Strict Positive Realness for Linear Time-invariant Systems with Time-delays

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Abstract

Strict positive realness (SPR) of linear time-invariant systems with multiple time-delays is discussed in this paper. We present sufficient conditions via linear matrix inequalities (LMIs) such that linear delayed system is strictly positive real. More generally, we present a memoryless state feedback controller via LMIs such that the resulting closed-loop system is SPR with α-asymptotic stability constraint (α-SPR) for a class of linear time-delay control system. Furthermore, we give an LMI approach to the optimization problem of computation of the maximal allowable bound on the time-delays such that the closed-loop system is α-SPR.

Keywords: Strictly positive real, Asymptotic stability, Linear system, Time-delay, Linear matrix inequality

1 Introduction

Motivated by an appeal to network theory, a definition of strictly positive real (SPR) transfer functions has been proposed for linear systems in [10]. That is, a SPR rational transfer function can be realized as the driving point impedance of a dissipative network. Conversely, a dissipative network has a driving point impedance that is rational and SPR. Recently many researchers have been interested in the positive real synthesis problem for linear time-invariant (LTI) systems [5, 7]. The relationship between SPR and the well-known Lefschetz-Kalman-Yacubovich lemma was shown in [10]. Different definitions of strict positive realness (SPR) have been proposed in [5] which depends on various extentions of the concept of positive real functions. In recent years, there have been many publications on the control problems for linear time-delay systems in [4, 9, 11]. In general, much good work has been done by regarding linear time-delay systems as an extention of LTI systems in [4].

It is known that time-delays often lead to poor performance and instability of many physical processes, and many problems still remain unsolved. Since the SPR plays an important role in studying LTI systems, it is worthwhile to discuss the SPR for linear time-delay systems, which is expected to present an useful approach to study linear time-delay systems. However there has not been any research in SPR problem or SPR synthesis problem for linear time-delay systems as discussed in this paper.

Extending the works on SPR for the LTI systems and the special requirements on the pole placement for the LTI systems such as α-stability in [1], we shall discuss the SPR problem and its related problem with an α-stability constraint for linear systems with multiple time-delays. We first discuss the stability problem and its related issues for linear time-delay systems. Noticing the fact that the stability of a given system is equivalent to the condition of non-singularity of its transfer function matrix in the closed right-half plane. Therefore our approach to study the stability of delay system is to propose sufficient conditions to guarantee the non-singularity. Using the Schur complement and the properties of positive definite Hermitian matrix, we then present the sufficient conditions via LMIs under which the linear time-delay system is SPR. Furthermore, we present sufficient conditions via LMIs to design a memoryless controller such that the resulting closed-loop system is SPR with an α-asymptotic stability constraint (α-SPR). Finally, we present an algorithm for suboptimal bound of time-delay to guarantee the α-SPR for the closed-loop systems.

The technique used in this paper is different from the traditional Lyapunov-Krasovskii approach in [6]. Here, the time-delay terms can be regarded as "uncertainties" or "perturbations". If such maximal "uncertainties" are regarded as vertices, see [2], the related LMI condition can be deduced from each "vertex", then we can obtain the sufficient conditions via LMIs from all possible vertices such that the system is SPR or α-SPR. Although there were some necessary and sufficient conditions to guarantee SPR for LTI systems in [7, 10], it is obvious that these conditions and approaches for SPR of LTI systems cannot be extended to those for SPR of linear time-delay systems. It is shown that our SPR results for linear time-delay systems extend the previous sufficient condition of SPR for linear systems.

The outline of the paper is as follows: Section 2 presents the problem statement, some preliminary lemmas and stability results; Section 3 presents the sufficient conditions via LMI such that linear time-delay system without control is SPR and α-SPR respectively; Section 4 presents a design of memoryless state feedback controllers via LMI to guarantee the SPR or α-SPR for the closed-loop system; Section 5 presents an algorithm to search the maximal time-delay, and conclusions are made in Section 6.

Throughout this paper, the following notation will be used: $j=\sqrt{-1}$; X' denotes the transpose of matrix X; X^* denotes the complex conjugate transpose of matrix X; $X^{\frac{1}{2}}$ denotes the square root of positive Hermitian matrix X; I denotes an identity matrix with appropriate dimensions; $A \le B$ ($A<B$) denotes that $A-B$ is negative semidefinite (negative definite), where $A, B \in C^{n\times n}$ are Hermitian matrices.

