Global Analytical Model Predictive Control with Input Constraints

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Abstract

We derive a closed-form global analytical solution for Model Predictive Control (MPC) of linear, discrete-time systems, subject to a quadratic performance index and hard magnitude constraints at the system input. The solution is shown to be a partition of the state space in regions for which an analytic expression is given for the corresponding control law. Both the regions and the control law are characterised in terms of the parameters of the open-loop optimal control problem that underlies MPC. The result exploits the geometric properties of quadratic programming.

1 Introduction

Model Predictive Control (MPC) is a control method that, at each sampling instant, computes the current control input by solving an open-loop optimal control problem. The initial state for the optimisation is taken to be the current system state, and future states are *predicted* using a model of the system. The optimal control sequence resulting from the optimisation is an open loop strategy. However, this is converted into a feedback strategy by applying only the first element of this sequence and then repeating the whole procedure at the next sampling instant when new measurements of the system states are obtained. This technique is known as receding-horizon control.

An essential feature of MPC is the ability to directly handle constraints, including constraints on the control inputs, system outputs and/or internal states. This feature has been one of the keys to its success in industrial applications [4]. Constraints are included as additional conditions to be satisfied in the optimisation problem that is solved at each sampling instant. In general, constrained optimisation problems do not have a closed-form analytical solution and hence they are typically solved on-line using numerical optimisation methods. Recently, however, there has been interest in studying the underlying structure of MPC, and in obtaining off-line solutions to MPC problems. Results pertaining to off-line solutions of MPC problems have been developed from two perspectives: (i) Analytical closed-form solutions [3, 6], and (ii) Numerical solutions [2]. In [3], the authors have shown that the receding-horizon control problem has an analytical closed-form solution in a local, but nontrivial, region of the state space. This solution is identical to clipping the unconstrained solution. Independently, [6] studied special cases of MPC that take analytical forms. In the particular case of horizon $N = 1$ they show that the receding-horizon control problem has an analytical closed-form solution that agrees with that obtained in [3]. In [2], the authors establish, using multi-parametric quadratic programming properties, that the MPC solution for linear systems subject to linear constraints and quadratic cost is a continuous and piecewise affine function of the state. They also develop an off-line algorithm that computes the solution numerically. The result in [2] is the first one that relies on the inherent MPC structure to provide an efficient controller parameterisation.

In this paper we delve deeper into the MPC structure. Our analysis exposes and exploits the geometric properties of constrained MPC to obtain global analytic solutions which can be easily precomputed off-line. For linear, time-invariant, discrete-time models with a quadratic performance index and hard magnitude constraints on the input, we derive a closed-form expression for the global, analytical solution to the MPC problem. The solution is obtained by transforming the underlying open-loop optimal control problem into an equivalent quadratic programme, and then using geometric arguments to solve the latter problem. The resulting solution consists of a partition of the state space into regions in which the corresponding control law has an affine analytic form. Both the regions and the control law are characterised in terms of the parameters of the open-loop optimal control problem that underlies MPC. In this way, MPC is presented as a piecewise affine switching strategy that can be *analytically* computed off line.

The remainder of the paper proceeds as follows. In §2 we formulate the problem under consideration. In §3 we describe the technique that we employ for its solution, namely, a geometric view of quadratic programming. The closed-form global analytical MPC solution is given in §4. Finally, conclusions are given in §5.

2 Problem Formulation

The system model is given by

$$x(k+1) = Ax(k) + Bu(k)\,, \qquad k = 0, 1, 2, \ldots \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector and $u(k) \in \mathbb{R}^m$ is the input vector. We assume that (A, B) is stabilisable.

