Constantly Scaled H_∞ Control Problems for Pseudo Full Information Problems

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Abstract

This paper deals with the constantly scaled H_∞ control problem. In general, the solvability condition for this problem is non-convex. However, it is known that the state feedback (SF) and the full information (FI) problems can be reduced to convex problems. In this paper, we consider a class of problems including the SF and the FI problems as the special cases. The class is characterized by an additional assumption called a pseudo full information (PFI) condition. Assuming the PFI condition, we first derive a solvability condition. This condition involves fewer number of variables than the standard solutions for the constantly scaled H_∞ control problem. Based on this condition, we further give a convex sufficient condition for the solvability. The convex solvability condition for the SF and the FI problems can be regarded as the extreme cases of the derived sufficient condition. Moreover, we show that there exists a simple formula of a possible controller, if the PFI condition is assumed.

1 Introduction

For the past two decades, much effort has been devoted to developing analysis and synthesis methods for robust control systems. One of the most important issues in robust control is conservatism of analysis and synthesis methods. In order to reduce the conservatism, scaled small gain conditions or passivity conditions with multipliers are often employed in establishing robust stability and/or robust performance [3, 5, 13]. Those measures make it possible to analyze robustness efficiently. In fact, many analysis results can be verified efficiently by casting them as linear matrix inequalities (LMIs) [4, 11, 15], where analysis problems are reduced to optimization problems in order to find scalings.

On the other hand, the corresponding synthesis problems aim to find both scalings and controllers. Unfortunately, even if scalings are restricted to being constant, i.e. being frequency independent, the synthesis problems are non-convex in general. Concerning the constantly scaled H_∞ control problem, several methods have been proposed to find global solutions [7, 16, 17]. However, those methods demand a large amount of computational effort due to the non-convexity of the problem.

In spite of the fact that the synthesis problem is non-convex in general, the state feedback (SF) and the full information (FI) problems can be equivalently reduced to convex problems [3, 14]. Recently it has been shown that a larger class of problems can be reduced to convex problems [1]. The class is characterized by an additional assumption with respect to a realization of $G_{21}(s)$. In addition, the class includes the SF and the FI problems as the special cases. However, the results in [1] have dealt with only the extreme cases in terms of D_{21}, i.e. the cases corresponding to $D_{21} = 0$ and $D_{21}^T D_{21} > 0$, and have not clarified whether the synthesis problem can be reduced to convex problems for general D_{21}. In addition, no formulae of controllers have been shown in [1].

Assuming an assumption similar to [1], this paper shows two solvability conditions for general D_{21}. The first condition is a necessary and sufficient condition which serves to derive an explicit formula of a possible controller. This condition involves fewer number of variables than the standard solutions for the constantly scaled H_∞ control problem. However, this is still non-convex. Based on the first condition, we further show that a convex *sufficient* condition can be derived for general D_{21}. Moreover, we show that the convex solvability conditions for the SF and the FI problems can be regarded as the extreme cases of the derived condition. In other words, the sufficient condition gives necessity in the extreme cases. The derived sufficient condition is less conservative than the pure H_∞ control problem, since scaling matrices are still involved.

This paper is organized as follows: In Section 2, we offer a generalized definition of stability and a robust stability criterion based on an integral quadratic constraint (IQC). Based on this robust stability criterion, the corresponding synthesis problem is formulated, and the assumption called a pseudo full information condition is presented in Section 3. The main results of this paper are shown in Section 4.

Notation of this paper is fairly standard. M^T, M^* and $M^\dagger$ are the transpose, the conjugate trans-

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pose and the pseudo inverse of a matrix M, respectively. $M^{\perp}$ is a full row rank matrix whose rows span the orthogonal complement of the range of M.

2 Robust stability analysis

In this paper, all results are stated in a generalized manner which unifies results for continuous and discrete time systems. To this aim, we define systems on a frequency-domain which may not be tied to a specific time-domain, as has been proposed in [8]. We also introduce a generalized notion of "stability" for those systems.

Systems treated in this paper are defined through state space realizations as follows:

$$\mathcal{P}(\varsigma) = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right] =: \mathcal{D} + \mathcal{C}(\varsigma I - \mathcal{A})^{-1}\mathcal{B} \quad (1)$$

where $\varsigma \in \mathbf{C}$ is employed to define the function in (1). In the usual cases such as continuous and discrete time systems, ς represents the Laplace and the Z transformation variables s and z, respectively.

For the above system, we define the stability. $\mathcal{P}(\varsigma)$ in (1) is said to be stable, if all the eigenvalues of $\mathcal{A}$ lie in the following region:

$$\mathbf{\Lambda} = \left\{ \lambda \in \mathbf{C} : \begin{bmatrix} \lambda^* \\ 1 \end{bmatrix}^* \begin{bmatrix} q & s \\ s & r \end{bmatrix} \begin{bmatrix} \lambda^* \\ 1 \end{bmatrix} < 0 \right\}$$

where $0 \le q \in \mathbf{R}$, $r \in \mathbf{R}$ and $s \in \mathbf{R}$ are given constants. Throughout this paper, we assume that the following matrix has a positive and a negative eigenvalues:

$$\begin{bmatrix} q & s \\ s & r \end{bmatrix}.$$

It follows that $r - sq^{-1}s < 0$ holds, if $q > 0$ holds.