2 Problem formulation and preliminary results

Consider the following system

$$\begin{aligned}\dot{x}(t) &= \sum_{k=0}^{r} A_k x(t-\tau_k) + Dw(t) + Bu(t),\\ z(t) &= \sum_{k=0}^{r} C_k x(t-\tau_k) + Fw(t) + Eu(t), \qquad (1)\\ x(t) &= \phi(t), \quad t\in[-\tau,0],\end{aligned}$$

where $x(t)\in R^n, z(t), w(t) \in R^q$ and $u(t)\in R^m$ are the state, the output, the disturbance input and control input, respectively. $\tau_0=0$, $\tau_k>0$, $k=1,2,\cdots,r$, are the time delays, $\tau=\max\{\tau_k, 1\le k\le r\}$ and $\phi(t)$ is a continuous vector-valued function defined on $[-\tau,0]$ and it is an initial-value function for the delayed system. A_k, D, B, C_k, F and E $k=0,1,2,\cdots,r$ are known real constant matrices with appropriate dimensions.

First we discuss the strict positive realness (SPR) for system (2) with $u(t)=0$. In this case, let the transfer function matrix from w to z be $T(s)$, then

$$\begin{aligned}T(s) &= Y(s)X(s)D+F,\\ X(s) &= (sI-\textstyle\sum_{k=0}^{r} A_k e^{-\tau_k s})^{-1}, \qquad (2)\\ Y(s) &= \textstyle\sum_{k=0}^{r} C_k e^{-\tau_k s}.\end{aligned}$$

Motivated by SPR definition for linear system in [10, 5], we give the definition of SPR for linear time-delay system (2) as follows:

Definition 1 *T(s) is termed to be strictly positive real (SPR) or system (2) with* $u=0$ *is SPR if*

a) all elements of $T(s)$ *are analytic in the closed right-half plane* $\mathrm{Re}(s)\ge 0$*; and*

b) there exists an $\epsilon>0$ *such that*

$$T(j\omega)+T'(-j\omega)>\epsilon I, \qquad \forall\omega\in(-\infty,\infty). \qquad (3)$$

If the pole placement such as α-stability is considered in Definition 1, then we have the definition of α-SPR as follows:

Definition 2 *Given a constant* $\alpha\ge 0$, *T(s) is said to be strictly positive real with an* α*-stability constraint (*α*-SPR for simplicity) or system (2) with* $u=0$ *is* α*-SPR if Condition b) in Definition 1 holds and*

c) all elements of $T(s)$ *are analytic in* $\mathrm{Re}(s)\ge-\alpha$.

Remark 3 *Condition c) implies that all poles of* $T(s)$ *lie in the set* $\{s\in\mathcal{C}: \ \mathrm{Re}(s)<-\alpha\}$. *If Condition c) holds, then we say that the corresponding system is* α*-asymptotically stable. It is evident that Condition c) is stronger than Condition a). Condition c) also implies that system (2) with* $w=0$ *and* $u=0$ *is asymptotically stable.*

Now our objective of this paper is to find sufficient conditions to guarantee system (2) with $u(t)=0$ is of SPR and α-SPR (see Definition 1 and 2) respectively. Furthermore, we shall design memoryless state feedback controllers such that the resulting closed-loop system of (2) is of SPR and α-SPR respectively.

As preliminaries, we have the following observations which are crucial to our main results in the sequel:

Lemma 4 *Let $T_k \in R^{n\times n}$ be symmetric matrices and c_k be positive constant ($k = 1, 2, \cdots, r$), then the following two conditions are equivalent:*

(i) For any 2^r different choice $\delta_k = -c_k$ or c_k ($k = 1, 2, \cdots, r$), $T_0 + \sum_{k=1}^{r} \delta_k T_k < 0$;

(ii) For any scalar $\rho_k \in [-c_k, c_k]$ ($k = 1, 2, \cdots, r$), $T_0 + \sum_{k=1}^{r} \rho_k T_k < 0$.

