, the study on controlled BSDEs is quite lacking in literature. To our best knowledge there are only a few papers dealing with optimal control of BSDEs, including [19] and [8] which established local and global maximum principles, respectively, and [11] in which a controlled BSDE with linear state drift is studied.

This paper is concerned with optimal control of a linear BSDE with a quadratic cost criteria, namely, a stochastic backward linear–quadratic (BLQ) problem. It is well-known that LQ control is one of the most important classes of optimal control and the solution of this problem has had a profound impact on many engineering applications. Stochastic forward LQ theory has been well established, especially with the recent development on the so-called *indefinite* stochastic LQ

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control ([1, 5, 6, 15]). However, stochastic BLQ control remains an almost completely unexplored area.

The main contribution of this paper is a complete solution of a general BLQ problem. As it turns out, the optimal control depends linearly on the entire past history of the state process $(x(\cdot), z(\cdot))$. In addition, explicit formulas for the optimal control and the optimal cost in terms of a pair of Riccati equations, a Lyapunov equation, an uncontrolled BSDE and an uncontrolled SDE are established. A key part of our derivation is a proof of existence and uniqueness of solutions of the Riccati equations mentioned above. Due to the constraints of space, however, we have not included the proof of this result, and refer the interested reader to the paper [13].

It is interesting to remark that our original approach to solving the BLQ problem was inspired by [12] where an (uncontrolled) BSDE is viewed as a *controlled forward* SDE. Extending this idea, we can show that the optimal control of the BLQ problem is the limit of a sequence of square integrable processes, obtained by solving a family of *forward* LQ problems. During this procedure, the key Riccati equations, along with other related equations, come out very naturally. What is more interesting is that once these equations are in place, one may forget about the forward formulation and limiting procedure, which is rather complicated, and instead use these equations *directly* along with the completion-of-square technique to obtain the optimal control for the original BLQ problem. Nevertheless, the forward formulation still represents an alternative, and insightful, approach to the backward control problem and for this reason, an outline of this procedure is also presented in this paper.

2 Problem formulation

We assume throughout that $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, P)$ is a given and fixed complete filtered probability space and that $W(\cdot)$ is a scalar-valued Brownian motion on this space. (Our assumption that $W(\cdot)$ is scalar-valued is for the sake of simplicity. No essential difficulties are encountered when extending our analysis to the case of vector-valued Brownian motions). In addition, we assume that $\mathcal{F}_t$ is the augmentation of $\sigma\{W(s) \mid 0 \leq s \leq t\}$ by all the P-null sets of $\mathcal{F}$.

Throughout this paper, we denote the set of symmetric $n \times n$ matrices with real elements by S^n. If $M \in S^n$ is positive (semi-)definite, we write $M > (\geq) \ 0$. Let X be a given Hilbert space. The set of X-valued continuous functions is denoted by $C(0, T; X)$. If $N(\cdot) \in C(0, T; S^n)$ and $N(t) > (\geq) \ 0$ for every $t \in [0, T]$, we say that $N(\cdot)$ is positive (semi-)definite, which is denoted by $N(\cdot) > (\geq) \ 0$. Suppose $\eta : \Omega \to \mathbf{R^n}$ is an $\mathcal{F}_T$-random variable. We write $\eta \in L^2_{\mathcal{F}_T}(\Omega; \mathbf{R^n})$ if η is square integrable (i.e. $E|\eta|^2 < \infty$). Consider now the case when $f : [0, T] \times \Omega \to \mathbf{R^n}$ is an $\{\mathcal{F}_t\}_{t\geq 0}$ adapted process. If $f(\cdot)$ is square integrable (i.e. $E\int_0^T |f(t)|^2\, dt < \infty$) we shall write $f(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbf{R^n})$; if $f(\cdot)$ is uniformly bounded (i.e. $E \operatorname{ess\,sup}_{t\in[0,T]} |f(t)| < \infty$) then $f(\cdot) \in L^\infty_{\mathcal{F}}(0, T; \mathbf{R^n})$. If $f(\cdot)$ has (P-a.s.) continuous sample paths and $E \sup_{t\in[0,T]} |f(t)|^2 < \infty$ we write $f(\cdot) \in L^2_{\mathcal{F}}(0, T; C(0, T; \mathbf{R^n}))$; if $E \sup_{t\in[0,T]} |f(t)| < \infty$ then $f(\cdot) \in L^\infty_{\mathcal{F}}(0, T; C(0, T; \mathbf{R^n}))$. These definitions generalize in the obvious way to the case when $f(\cdot)$ is $\mathbf{R^{n\times m}}$- or S^n-valued. Finally, in cases where we are restricting ourselves to deterministic Borel measurable functions $f : [0, T] \to \mathbf{R^n}$, we shall drop the subscript $\mathcal{F}$ in the notation; for example $L^\infty(0, T; \mathbf{R^n})$.