For this model we pose the following *finite-horizon open-loop optimal control problem*: given the current state measurement $x(k) = x$, find the N-move control sequence

$$\mathcal{U} = \{u(k), u(k+1), \ldots, u(k+N-1)\}$$

that minimises the performance index:

$$V_N(x, \mathcal{U}) = \sum_{\ell=k}^{k+N-1} [x^T(\ell)Qx(\ell) + u^T(\ell)Ru(\ell)] + x^T(k+N)Px(k+N)\,. \tag{2}$$

In (2), N is the prediction horizon; $Q \geq 0$ and $R > 0$ are the state and control weighting matrices, respectively, and $x^T Px$, $P > 0$, is the terminal cost function. We will express (1) and (2) in a more convenient form. To this end, we collect N state and input vectors in the following $(Nn \times 1)$ and $(Nm \times 1)$ vectors, respectively:

$$\mathbf{x} = \begin{bmatrix} x(k+1) \\ x(k+2) \\ \vdots \\ x(k+N) \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+N-1) \end{bmatrix}. \tag{3}$$

Then, from (1), we can write

$$\mathbf{x} = \Phi\,\mathbf{u} + \Lambda\,x\,, \tag{4}$$

where $x(k) = x$ and

$$\Phi = \begin{bmatrix} B & 0 & \ldots & 0 & 0 \\ AB & B & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A^{N-1}B & A^{N-2}B & \ldots & AB & B \end{bmatrix}, \Lambda = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}. \tag{5}$$

Using (4), (5) and

$$\mathbf{Q} = \text{diag}[Q, \ldots, Q, P]\,,$$
$$\mathbf{R} = \text{diag}[R, \ldots, R]\,,$$

we can express the performance index (2) as

$$\begin{aligned} V_N(x, \mathbf{u}) &= x^T Qx + \mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u} \\ &= x^T Qx + [\Phi\,\mathbf{u} + \Lambda\,x]^T\mathbf{Q}[\Phi\,\mathbf{u} + \Lambda\,x] + \mathbf{u}^T\mathbf{R}\mathbf{u} \\ &= \overline{V} + \mathbf{u}^T W\mathbf{u} + 2\mathbf{u}^T F\,x\,. \end{aligned} \tag{6}$$

In (6), $\overline{V}$ is independent of $\mathbf{u}$ and

$$W = \Phi^T\mathbf{Q}\Phi + \mathbf{R}\,, \qquad F = \Phi^T\mathbf{Q}\,\Lambda\,. \tag{7}$$

We will consider the *constrained* minimisation of (2) for (1) (equivalently, (6) for (4)) under *magnitude constraints* on the input of the form

$$u(\ell) \in \Omega^m\,, \qquad \ell = k, k+1, \ldots, k+N-1\,, \tag{8}$$

where $\Omega \triangleq [-\Delta, \Delta]$, $\Delta > 0$. Constraints of the form (8) can be expressed as linear constraints on $\mathbf{u}$ of the form

$$L\mathbf{u} \leq M\,, \tag{9}$$

where the matrix L and the vector M are easily constructed.

The problem of minimising (6) for (1) (equivalently, (4)) subject to (9) is denoted in compact form by:

$$\mathcal{P}_N(x) : \begin{cases} V_N^{\text{OPT}}(x) &= \min\limits_{L\mathbf{u} \leq M} V_N(x, \mathbf{u})\,, \\ \mathbf{u}^{\text{OPT}}(x) &= \operatorname*{argmin}\limits_{L\mathbf{u} \leq M} V_N(x, \mathbf{u})\,, \end{cases} \tag{10}$$

where

$$\mathbf{u}^{\text{OPT}}(x) = \begin{bmatrix} u^{\text{OPT}}(k; x) \\ u^{\text{OPT}}(k+1; x) \\ \vdots \\ u^{\text{OPT}}(k+N-1; x) \end{bmatrix} \tag{11}$$

is the resulting optimal control vector, and $V_N^{\text{OPT}}(x) \triangleq V_N(x, \mathbf{u}^{\text{OPT}}(x))$ is the *optimal value function*, that is, the value of (2) at $\mathbf{u} = \mathbf{u}^{\text{OPT}}(x)$.