Lemma 5 *Let $M, N \in R^{n\times n}$, then $Q = M + jN \in C^{n\times n}$ be a positive (negative) definite Hermitian matrix if and only if real matrix $R_0 := \begin{pmatrix} M & -N \\ N & M \end{pmatrix} \in R^{2n\times 2n}$ is a positive (negative) definite matrix.*

Before we discuss SPR and α-SPR for system (2), we need the stability results for system (2). First we consider system (2) with $u(t) = 0$ and $w(t) = 0$ as follows:

$$\dot{x}(t) = \sum_{k=0}^{r} A_k x(t - \tau_k). \tag{4}$$

From Lemmas 4-5, we have the following stability result for system (4), which presents sufficient conditions via LMIs to guarantee the asymptotic stability for system (4).

Proposition 6 *System (4) is asymptotically stable if there exists a positive definite matrix $P \in R^{n\times n}$ such that the following 2^{2r} different LMIs hold:*

$$\begin{pmatrix} M_0(\delta_{11}, \cdots, \delta_{1r}) & -N_0(\delta_{21}, \cdots, \delta_{2r}) \\ N_0(\delta_{21}, \cdots, \delta_{2r}) & M_0(\delta_{11}, \cdots, \delta_{1r}) \end{pmatrix} < 0, \tag{5}$$

where

$$\begin{array}{rl} & M_0(\delta_{11}, \cdots, \delta_{1r}) \\ = & P(A_0 + \sum_{k=1}^{r} \delta_{1k} A_k) + (A_0 + \sum_{k=1}^{r} \delta_{1k} A_k)' P, \\ & N_0(\delta_{21}, \cdots, \delta_{2r}) \\ = & P(\sum_{k=1}^{r} \delta_{2k} A_k) - (\sum_{k=1}^{r} \delta_{2k} A_k)' P, \end{array} \tag{6}$$

and $\delta_{1k}, \delta_{2k} = -1$ or 1; $k = 1, 2, \cdots, r$.

Proof: Let $U(\lambda) = A_0 + \sum_{k=1}^{r} A_k e^{-\lambda\tau_k}$. First we show that for the given positive definite matrix $P \in R^{n\times n}$, the following inequality holds for any $\lambda = \alpha + j\omega \in \bar{C}^+$.

$$PU(\lambda) + U(\lambda)^* P < 0. \tag{7}$$

To this end, for $k = 1, 2, \cdots, r$, let

$$\theta_{1k} := \cos(\tau_k\omega)e^{-\alpha\tau_k}, \quad \theta_{2k} := -\sin(\tau_k\omega)e^{-\alpha\tau_k}. \tag{8}$$

Obviously, $\theta_{1k}, \theta_{2k} \in [-1, 1]$, ($k = 1, 2, \cdots, r$). Rewrite the left-hand side of (7) as follows:

$$\begin{array}{rl} & PU(\lambda) + U(\lambda)^* P \\ = & M_0(\theta_{11}, \cdots, \theta_{1r}) + jN_0(\theta_{11}, \cdots, \theta_{1r}). \end{array} \tag{9}$$

In addition, it follows from Lemma 4 that (5) implies

$$\begin{pmatrix} M_0(\theta_{11}, \cdots, \theta_{1r}) & -N_0(\theta_{21}, \cdots, \theta_{2r}) \\ N_0(\theta_{21}, \cdots, \theta_{2r}) & M_0(\theta_{11}, \cdots, \theta_{1r}) \end{pmatrix} < 0. \tag{10}$$

Therefore it follows from Lemma 5 that (9) and (10) imply that (7) holds.

It is well-known that the system is asymptotically stable if and only if the corresponding characteristic function is analytic in the closed right-half complex plane. Therefore we only need to show that the matrix $\lambda I - U(\lambda)$ is non-singular for any λ with $\mathrm{Re}\lambda \geq 0$. If it is not true, then there exists $\lambda_0 = \alpha_0 + j\omega_0$ with $\alpha_0, \omega_0 \in R$, $\alpha_0 \geq 0$ and a nonzero vector $\xi \in C^n$ such that $(\lambda_0 I - U(\lambda_0))\xi = 0$, that is, $U(\lambda_0)\xi = \lambda_0\xi$. Noting that P is positive definite and (7) holds with $\lambda = \lambda_0$, we have

$$2\alpha_0 = \frac{\xi^*(PU(\lambda_0) + U(\lambda_0)^* P)\xi}{\xi^* P\xi} < 0, \tag{11}$$

which is a contradiction, therefore the proof is completed.