Consider the BSDE:

$$\begin{cases} dx(t) = \Big\{A(t)\,x(t) + B(t)\,u(t) + C(t)z(t)\Big\}\,dt \\ \quad +z(t)\,dW(t), \\ x(T) = \xi, \end{cases} \tag{1}$$

where $u(\cdot)$ is the control process. The class of *admissible controls* for (1) is:

$$\mathcal{U} = L^2_{\mathcal{F}}(0, T; \mathbf{R^n}). \tag{2}$$

Later, we shall state assumptions on the coefficients $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ and the terminal condition ξ so as to guarantee the existence of a unique solution pair $(x(\cdot), z(\cdot)) \in L^2_{\mathcal{F}}(\Omega; C(0, T; \mathbf{R^n})) \times \mathbf{L^2_{\mathcal{F}}(0, T; R^n)}$ of the BSDE (1) for every admissible control $u(\cdot) \in \mathcal{U}$. We refer to such a 3-tuple $(x(\cdot), z(\cdot); u(\cdot))$ as an *admissible triple*. The cost associated with an admissible triple $(x(\cdot), z(\cdot); u(\cdot))$ is given by:

$$\begin{aligned} J(\xi; u(\cdot)) := E\,\frac{1}{2}\,\Big[&x(0)'Hx(0) \\ &+ \int_0^T \Big(x(t)'Q(t)x(t) + z(t)'S(t)z(t) \\ &+ u(t)'R(t)u(t)\Big)\,dt\Big]. \end{aligned} \tag{3}$$

The *backward linear–quadratic (BLQ) control problem* can be stated as follows:

$$\begin{cases} \min J(\xi; u(\cdot)) \\ \text{subject to:} \\ \quad u(\cdot) \in \mathcal{U}, \\ \quad (x(\cdot), z(\cdot); u(\cdot)) \text{ satisfies (1).} \end{cases} \tag{4}$$

Throughout this paper, we shall assume the following:

Assumption:

(**A1**)

$$\begin{cases} A,\, C & \in & L^\infty(0,\, T;\, \mathbf{R^{n\times n}}), \\ B & \in & L^\infty(0,\, T;\, \mathbf{R^{n\times m}}), \\ Q,\, S & \in & L^\infty(0,\, T;\, S^n),\ Q,\ S \geq\ 0, \\ R & \in & L^\infty(0,\, T; S^m),\ R\ >\ 0, \\ H & \in & S^n,\ \ H\ >\ 0, \\ \xi & \in & L^2_{\mathcal{F}_T}(\Omega;\, \mathbf{R^n}). \end{cases}$$

In particular, Assumption (**A1**) is sufficient to guarantee the existence of a unique solution pair $(x(\cdot),\, z(\cdot)) \in L^2_{\mathcal{F}}(\Omega; C(0,\, T;\, \mathbf{R^n})) \times \mathbf{L^2_{\mathcal{F}}(0,\, T;\, R^n)}$ of (1) for every admissible control $u(\cdot) \in \mathcal{U}$; see [20, Chapter 7].