At time k, and with initial state $x(k) = x$, $\mathcal{P}_N(x)$ is solved. The first control move $u^{\text{OPT}}(k; x)$ in (11) is the control applied to the system (1) at time k, that is,

$$u(k) \triangleq \mathcal{K}_N(x) = u^{\text{OPT}}(k; x)\,, \tag{12}$$

and then the whole procedure is repeated at time $k+1$ with the new initial state $x(k+1) = x$. In this way MPC defines a state feedback *receding-horizon* control law $\mathcal{K}_N(x)$ given by (12). Since the model (1) and the performance index (2) are time invariant, the resulting MPC feedback law is also time invariant. Hence, the initial time in the open-loop optimal control problem may be taken to be $k = 0$, that is,

$$\mathcal{K}_N(x) = u^{\text{OPT}}(0; x)\,. \tag{13}$$

Note that MPC is given by the mapping $\mathcal{K}_N(x) : \mathbb{R}^n \to \Omega$ in an *implicit* form. It is, in general, impossible to pre-compute the mapping $\mathcal{K}_N(\cdot)$ analytically. Thus, common implementations of MPC compute *numerically* and *on-line*, at time k and for the initial state $x(k) = x$, the optimal control move $\mathcal{K}_N(x)$ rather than pre-computing the control law $\mathcal{K}_N(\cdot)$. In §4 we will show that it is possible to derive, *analytically*, an explicit expression for the mapping $\mathcal{K}_N(\cdot)$ in the case described above.

3 The Geometry of Quadratic Programming

The finite-horizon optimal control problem $\mathcal{P}_N(x)$ in (10) is a *non-dynamic quadratic programme (QP)*, that is, a problem that comprises the minimisation of a quadratic cost function (cf. (6)) subject to linear constraints (cf. (9)). Numerical solutions to MPC typically use active set or interior point methods to solve this problem. In this paper, instead, we will exploit the geometry of QP to obtain an analytical solution to the problem.

Since (6) is a quadratic function of $\mathbf{u}$, its minimum, in the *unconstrained* case, is obtained by differentiating (6) with respect to $\mathbf{u}$ and equating the result to zero. This yields

$$\mathbf{u}_{\text{UC}}^{\text{OPT}}(x) = -W^{-1} F x\,, \tag{14}$$

where the subscript UC stands for unconstrained. The control (14) is the vector form of the standard finite horizon linear quadratic optimal control (e.g., [1]), whose solution is typically obtained by solving a discrete-time difference Riccati equation. We observe that (14) defines a transformation

$$\mathbf{u} = -W^{-1} F x\,, \tag{15}$$

between the state-space coordinates $x \in \mathbb{R}^n$ and the space of Nm-vector control coordinates $\mathbf{u} \in \mathbb{R}^{Nm}$.

In the constrained case, the geometry of QP is the key to obtaining an analytical solution of (10). Consider the equation

$$\mathbf{u}^{\mathrm{T}} W \mathbf{u} + 2\mathbf{u}^{\mathrm{T}} F x = c \tag{16}$$

where c is a constant. This defines ellipsoids in $\mathbb{R}^{Nm}$ centred at $\mathbf{u}_{\text{UC}}^{\text{OPT}}(x) = -W^{-1} F x$. Also, (9) defines a *constraint volume* in $\mathbb{R}^{Nm}$, R_{UC} say, inside which the optimal constrained solution $\mathbf{u}^{\text{OPT}}(x)$ must lie. Then (10) can be regarded as finding the smallest ellipsoid that "makes contact with" the boundary of R_{UC}, and $\mathbf{u}^{\text{OPT}}(x)$ is the point of contact. This is illustrated in Figure 1 for the case $N = 2$, $m = 1$ (single input) and the constraint set given by (8). In this case the constraint volume is a square in $\mathbb{R}^2$ centred at the origin.

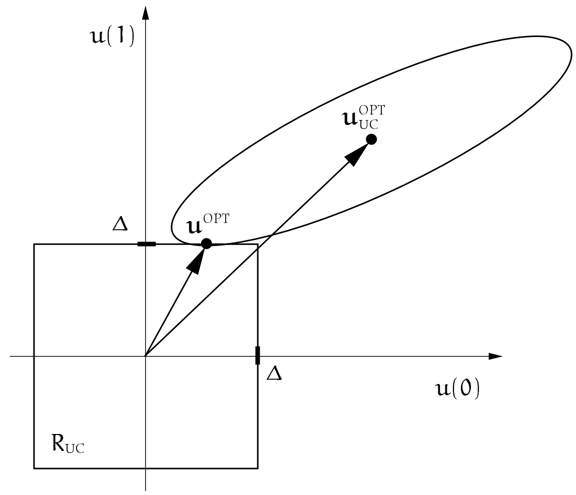

Figure 1: Geometric interpretation of QP.