As an application of Proposition 6, we can obtain the α-asymptotic stability of the system (4) for a given constant scalar $\alpha \geq 0$. Take the following state transformation:

$$x(t) = e^{-\alpha(t+\tau)} y(t), \quad t \geq \tau. \tag{12}$$

Then (4) is

$$\dot{y}(t) = (A_0 + \alpha I)y(t) + \sum_{k=1}^{r} A_k e^{\alpha\tau_k} y(t - \tau_k). \tag{13}$$

Then the asymptotic stability of the trivial solution for (13) is equivalent to α-asymptotic stability of (4). Hence it follows from Proposition 6 that we have the following result.

Proposition 7 *System (4) is α-asymptotically stable if there exists a positive definite matrix $P \in R^{n\times n}$ such that the following 2^{2r} different LMIs hold:*

$$\begin{pmatrix} M_\alpha(\delta_{11}, \cdots, \delta_{1r}) & -N_\alpha(\delta_{21}, \cdots, \delta_{2r}) \\ N_\alpha(\delta_{21}, \cdots, \delta_{2r}) & M_\alpha(\delta_{11}, \cdots, \delta_{1r}) \end{pmatrix} < 0, \tag{14}$$

where

$$\begin{array}{rl} & M_\alpha(\delta_{11}, \cdots, \delta_{1r}) \\ = & P(A_0 + \alpha I + \sum_{k=1}^{r} \delta_{1k} e^{\alpha\tau_k} A_k) \\ & +(A_0 + \alpha I + \sum_{k=1}^{r} \delta_{1k} e^{\alpha\tau_k} A_k)' P, \\ & N_\alpha(\delta_{21}, \cdots, \delta_{2r}) \\ = & P(\sum_{k=1}^{r} \delta_{2k} e^{\alpha\tau_k} A_k) - (\sum_{k=1}^{r} \delta_{2k} e^{\alpha\tau_k} A_k)' P, \end{array} \tag{15}$$

and $\delta_{1k}, \delta_{2k} = -1$ *or* 1*;* $k = 1, 2, \cdots, r$*.*

3 SPR and α-SPR for delayed systems

In this section, we extend our discussion to SPR and α-SPR for system (2) with $u(t) = 0$. We first present sufficient conditions such that system (2) with $u(t) = 0$ is of SPR. These conditions are presented in terms of LMIs. Then we extend our results to α-SPR case.

Theorem 8 *System (2) with* $u = 0$ *is SPR if there exists a positive definite matrix* $P \in R^{n\times n}$ *such that the following* 2^{2r} *LMIs on* P *holds:*

$$\begin{pmatrix} M(\delta_{11}, \cdots, \delta_{1r}) & -N(\delta_{21}, \cdots, \delta_{2r}) \\ N(\delta_{21}, \cdots, \delta_{2r}) & M(\delta_{11}, , \cdots, \delta_{1r}) \end{pmatrix} < 0, \tag{16}$$

where

$$\begin{aligned} & M(\delta_{11}, \cdots, \delta_{1r}) \\ = & \begin{pmatrix} M_0(\delta_{11}, \cdots, \delta_{1r}) & M_{12}(\delta_{11}, \cdots, \delta_{1r}) \\ M_{12}'(\delta_{11}, \cdots, \delta_{1r}) & -F - F' \end{pmatrix}, \\ & N(\delta_{21}, \cdots, \delta_{2r}) \\ = & \begin{pmatrix} N_0(\delta_{11}, \cdots, \delta_{1r}) & \sum_{k=1}^r \delta_{2k} C_k' \\ -\sum_{k=1}^r \delta_{2k} C_k & 0 \end{pmatrix}, \\ & M_{12}(\delta_{11}, \cdots, \delta_{1r}) \\ = & PD - C_0' - \sum_{k=1}^r \delta_{1k} C_k', \end{aligned} \tag{17}$$