3 Main result

In this section, we present two alternative though equivalent expressions for the optimal BLQ control. The first one gives an explicit formula via a pair of Riccati equations, an uncontrolled BSDE and an uncontrolled SDE:

$$\begin{cases} \dot\Sigma(t) - A(t)\Sigma(t) - \Sigma(t)A(t)' \\ \quad -\Sigma(t)Q(t)\Sigma(t) + B(t)R(t)^{-1}B(t)' \\ \quad +C(t)\Sigma(t)\Big(S(t)\Sigma(t)+I\Big)^{-1}C(t)' = 0, \\ \Sigma(T) = 0. \end{cases} \tag{5}$$

$$\begin{cases} \dot Z(t) + Z(t)A(t) + A(t)'Z(t) \\ \quad +Z(t)\Big[B(t)R(t)^{-1}B(t)' \\ \quad +C(t)\Sigma(t)(I+S(t)\Sigma(t))^{-1}C(t)'\Big]Z(t) \\ \quad -Q(t) = 0, \\ Z(0) = H, \end{cases} \tag{6}$$

$$\begin{cases} dh(t) = \Big\{\Big(A(t)+\Sigma(t)Q(t)\Big)h(t) \\ \quad +C(t)\Big(I+\Sigma(t)S(t)\Big)^{-1}\eta(t)\Big\}\,dt \\ \quad +\eta(t)\,dW(t), \\ h(T) = -\xi. \end{cases} \tag{7}$$

$$\begin{cases} dq(t) = \Big\{-\Big[A(t)+B(t)R(t)^{-1}B(t)'Z(t) \\ \quad +C(t)(I+\Sigma(t)S(t))^{-1}\Sigma(t)C(t)'Z(t)\Big]'\,q(t) \\ \quad +Z(t)C(t)(I+\Sigma(t)S(t))^{-1}\eta(t)\Big\}dt \\ \quad +\Big\{(Z(t)-S(t))(I+\Sigma(t)S(t))^{-1}\eta(t) \\ \quad +(I+Z(t)\Sigma(t))(I+S(t)\Sigma(t))^{-1}C(t)' \\ \quad \times(I+Z(t)\Sigma(t))^{-1}(Z(t)h(t)-q(t))\Big\}\,dW(t), \\ q(0) = 0. \end{cases} \tag{8}$$

In addition, the optimal cost will involve the solution of the following Lyapunov equation:

$$\begin{cases} \dot N(t) + N(t)(A(t)+\Sigma(t)Q(t)) \\ \quad +(A(t)+\Sigma(t)Q(t))'N(t) - Q(t) = 0, \\ N(0) = (H^{-1}+\Sigma(0))^{-1}, \end{cases} \tag{9}$$

Theorem 3.1 *The BLQ problem (4) is uniquely solvable. Moreover, the following control*

$$u(t) = R(t)^{-1}B(t)'(Z(t)x(t)+q(t)) \tag{10}$$

is optimal, where $Z(\cdot)$ and $q(\cdot)$ are the solutions of (6) and (8), respectively. The optimal state trajectory $(x(\cdot),\, z(\cdot))$ is the unique solution of the BSDE:

$$\begin{cases} dx(t) = \Big\{\Big(A(t)+B(t)\,R(t)^{-1}B(t)'Z(t)\Big)\,x(t) \\ \quad +C(t)z(t) + B(t)\,R(t)^{-1}\,B(t)'\,q(t)\Big\}\,dt \\ \quad +z(t)\,dW(t), \\ x(T) = \xi, \end{cases} \tag{11}$$

and the optimal cost is

$$\begin{aligned} J^*(\xi) := E\Big\{&\xi' N(T)\xi \\ &+\int_0^T\Big\{\eta(t)'\Big[(S(t)\Sigma(t)+I)^{-1}S(t)-N(t)\Big]\eta(t) \\ &-2\eta(t)'(I+S(t)\Sigma(t))^{-1}C(t)'N(t)h(t)\Big\}\,dt \end{aligned} \tag{12}$$

where $N(\cdot)$ is the unique solution of (9).