Consider now the following transformation

$$\tilde{\mathbf{u}} = W^{1/2}\mathbf{u}\,. \tag{17}$$

In the new coordinates defined by (17), the constraint volume R_{UC} is mapped into another volume, denoted also by R_{UC} for simplicity of notation, and the ellipsoids (16) take the form of spheres centred at $\tilde{\mathbf{u}}_{\text{UC}}^{\text{OPT}}(x) = -W^{-1/2} F x$. Thus (10) is transformed into the problem of finding the point in R_{UC} that is closest to $\tilde{\mathbf{u}}_{\text{UC}}^{\text{OPT}}(x)$ in Euclidean distance. This is qualitatively illustrated in Figure 2 for the case $N = 2$, $m = 1$ (single input) and the input constraint set given by (8). In this case, the solution is obtained by partitioning $\mathbb{R}^2$ in nine regions; the first region is the parallelogram $R_0 = R_{\text{UC}}$, and the remaining regions, denoted by R_1 to R_8, are delimited by lines that are normal to the faces of the parallelogram and pass through its vertices, as shown in Figure 2. The optimal constrained solution $\tilde{\mathbf{u}}^{\text{OPT}}(x)$ is affine in x and determined by the region in which the optimal unconstrained solution $\tilde{\mathbf{u}}_{\text{UC}}^{\text{OPT}}(x)$ lies, in the following way: First, it is clear that $\tilde{\mathbf{u}}^{\text{OPT}}(x) = \tilde{\mathbf{u}}_{\text{UC}}^{\text{OPT}}(x)$ if $\tilde{\mathbf{u}}_{\text{UC}}^{\text{OPT}}(x) \in R_{\text{UC}}$, that is, the optimal constrained solution coincides with the optimal unconstrained solution in R_{UC}. Next, the optimal constrained solution in each of the regions R_1, R_3, R_5 and R_7 is simply equal to the vertex that is contained in the region. Finally, the optimal constrained solution in the regions R_2, R_4, R_6 and R_8 is defined by the orthogonal projection of $\tilde{\mathbf{u}}_{\text{UC}}^{\text{OPT}}(x)$ onto the faces of the parallelogram. This can be seen from Figure 2, where a case in which the solution falls in R_8 is illustrated. In the following section the procedure is generalised to $\mathbb{R}^{Nm}$.

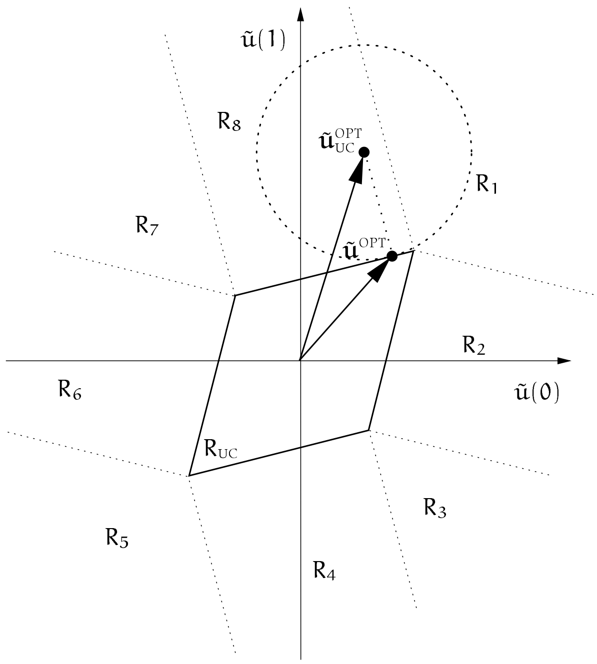

Figure 2: Geometry of QP as a minimum Euclidean distance problem.