and $\delta_{1k}, \delta_{2k} = -1$ *or* 1*;* $k = 1, 2, \cdots, r$*.*

Proof: From SPR of Definition 1, we first show that all elements of $T(s)$ are analytic in the closed right-half plane $\mathrm{Re}(s) \geq 0$. Note that (16) implies (5). It follows from the proof of Theorem 6 that $U(\lambda)$ is non-singular in the closed right-half plane $\mathrm{Re}(\lambda) \geq 0$. Then $X(s)$ is analytic in $\mathrm{Re}(s) \geq 0$, so is $T(s)$ in $\mathrm{Re}(s) \geq 0$. Now we show that (3) holds. Let $\Omega(j\omega) := -PA_0 - A_0'P - \sum_{k=1}^r (e^{-j\tau_k\omega} PA_k + e^{j\tau_k\omega} A_k'P)$. It is evident that $\Omega(j\omega)$ is Hermitian matrix. First, we show that

$$\Omega(j\omega) > 0 \tag{18}$$

holds for any $\omega \in (-\infty, \infty)$. Let

$$\theta_k := \cos(\tau_k\omega), \quad \beta_k := -\sin(\tau_k\omega) \tag{19}$$

for $k = 1, 2, \cdots, r$.

Obviously, $\theta_k, \beta_k \in [-1, 1]$ $(k = 1, 2, \cdots, r)$, and

$$\begin{aligned} \Omega(j\omega) = & -PA_0 - A_0'P - \textstyle\sum_{k=1}^r \theta_k (PA_k + A_k'P) \\ & -j \textstyle\sum_{k=1}^r \beta_k (-PA_k + A_k'P). \end{aligned} \tag{20}$$

It follows from (16) that (5) holds. Similar to the proof of Theorem 6, (5) implies that (10) holds for any $\omega \in (-\infty, \infty)$. Therefore (18) holds.

Next, let

$$\Delta(j\omega) := \Omega^{-\frac{1}{2}}(j\omega)(Y'(-j\omega) - PD) + \Omega^{\frac{1}{2}}(j\omega)X(j\omega)D,$$

then we have

$$\begin{aligned} & T(j\omega) + T'(-j\omega) \\ = & F + F' + Y(j\omega)X(j\omega)D + D'X'(-j\omega)Y'(-j\omega) \\ = & F + F' + D'PX(j\omega)D + D'X'(-j\omega)PD \\ & +[Y(j\omega) - D'P]X(j\omega)D \\ & +D'X'(-j\omega)[Y'(-j\omega) - PD] \\ = & F + F' + D'X'(-j\omega)\Omega(j\omega)X(j\omega)D \\ & +[Y'(-j\omega) - D'P]X(j\omega)D \\ & +D'X'(-j\omega)[Y'(-j\omega) - PD] \\ = & F + F' - [Y'(-j\omega) - PD]^*\Omega^{-1}(j\omega)[Y'(-j\omega) \\ & -PD] + \Delta(j\omega)^*\Delta(j\omega) \\ \geq & F + F' \\ & -[Y'(-j\omega) - PD]^*\Omega^{-1}(j\omega)[Y'(-j\omega) - PD] \end{aligned}$$

Finally, we show that there exists a sufficiently small $\epsilon > 0$ such that

$$F + F' - [Y'(-j\omega) - PD]^*\Omega^{-1}(j\omega)[Y'(-j\omega) - PD] > \epsilon I \tag{21}$$

holds for any $\omega \in (-\infty, \infty)$. From the Schur complement, we have that (21) holds if the following inequality

$$\begin{pmatrix} \Omega(j\omega) & Y'(-j\omega) - PD \\ Y(j\omega) - D'P & F + F' - \epsilon I \end{pmatrix} > 0 \tag{22}$$

holds for any $\omega \in (-\infty, \infty)$.