The second form of the optimal control we will present is in terms of the Hamiltonian system:

$$\begin{cases} dx(t) = \Big\{A(t)x(t) - B(t)R(t)^{-1}B(t)'y(t) \\ \quad +C(t)z(t)\Big\}dt + z(t)dW(t), \\ x(T) = \xi, \end{cases} \tag{13}$$

$$\begin{cases} dy(t) = \Big\{-A(t)'y(t) - Q(t)x(t)\Big\}dt \\ \quad +\Big\{-C(t)'y(t) - S(t)z(t)\Big\}dW(t), \\ y(0) = -Hx(0). \end{cases} \tag{14}$$

Notice that the combination of (13)–(14) does not qualify as a conventional forward–backward stochastic differential equation (FBSDE) as defined in, say, [16, 20]. The subtle difference is that the forward and backward variables in (13)–(14) are directly related at the *initial* time, while those in the FBSDE are related at the *terminal* time. Moreover, one cannot transform between these two types of equations by reversing the time, due to the required adaptiveness. In the sequel, we shall refer to any three-tuple of square-integrable processes $(x(\cdot),\, z(\cdot),\, y(\cdot))$ which satisfies the equations (13)–(14) as a *solution of the Hamiltonian system (13)–(14)*.

Theorem 3.2 *The Hamiltonian system (13)–(14) has a unique solution $(x(\cdot),\ z(\cdot),\ y(\cdot))$. Moreover, the BLQ problem (4) is uniquely solvable, with the optimal control*

$$u(t) = -R(t)^{-1}B(t)'y(t), \tag{15}$$

and $(x(\cdot),\ z(\cdot))$ the corresponding optimal state process. The optimal cost is (12).

Remark 3.1 If (15) is optimal, then (13)–(14) are exactly the corresponding state equation and adjoint equation; see [8]. This is the reason why we call (13)–(14) the Hamiltonian system.

Theorem 3.2 shows that the optimal control is linear in the process $y(\cdot)$. The following simple result further reveals that the optimal control is a linear feedback of the past and current values of the state process $(x(\cdot),\ z(\cdot))$.

Proposition 3.1 *Let $y(\cdot)$ be the process obtained from the Hamiltonian system (13)–(14). Then:*

$$\begin{aligned} y(t) = \Phi(t)\Big\{ & -Hx(0) \\ & + \int_0^t \Phi(s)^{-1}\Big[Q(s)x(s) + C(s)'S(s)z(s)\Big]\,ds \\ & - \int_0^t \Phi(s)^{-1}S(s)z(s)dW(s)\Big\} \end{aligned}$$

where $\Phi(\cdot)$ is the unique solution of the matrix SDE:

$$\left\{\begin{array}{rcl} d\Phi(t) & = & -A(t)'\Phi(t)dt - C(t)'\Phi(t)dW(t), \\ \Phi(0) & = & I. \end{array}\right.$$

4 Proofs of Theorems 3.1 and 3.2

In this section we give an outline of the proofs of the main results of the paper, Theorems 3.1 and 3.2. For further details, we refer the reader to [13]. The basic idea is first to find a lower bound of the cost function (3) (see Lemma 4.1), and then to identify a control which achieves *exactly* this lower bound (see Proposition 4.2).

Lemma 4.1 *For every $u(\cdot) \in \mathcal{U}$, we have*

$$\begin{aligned} J(\xi; u(\cdot)) \geq\ & h(0)'\Big[H\Sigma(0) + I\Big]^{-1}Hh(0) \\ & + E\int_0^T \Big\{h'Qh + \eta'(S\Sigma + I)^{-1}S\eta\Big\}dt, \end{aligned} \tag{16}$$

where $\Sigma(\cdot)$ and $(h(\cdot), \eta(\cdot))$ are the solutions of (5) and (7), respectively.

Our next step involves finding a control that achieves this lower bound in Lemma 4.1. To this end, recall the Hamiltonian system (13)–(14).