4 Global Analytical Solution

To obtain an analytic solution of (10), we proceed as follows. We first use the transformation (17) and derive the solution in the $\tilde{\mathbf{u}}$-coordinates using geometric tools. Then we employ the transformations (15), (17), that is

$$\tilde{\mathbf{u}} = -W^{-1/2} F x \qquad (18)$$

to retrieve the solution in the state space.

Note that the hypercube that defines the constraint volume in the $\mathbf{u}$-coordinates is mapped into a convex constraint volume in the $\tilde{\mathbf{u}}$-coordinates. In these coordinates, the solution is obtained by partitioning $\mathbb{R}^{Nm}$ into a region inside the constraint volume, where the optimal constrained solution coincides with the optimal unconstrained solution, and, outside the constraint volume, into regions which we denote $\bar{N}$-*constrained regions*. These regions correspond to $\bar{N}$, $\bar{N} = 1, \dots, Nm$, elements of the optimal control vector $\mathbf{u}^{\text{OPT}}(x)$ equal to Δ or $-\Delta$. For example, in Figure 2, regions R_2, R_4, R_6 and R_8 are 1-constrained regions, and regions R_1, R_3, R_5 and R_7 are 2-constrained regions. In the $\mathbf{u}$-coordinates, the optimal control vector $\tilde{\mathbf{u}}^{\text{OPT}}$ in each of the $\bar{N}$-constrained regions is then given by the point on the corresponding $\bar{N}$-hyperface of the constraint volume that is closest, in Euclidean distance, to the unconstrained solution $\tilde{\mathbf{u}}^{\text{OPT}}_{\text{UC}}(x) = -W^{-1/2} F x$ (cf. (14) and (17)).

We will parameterise each $\bar{N}$-constrained region by a triple (ℓ, s, v), where ℓ and s are sets of indices, and v is a vector. We describe below how the elements of the triple are generated as well as several matrices and vectors that are constructed with the aid of these elements.

Notation and Definitions
We will use the following sets of indices:

- The set of the first Nm natural numbers:
$$\mathcal{I}_{Nm} = \{1, \quad 2, \quad \dots, \quad Nm\} \; .$$
- The ordered set ℓ of $\bar{N}$, $1 \leq \bar{N} \leq Nm$ indices selected from $\mathcal{I}_{Nm}$:
$$\ell = \{\ell_1, \quad \ell_2, \quad \dots, \quad \ell_{\bar{N}}\} \qquad (19)$$
where
$$\begin{aligned} \ell_1 &\in \{1, \quad \dots, \quad Nm - (\bar{N} - 1)\} \;, \\ \ell_2 &\in \{\ell_1 + 1, \quad \dots, \quad Nm - (\bar{N} - 2)\} \;, \\ &\vdots \\ \ell_k &\in \{\ell_{k-1} + 1, \quad \dots, \quad Nm - (\bar{N} - k)\} \;, \\ &\vdots \\ \ell_{\bar{N}} &\in \{\ell_{\bar{N}-1} + 1, \quad \dots, \quad Nm\} \;. \end{aligned}$$
- The set difference
$$s = \mathcal{I}_{Nm} - \ell = \{s_1, \quad s_2, \quad \dots, \quad s_{Nm-\bar{N}} : s_k \in \mathcal{I}_{Nm} \text{ and } s_k \notin \ell\} \; . \qquad (20)$$

For example, let $Nm = 4$ and $\bar{N} = 3$, then $\mathcal{I}_{Nm} = \{1, 2, 3, 4\}$, the sets ℓ, constructed as in (19), are $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$ and $\{2, 3, 4\}$, and the corresponding sets s, constructed as in (20), are $\{4\}$, $\{3\}$, $\{2\}$, $\{1\}$.