Let the left-hand side of (22) be $\Gamma(j\omega)$, then

$$\begin{aligned} \Gamma(j\omega) = & -\begin{pmatrix} PA_0 + A_0'P & PD - C_0' \\ D'P - C_0 & -F - F' + \epsilon I \end{pmatrix} \\ & -\textstyle\sum_{k=1}^r \theta_k \begin{pmatrix} PA_k + A_k'P & -C_k' \\ -C_k & 0 \end{pmatrix} \\ & +j \textstyle\sum_{k=1}^r \beta_k \begin{pmatrix} PA_k - A_k'P & C_k' \\ -C_k & 0 \end{pmatrix}. \end{aligned} \tag{23}$$

From the continuity of matrix, there exists a sufficiently small $\epsilon > 0$ such that (16) still holds if $-F - F'$ is replaced by $-F - F' + \epsilon I$ in (17). In this case, we denote (16) as $(16)_\epsilon$ for simplicity. Similar to the proof of Theorem 6, we can also show that (22) holds for any $\omega \in (-\infty, \infty)$ if $(16)_\epsilon$ holds, that implies that (3) holds and the proof is omitted here.

Remark 9 *Those parameters* θ_k *and* $\beta_k \in [-1, 1]$ *in (19) can be regarded as "uncertainties" or "perturbations" in the proof of Theorem 8. The* r *"uncertainties" of* θ_k *have their* 2^r *vertices (Boyd et al. 1994) represented as* $(\delta_{11}, \delta_{12}, \cdots, \delta_{1r})$*.* β_k *are similar to* θ_k*. It follows from Lemma 4 that the related LMI condition can be deduced from each vertex, then we can obtain the sufficient conditions via LMIs from all possible* 2^{2r} *vertices such that the system is SPR or* α*-SPR. It can also be seen from the formulations of LMIs (16)-(17).*

Remark 10 *Consider the special case of system (2) with* $r = 0$ *and* $u = 0$*, that is, the following system:*

$$\dot{x}(t) = A_0 x(t) + Dw(t), \quad z(t) = C_0 x(t) + Fw(t). \tag{24}$$

Then it follows from Theorem 8 that system (24) is SPR if there exists a positive definite matrix $P \in R^{n\times n}$ *such that the following LMI on* P *holds:*

$$\begin{pmatrix} PA_0 + A_0'P & PD - C_0' \\ D'P - C_0 & -F - F' \end{pmatrix} < 0. \tag{25}$$

Compared with Condition 1) in [8], we find that it only guarantees the semi-negative definite of LHS of (25) at the expense of additional constraint that a transfer function matrix has no zeros on the $j\omega$*-axis. In our work, SPR for linear system (24) can be guaranteed via a single LMI (25) without such constraint.*

Based on Theorem 8 and Proposition 7, the following theorem presents sufficient conditions via LMIs such that system (2) with $u(t) = 0$ is α-SPR.

Theorem 11 *For a given constant* $\alpha > 0$*, system (2) with* $u = 0$ *is* α*-SPR if there exists a positive definite matrix* $P \in R^{n\times n}$ *such that the following* 2^{2r} *LMIs on* P *holds:*

$$\begin{pmatrix} M_{\alpha-spr}(\delta_{11},\cdots,\delta_{1r}) & -N_{\alpha-spr}(\delta_{21},\cdots,\delta_{2r}) \\ N_{\alpha-spr}(\delta_{21},\cdots,\delta_{2r}) & M_{\alpha-spr}(\delta_{11},\cdots,\delta_{1r}) \end{pmatrix} < 0, \tag{26}$$

where

$$\begin{array}{cl} & M_{\alpha-spr}(\delta_{11},\cdots,\delta_{1r}) \\ = & \begin{pmatrix} M_\alpha(\delta_{11},\cdots,\delta_{1r}) & M_{12}(\delta_{11},\cdots,\delta_{1r}) \\ M_{12}'(\delta_{11},\cdots,\delta_{1r}) & -F - F' \end{pmatrix}, \\ & N_{\alpha-spr}(\delta_{21},\cdots,\delta_{2r}) \\ = & \begin{pmatrix} N_\alpha(\delta_{11},\cdots,\delta_{1r}) & \sum_{k=1}^r \delta_{2k}C_k' \\ -\sum_{k=1}^r \delta_{2k}C_k & 0 \end{pmatrix}, \end{array} \tag{27}$$

and $\delta_{1k}, \delta_{2k} = -1$ *or* 1*;* $k = 1, 2, \cdots, r$*.*

Proof: It follows from (26)-(27) that (14) holds, then from Proposition 7, we have that Condition c) of Definition 2 holds. In addition, it follows from Lemma 4 that (26)-(27) imply that LMI condition (16) holds in Theorem 8. Therefore Condition b) holds. This completes the proof.