Proposition 4.1 *The Hamiltonian system (13)–(14) has a unique solution $(x(\cdot),\ z(\cdot),\ y(\cdot))$. Moreover, the following relations are satisfied:*

$$\left\{\begin{array}{l} x(t) = \Sigma(t)y(t) - h(t), \\ z(t) = -\Sigma(t)(S(t)\Sigma(t) + I)^{-1}C(t)'y(t) \\ \quad -(\Sigma(t)S(t) + I)^{-1}\eta(t), \\ x(0) = -(\Sigma(0)H + I)^{-1}h(0), \end{array}\right. \tag{17}$$

where $\Sigma(\cdot)$ and $(h(\cdot), \eta(\cdot))$ are the solutions of (5) and (7), respectively.

The following result gives us a control which achieves the lower bound in Lemma 4.1.

Proposition 4.2 *Let $(x(\cdot),\ z(\cdot),\ y(\cdot))$ be the solution of the Hamiltonian system (13)–(14) and $u(\cdot)$ be given by*

$$u(t) = -R(t)^{-1}B(t)'y(t). \tag{18}$$

Then $(x(\cdot),\ z(\cdot))$ is the solution of the BSDE (1) corresponding to (18) and

$$\begin{aligned} J(\xi; u(\cdot)) =\ & h(0)'\Big[H\Sigma(0) + I\Big]^{-1}Hh(0) \\ & + E\int_0^T \Big\{h'Qh + \eta'(S\Sigma + I)^{-1}S\eta\Big\}\,dt. \end{aligned} \tag{19}$$

is the associated cost.

Proof of Theorem 3.2: The unique solvability of the Hamiltonian system (13)–(14) has been proved in Proposition 4.1. The optimality of (18) follows from the fact that the cost (19) associated with the control (18) is equal to a lower bound to the optimal cost; see Lemma 4.1. The expression (12) for the optimal cost can be obtained by applying Ito's formula to $h(t)'N(t)h(t)$. Finally, we are able to conclude that the control (18) is unique because the BLQ problem (4) is a (strictly) convex optimization problem: The set of admissible triples $(x(\cdot),\ z(\cdot),\ u(\cdot))$ associated with (1) is a convex set, and the cost (3) is a strictly convex function on this set. ∎

The following lemma is important in the proof of Theorem 3.1:

Lemma 4.2 *Let $(x(\cdot),\ z(\cdot),\ y(\cdot))$ be the solution of the Hamiltonian system (13)–(14) and $q(\cdot)$ be the solution of the SDE (8). Then*

$$y(t) = -Z(t)x(t) - q(t). \tag{20}$$

Proof of Theorem 3.1: It follows immediately from Proposition 4.2 and Lemma 4.2. ∎

5 Origin of idea: Forward formulation

In Section 4, we obtained the solution of the BLQ problem (4) by showing that the control (15) or (10) achieves a lower bound to the cost function. In showing this result, equations (5)–(8), especially the Riccati equations (5) and (6), play a crucial role. In other words, once these equations are in place, then the whole derivation, albeit quite tedious, is essentially in the same spirit as the completion-of-square technique commonly used in tackling forward LQ problems. However, the reader may be puzzled about how these (rather complicated) equations were obtained in the first place. This section serves to unfold the origin of those equations by outlining an alternative, and intuitively appealing, approach to the BLQ problem (4). The idea is basically inspired by [12] where an (uncontrolled) BSDE is regarded as a *controlled forward* SDE. Here we go one step further to show that the BLQ problem can also be viewed as a (constrained) forward LQ problem. Moreover, it is shown in [13] that the solution (18) of the BLQ problem and the relationships (17) coincide with the limiting solution of a sequence of unconstrained forward LQ problems. In this process, the Riccati equations (5) and (6), along with other related equations, come out very naturally. Finally, for the sake of notational convenience, we shall assume throughout this section that $S = 0$. The extension to the case $S \geq 0$ can be obtained in a similar way.