Given a matrix $A \in \mathbb{R}^{t \times r}$, and sets of indices p, with $\bar{t} \leq t$ elements, and q, with $\bar{r} \leq r$ elements, the notation $A(p, q)$ identifies the submatrix of A formed by selecting the rows of indices given by the elements of p and the columns of indices given by the elements of q. When a colon replaces p (or q), that is $A(:, q)$ (or $A(p, :)$), then all rows (or columns) are selected. For example, for the matrix W defined in (7) and the sets (19) and (20), $W(s, \ell)$ denotes the submatrix of W formed by selecting the rows with indices in s and the columns with indices in ℓ. Using this notation, we introduce the matrices

$$L_s \triangleq \begin{cases} [W(s, s)]^{-1} W(s, :) & \text{if} \quad s \neq \emptyset \,, \\ [\,] & \text{if} \quad s = \emptyset \,, \end{cases} \qquad (21)$$

$$L_{s \cup \ell_k} \triangleq [W(s \cup \ell_k, s \cup \ell_k)]^{-1} W(s \cup \ell_k, :) \,, \qquad (22)$$

where $[\,]$ is the empty matrix and where $s \cup \ell_k$ is the union of set s and element $\ell_k \in \ell$. Note that $L_{\mathcal{I}_{Nm}} = I_{Nm}$, and, from its definition

$$L_s(:, s) = I_{Nm-\bar{N}}\,, \tag{23}$$

where I_r denotes the identity matrix of dimension $r \times r$. When used without subscript, I denotes the identity matrix of dimension $Nm \times Nm$, and $I(\ell, :)$ identifies the submatrix of I formed by selecting the rows of indices given by the elements of ℓ.

Let $V(\bar{N})$, $\bar{N} \le Nm$ be the set of vertices in $\mathbb{R}^{\bar{N}}$ of the hypercube $\Omega^{\bar{N}}$, $\Omega = [-\Delta, \Delta]$. For example

$$V(2) = \left\{ \begin{bmatrix}\Delta\\ \Delta\end{bmatrix}, \begin{bmatrix}\Delta\\ -\Delta\end{bmatrix}, \begin{bmatrix}-\Delta\\ -\Delta\end{bmatrix}, \begin{bmatrix}-\Delta\\ \Delta\end{bmatrix} \right\}.$$

Then, given ℓ and s defined in (19) and (20), respectively, and a vertex $\nu \in V(\bar{N})$, we define the $(Nm \times 1)$ vectors

$$\Delta_\ell^+ = \begin{bmatrix} I(\ell,:)\\ I(s,:)\end{bmatrix}^{-1} \begin{bmatrix}\nu\\ \Delta_{Nm-\bar{N}}\end{bmatrix}, \tag{24}$$

$$\Delta_\ell^- = \begin{bmatrix} I(\ell,:)\\ I(s,:)\end{bmatrix}^{-1} \begin{bmatrix}\nu\\ -\Delta_{Nm-\bar{N}}\end{bmatrix}, \tag{25}$$

where $\Delta_{Nm-\bar{N}}$ denotes an $(Nm - \bar{N}) \times 1$ vector having all entries equal to Δ. For example, let $Nm = 5$, $\bar{N} = 3$, $\ell = \{2,\ 3,\ 5\}$, $\nu = \begin{bmatrix}\Delta & -\Delta & -\Delta\end{bmatrix}^{\mathrm{T}} \in V(3)$. Then $I_{Nm} = \{1,\ 2,\ 3,\ 4,\ 5\}$, $s = \{1,\ 4\}$, and

$$\Delta_\ell^+ = \begin{bmatrix}\Delta\\ \Delta\\ -\Delta\\ \Delta\\ -\Delta\end{bmatrix}, \qquad \Delta_\ell^- = \begin{bmatrix}-\Delta\\ \Delta\\ -\Delta\\ -\Delta\\ -\Delta\end{bmatrix}.$$

As mentioned before, our procedure gives the solution of $\mathcal{P}_N(x)$ in (10) by partitioning $\mathbb{R}^{Nm}$ into $\bar{N}$-constrained regions characterised by the elements just defined. This is done in the following theorem.