4 Memoryless state feedback controller

In this section, we apply Theorem 11 to design memoryless state feedback controller such that the resulting closed-loop system of (2) is internal α-asymptotically stable ($w = 0$), and meanwhile, the resulting closed-loop system is also of SPR. The following result presents sufficient conditions, under which we can find a memoryless controller such that the closed-loop system of (2) is of α-SPR. The conditions are presented via 2^{2r} LMIs on two matrices.

Theorem 12 *There exists a memoryless controller such that the resulting closed-loop system of (2) is* α*-SPR if there exist a pair of solutions* $X \in R^{n\times n}, Y \in R^{m\times n}$ *with* $X > 0$ *for the following* 2^{2r} *LMIs (28) on* X, Y*:*

$$\begin{pmatrix} \tilde{M}_{\alpha-spr}(\delta_{11},\cdots,\delta_{1r}) & -\tilde{N}_{\alpha-spr}(\delta_{21},\cdots,\delta_{2r}) \\ \tilde{N}_{\alpha-spr}(\delta_{21},\cdots,\delta_{2r}) & \tilde{M}_{\alpha-spr}(\delta_{11},\cdots,\delta_{1r}) \end{pmatrix} < 0, \tag{28}$$

where

$$\begin{array}{cl} & \tilde{M}_{\alpha-spr}(\delta_{11},\cdots,\delta_{1r}) \\ = & \begin{pmatrix} \tilde{M}_\alpha(\delta_{11},\cdots,\delta_{1r}) & \tilde{M}_{12}(\delta_{11},\cdots,\delta_{1r}) \\ \tilde{M}_{12}'(\delta_{11},\cdots,\delta_{1r}) & -F - F' \end{pmatrix}, \\ & \tilde{N}_{\alpha-spr}(\delta_{21},\cdots,\delta_{2r}) \\ = & \begin{pmatrix} \tilde{N}_\alpha(\delta_{21},\cdots,\delta_{2r}) & X\sum_{k=1}^r \delta_{2k}C_k' \\ -\sum_{k=1}^r \delta_{2k}C_kX & 0 \end{pmatrix}, \\ & \tilde{M}_{12}(\delta_{11},\cdots,\delta_{1r}) \\ = & D - Y'E' - X\sum_{k=0}^r \delta_{1k}C_k', \\ & \tilde{M}_\alpha(\delta_{11},\cdots,\delta_{1r}) \\ = & BY + Y'B' + (A_0 + \alpha I + \sum_{k=1}^r \delta_{1k}e^{\alpha\tau_k}A_k)X \\ & +X(A_0 + \alpha I + \sum_{k=1}^r \delta_{1k}e^{\alpha\tau_k}A_k)', \\ & \tilde{N}_\alpha(\delta_{21},\cdots,\delta_{2r}) \\ = & (\sum_{k=1}^r \delta_{2k}e^{\alpha\tau_k}A_k)X - X(\sum_{k=1}^r \delta_{2k}e^{\alpha\tau_k}A_k)', \end{array} \tag{29}$$

and $\delta_{1k}, \delta_{2k} = -1$ *or* 1*;* $k = 1, 2, \cdots, r$*.*

Moreover, a suitable memoryless state feedback controller can be given by $u(t) = YX^{-1}x(t)$*.*

Proof: For the reason of limitation of space, we omit the proof.

Remark 13 *Note that LMIs (28) are solvable only if* $(A_0 + \alpha I, B)$ *is stabilizable. For any given scalar* $\alpha > 0$*, (28)-(29) seem to be complicated to implement, the LMIs, however, can be efficiently solved numerically as shown in [2], and no tuning of parameters is required. In terms of LMIs, the controller with* α*-SPR and the* α*-asymptotically stabilized controller can easily be designed respectively.*

5 Maximal allowable time-delay

In this section, we shall present an algorithm to search maximal allowable time-delay for any given $\alpha > 0$. For simplicity of illustration, we only consider system (2) with single time-delay, nevertheless, the multiple time-delay can be discussed by using similar method.