Forward LQ problem:

Consider the following SDE:

$$\begin{cases} dx(t) = \Big\{A(t)x(t) + B(t)u(t) + C(t)v(t)\Big\}dt \\ \quad +v(t)\,dW(t), \\ x(0) = x^0, \end{cases} \tag{21}$$

We assume throughout that $x^0 \in \mathbf{R^n}$ and $(u(\cdot),\, v(\cdot)) \in \bar{\mathcal{U}}$ where

$$\bar{\mathcal{U}} = L^2_{\mathcal{F}}(0,\, T;\, \mathbf{R^m}) \times \mathbf{L^2_{\mathcal{F}}(0,\, T;\, R^n)}.$$

For every $i \in \mathbf{Z}^+$ let

$$\begin{aligned} J(x^0,\, u(\cdot),\, v(\cdot);\, i) := E\frac{1}{2}\Big\{&x^{0\prime}Hx^0 \\ &+ \int_0^T \Big(x(t)'Q(t)x(t) + u(t)'R(t)u(t)\Big)\,dt \\ &+ i\,|x(T) - \xi|^2\Big\}. \end{aligned} \tag{22}$$

The family of LQ problems, parameterized by i, is defined by:

$$\begin{cases} \min_{x^0,\,(u(\cdot),\,v(\cdot))} J(x^0,\, u(\cdot),\, v(\cdot);\, i), \\ \text{Subject to:} \\ \quad (u(\cdot),\, v(\cdot)) \in \bar{\mathcal{U}},\; x^0 \in \mathbf{R^n}, \\ \quad (x^0,\, x(\cdot),\, u(\cdot),\, v(\cdot)) \text{ satisfies (21).} \end{cases} \tag{23}$$

Comparing (23) with the BLQ problem (4) it is clear that the control $v(\cdot)$ replaces the process $z(\cdot)$ in the BSDE, while the terminal condition $x(T) = \xi$ in (4) is replaced by a penalty term in the cost of the forward problem (23). One fundamental differences between (4) and (23) should be recognized. In the BLQ problem (4), the initial condition $x(0)$ and the process $z(\cdot)$ are part of the state process $(x(\cdot),\, z(\cdot))$; that is, once $u(\cdot)$ has been chosen, the pair $(x(\cdot),\, z(\cdot))$ (and hence, $x(0)$) is uniquely determined. On the other hand, the pair $(u(\cdot),\, v(\cdot))$ and the initial condition $x(0)$ are decision variables in the forward problem (23). This additional degree of freedom is possible because the forward problem (23) does not involve a terminal condition on the state $x(\cdot)$.

In [13], it is shown that the optimal solution of the BLQ problem (4), as stated in Theorems 3.2 and 3.1, can be obtained by solving the forward problem (23) and letting $i \uparrow \infty$. On the other hand, it is not euntirely surprising that the forward approach recovers the solution of the BLQ problem. In particular, it is clear that if

$$\begin{aligned} J(x^0,\, u(\cdot),\, v(\cdot)) = E\frac{1}{2}\Big\{&x^{0\prime}Hx^0 \\ &+ \int_0^T \Big(x(t)'Q(t)x(t) + u(t)'R(t)u(t)\Big)\,dt\Big\}, \end{aligned} \tag{24}$$

then the following problem

$$\begin{cases} J^* := \min_{x^0,\,(u(\cdot),\,v(\cdot))} J(x^0,\, u(\cdot),\, v(\cdot)), \\ \text{Subject to:} \\ \quad E\frac{1}{2}|x(T) - \xi|^2 = 0, \\ \quad (u(\cdot),\, v(\cdot)) \in \bar{\mathcal{U}},\; x^0 \in \mathbf{R^n}, \\ \quad (x^0,\, x(\cdot),\, u(\cdot),\, v(\cdot)) \text{ is admissible for (21),} \end{cases} \tag{25}$$

is equivalent to the BLQ problem (4). Moreover, the solution of (25) can be obtained by using a penalty function approach which coincides precisely with the unconstrained problem (23). This provides an alternative approach to (4) which, as mentioned, was our original idea for solving the BLQ problem. The details are left to the interested readers.

6 Conclusion

In this paper, the optimal control for the BLQ control problem is derived explicitly in terms of a pair of

Riccati equations, a forward SDE and a BSDE. Moreover, this optimal control coincides with the solution of a constrained forward LQ problem, and is the limiting solution of a family of unconstrained forward LQ problems.

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