Theorem 4.1 (Global Solution of $\mathcal{P}_N(x)$). Consider the matrices W and F defined in (7). For $\bar{N} = 1, 2, \ldots, Nm$, form the sets ℓ and s as in (19) and (20), respectively; for each pair (ℓ, s) consider all vertices $\nu = \begin{bmatrix}\nu_1, \ldots, \nu_k, \ldots, \nu_{\bar{N}}\end{bmatrix}^{\mathrm{T}} \in V(\bar{N})$; for each triple (ℓ, s, ν) form Δ_ℓ^+ and Δ_ℓ^- as in (24) and (25). Then the triple (ℓ, s, ν) defines the region

$$\begin{cases} L_s^k\,(-W^{-1}F)\,x \begin{cases} \ge L_s^k\,\Delta_\ell^+ & \text{if}\, \nu_k > 0\,,\\ \le L_s^k\,\Delta_\ell^- & \text{if}\, \nu_k < 0\,,\end{cases} & k = 1, 2, \ldots, \bar{N}\,,\\ L_s\,\Delta_\ell^- \le L_s\,(-W^{-1}F)\,x \le L_s\,\Delta_\ell^+\,, \end{cases} \tag{26}$$

where

$$L_s^k \triangleq L_{s\cup\ell_k}(\ell_k - k + 1, :)\,,$$

in which the optimal constrained control $\mathbf{u}^{\mathrm{OPT}}(x)$ in (10) takes the form

$$\mathbf{u}^{\mathrm{OPT}}(x) = \begin{cases} \nu & \text{if} \quad \bar{N} = Nm\,,\\ \begin{bmatrix} I(\ell,:)\\ I(s,:)\end{bmatrix}^{-1} \begin{bmatrix}\nu\\ -G_s\,x + H_s\end{bmatrix} & \text{if} \quad \bar{N} < Nm\,,\end{cases} \tag{27}$$

where

$$G_s \triangleq [W(s,s)]^{-1}\,F(s,:)\,,$$
$$H_s \triangleq -[W(s,s)]^{-1}\,W(s,l)\,\nu$$

The global solution of $\mathcal{P}_N(x)$ in (10) is then given: (i) by all $\sum_{\bar{N}=1}^{Nm} \binom{Nm}{\bar{N}} 2^{\bar{N}}$ regions outside the constraint volume defined by (26) with optimal control (27); and (ii) by

$$\mathbf{u}^{\mathrm{OPT}}(x) = -W^{-1}F\,x\,, \tag{28}$$

inside the constraint volume defined by

$$-\Delta \le -W^{-1}(k,:)\,F\,x \le \Delta\,, \qquad k = 1, 2, \ldots, Nm\,. \tag{29}$$

Proof. See [5]. □

Theorem 4.1 gives the solution to the finite-horizon optimal control problem (10) that MPC solves at each sampling instant. The MPC law (13) is then simply obtained by selecting the first element of $\mathbf{u}^{\mathrm{OPT}}(x)$ in (27) in each region of the form (26).

Example 4.1. Consider the system (1) with

$$A = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}, \qquad B = \begin{bmatrix}0.5\\ 1\end{bmatrix},$$

which is the zero-order hold discretisation, with a sampling period of 1, of the double integrator. The input constraint is taken as $\Delta = 1$.

In the finite-horizon performance index (2) we take $N = 5$, $Q = C^{\mathrm{T}}C$, with $C = [1 \quad 0]$, and $R = 0.1$. The terminal cost matrix P is chosen as the solution of the algebraic Riccati equation $P = A^{\mathrm{T}}PA + Q - K^{\mathrm{T}}\bar{R}K$, where $K \triangleq \bar{R}^{-1}B^{\mathrm{T}}PA$, and $\bar{R} \triangleq R + B^{\mathrm{T}}PB$. The state-space partition for this case, computed from Theorem 4.1, are shown in Figure 4. The regions denoted by R_1, R_2, R_3 and R_4 correspond to a 1, 2, 3, and 4-constrained regions, respectively. Regions R_5 and R_6 correspond to unions of 1, 2, 3, 4, and 5-constrained regions.