Let $\gamma = e^{\alpha\tau}$, that is, $\tau = \frac{1}{\alpha}\ln\gamma$, and let

$$\begin{aligned} \tilde{M}_{\alpha-spr}(\delta_1) &= \begin{pmatrix} \tilde{M}_\alpha(\delta_1) & \tilde{M}_{12}(\delta_1) \\ \tilde{M}'_{12}(\delta_1) & -F-F' \end{pmatrix}, \\ \tilde{N}_{\alpha-spr} &= \begin{pmatrix} \gamma(A_1X - XA'_1) & XC'_1 \\ -C_1X & 0 \end{pmatrix}, \end{aligned} \tag{30}$$

where $\delta_1 = -1$ or 1 and

$$\begin{aligned} \tilde{M}_\alpha(\delta_1) &= BY + Y'B' + (A_0 + \alpha I + \delta_1\gamma A_1)X \\ &\quad + X(A_0 + \alpha I + \delta_1\gamma A_1)', \\ \tilde{M}_{12}(\delta_1) &= D - Y'E' - X\delta_1 C'_1. \end{aligned}$$

In this case, we can show that (28) is equivalent to the following two LMIs:

$$\begin{pmatrix} \tilde{M}_{\alpha-spr}(\delta_1) & -\tilde{N}_{\alpha-spr} \\ \tilde{N}_{\alpha-spr} & \tilde{M}_{\alpha-spr}(\delta_1) \end{pmatrix} < 0. \tag{31}$$

Then we can solve the following optimization problem formulated in terms of LMIs:

$$\max_{X>0,Y} \gamma(X,Y); \quad \text{subject to (31) and (30).} \tag{32}$$

If $\gamma_{opt}(X_{opt}, Y_{opt})$ is the corresponding maximum by (32), then the maximal bound for allowable time-delay can be given as follows:

$$\tau_{opt}(X_{opt}, Y_{opt}) = \frac{1}{\alpha}\ln[\gamma_{opt}(X_{opt}, Y_{opt})]. \tag{33}$$

The algorithm to search γ_{opt} is given as follows:

Step 1: Find the initial data $X = X_1 > 0, Y = Y_1, \gamma = \gamma_1 > 1$ of (31) and (30);

Step 2: Find $\gamma = \gamma_2$, $Y = Y_2$ from the following convex optimization problem:

$$\max_{Y} \gamma(X_1,Y); \quad \text{subject to (31) and (30);} \tag{34}$$

Step 3: Find $\gamma = \gamma_3$, $X = X_2$ from the following quasiconvex optimization problem:

$$\max_{X>0} \gamma(X,Y_2); \quad \text{subject to (31) and (30);} \tag{35}$$

then go back to Step 1 with $\gamma = \gamma_3$, $Y = Y_2$, $X = X_2$ till γ is convergent with a desired precision.

From the above algorithm, we have that

$$\gamma_1 \le \gamma(X_1, Y_2) \le \gamma(X_2, Y_2) \le \cdots. \tag{36}$$

It is evident that the series (36) is convergent. Let its limiting value be γ_*, then $\gamma_* \le \gamma_{opt}$. Since (36) may depend on the initial solution (γ_1, X_1, Y_1), that is, γ_* obtained from Step 1 to Step 3 may vary with the initial data in Step 1, therefore we only obtain sub-optimal solution γ_*. The sub-optimal allowable bound of time-delay τ is given by $\tau_*(X_*, Y_*) = \frac{1}{\alpha}\ln[\gamma_*(X_*, Y_*)]$.

6 Conclusions

This paper has discussed the stability problem and SPR problem for linear time-delay system, respectively. Our approaches are different from the traditional quadratic Lyapunov functional approach for linear time-delay system. Based on the frequency domain approach, the LMI method is applied to obtain the sufficient conditions to guarantee non-singularity of the transfer function matrix in the closed right-half complex plane. Then we develop the technique to solve both stability and SPR problem via LMIs. Furthermore, we also consider a particular optimization problem: the maximal bound allowed on time-delays such that the system is SPR subject to α-asymptotic stability constraint.

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