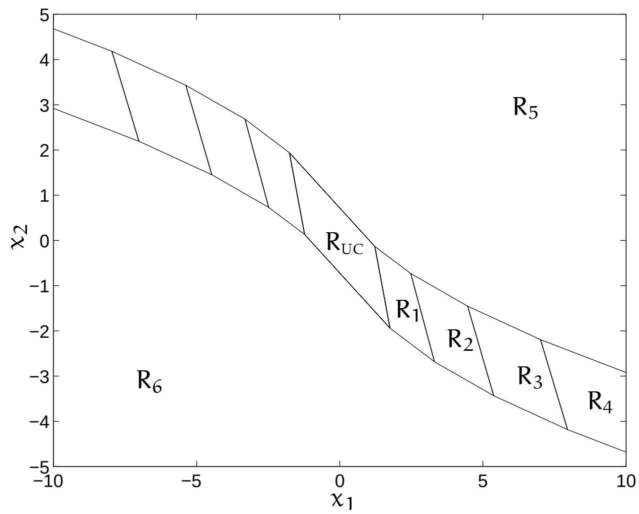

Figure 3: State-space partition for Example 4.1 for $N = 5$.

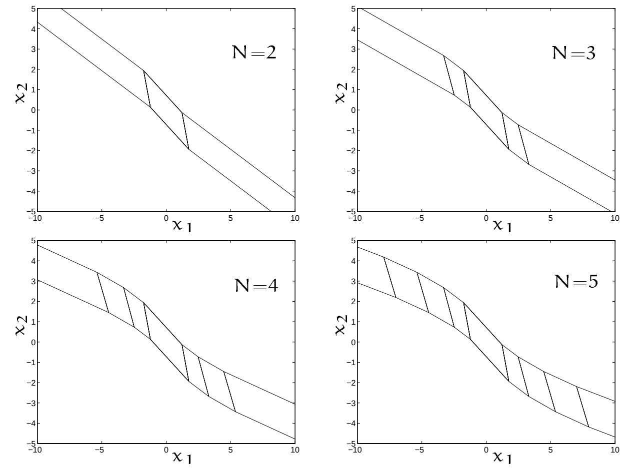

Figure 4: State-space partitions for Example 4.1 for $N = 2$, $N = 3$, $N = 4$ and $N = 5$.

The resulting MPC law (13), obtained as the first element of $\mathbf{u}^{\text{OPT}}(x)$ in (27) is

$$\mathcal{K}_5(x) = \begin{cases} -K\,x & \text{if } x \in R_{\text{UC}} \\ -G_1\,x + h_1 & \text{if } x \in R_1 \\ -G_2\,x + h_2 & \text{if } x \in R_2 \\ -G_3\,x + h_3 & \text{if } x \in R_3 \\ -G_4\,x + h_4 & \text{if } x \in R_4 \\ -\Delta & \text{if } x \in R_5 \\ \Delta & \text{if } x \in R_6 \end{cases} \tag{30}$$

where

$$\begin{aligned} K &= [0.9653 \;\; 1.3895] \\ G_1 &= [0.6154 \;\; 1.2870]\,, & h_1 &= -0.4156 \\ G_2 &= [0.4390 \;\; 1.2121]\,, & h_2 &= -0.7982 \\ G_3 &= [0.3399 \;\; 1.1665]\,, & h_3 &= -1.1746 \\ G_4 &= [0.2771 \;\; 1.1367]\,, & h_4 &= -1.5495 \end{aligned}$$

and similar expressions in the remaining unlabeled regions, where the gains G_i are preserved but the constants h_i change signs. Numerical simulations carried out using quadratic programming (not reproduced here) showed the validity of the control law obtained in Theorem 4.1.

Next we take, successively, $N = 2$, $N = 3$, $N = 4$ and $N = 5$ in the performance index (2). The state-space partitions corresponding to each value of N are shown in Figure 4. These results qualitatively coincide with the results of Example 5.3 in [2] for the double integrator (the parameters used in [2] were different).

5 Conclusions

For linear, time-invariant, discrete-time models with a quadratic performance index and hard magnitude constraints on the input, we have presented a closed-form expression for the global, analytical solution to the MPC problem. The resulting solution consists of a partition of the state space into regions in which the corresponding control law has an affine analytic form. Both the regions and the control law are characterised in terms of the parameters of the open-loop optimal control problem that underlies MPC.

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