$\mathbf{\Lambda}$ represents the generalized domain of stability. In the case of $q = r = 0$ and $s = 1$, $\mathbf{\Lambda}$ represents the open left half plane which denotes the stability of continuous time systems. On the other hand, in the case of $q = 1$, $r = -1$ and $s = 0$, $\mathbf{\Lambda}$ corresponds to the stability of discrete time systems, since $\mathbf{\Lambda}$ gives the open unit disk. $\mathbf{\Lambda}$ can represent more general domains of stability such as a circular region in the open left half plane, provided that q, r and s are appropriately defined.

As the cases of standard time-domains, the stability of $\mathcal{P}(\varsigma)$ can be verified by using a Lyapunov inequality.

Proposition 1 [8] *Let $\mathcal{A} \in \mathbf{R}^{n\times n}$ be a given matrix. Then, all the eigenvalues of $\mathcal{A}$ lie in $\mathbf{\Lambda}$, iff there exists a positive definite matrix $\mathcal{X} \in \mathbf{R}^{n\times n}$ such that $\mathcal{F}(\mathcal{X};\mathcal{A}) < 0$ holds, where*

$$\mathcal{F}(\mathcal{X};\mathcal{A}) = \begin{bmatrix} \mathcal{A}^T \\ I \end{bmatrix}^T \begin{bmatrix} q\mathcal{X} & s\mathcal{X} \\ s\mathcal{X} & r\mathcal{X} \end{bmatrix} \begin{bmatrix} \mathcal{A}^T \\ I \end{bmatrix}$$

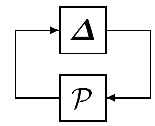

Figure 1: Robustness analysis

$\mathcal{F}(\mathcal{X};\mathcal{A}) < 0$ is the generalized Lyapunov inequality. We here define the following set for the sake of brief notation:

$$\mathbf{A}(n) = \{\mathcal{A} \in \mathbf{R}^{n\times n} : \exists \mathcal{X} > 0 \quad \mathcal{F}(\mathcal{X};\mathcal{A}) < 0\}.$$

Using this notation, Proposition 1 can be stated as that $\mathcal{P}(\varsigma)$ is stable, iff $\mathcal{A} \in \mathbf{A}(n)$ holds, where n is the size of $\mathcal{A}$.

This paper primarily concerns a robust stabilization problem based on the above definition of the stability. In order to formulate the robust stabilization problem, we show a robust stability analysis result in the rest of this section. We deal with the robust stability analysis problem for a feedback system depicted in Fig. 1. $\boldsymbol{\Delta} \subseteq \mathbf{R}^{m_1\times p_1}$ is a given set which represents uncertainty. $\mathcal{P}(\varsigma)$ is a closed-loop system constructed by a generalized plant and a controller. Then, the robust stability can be assured by the following proposition [3, 9, 12]:

Proposition 2 *Let $\boldsymbol{\Delta} \subseteq \mathbf{R}^{m_1\times p_1}$ be a given set, and $\mathcal{A} \in \mathbf{R}^{n\times n}$, $\mathcal{B} \in \mathbf{R}^{n\times m_1}$, $\mathcal{C} \in \mathbf{R}^{p_1\times n}$ and $\mathcal{D} \in \mathbf{R}^{p_1\times m_1}$ be given matrices. Suppose that there exist $0 < \mathcal{X} \in \mathbf{R}^{n\times n}$, $Q = Q^T \in \mathbf{R}^{m_1\times m_1}$, $R = R^T \in \mathbf{R}^{p_1\times p_1}$ and $S \in \mathbf{R}^{p_1\times m_1}$ such that the following inequalities hold:*

$$\begin{bmatrix} I \\ \Delta^T \end{bmatrix}^T \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} \begin{bmatrix} I \\ \Delta^T \end{bmatrix} \ge 0 \quad \forall \Delta \in \boldsymbol{\Delta}, \quad (2)$$

$$\begin{bmatrix} \mathcal{A} & I \\ \mathcal{C} & 0 \end{bmatrix} \begin{bmatrix} q\mathcal{X} & s\mathcal{X} \\ s\mathcal{X} & r\mathcal{X} \end{bmatrix} \begin{bmatrix} \mathcal{A} & I \\ \mathcal{C} & 0 \end{bmatrix}^T + \begin{bmatrix} \mathcal{B} & I \\ \mathcal{D} & 0 \end{bmatrix} \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} \begin{bmatrix} \mathcal{B} & I \\ \mathcal{D} & 0 \end{bmatrix}^T < 0. \quad (3)$$

Then, $\det(I - \mathcal{D}\Delta) \neq 0$ holds and there exists $\epsilon > 0$ such that $\mathcal{F}(\mathcal{X};\mathcal{A}_\Delta) \le -\epsilon I$ holds for all $\Delta \in \boldsymbol{\Delta}$, where $\mathcal{A}_\Delta = \mathcal{A} + \mathcal{B}\Delta(I - \mathcal{D}\Delta)^{-1}\mathcal{C}$.

The robust stability is ensured by IQCs [12] in Proposition 2. Concerning the IQCs, it has been shown in [3, 9] that the existence of Q, R and S in (2) and (3) implies the existence of a constant scaling, i.e. a scaling independent of ς, such that the robust stability is attained through the scaled small gain condition. The small gain condition in the continuous and the discrete time cases can be reduced to an H_∞ norm condition. It follows that the constantly scaled H_∞ norm condition ensures the robust stability in Proposition 2. This scaled small gain condition can offer a less conservative robust stability analysis than the D and the (D, G) scalings [5, 13] employed in the μ analysis.

Inequalities (2) and (3) are convex constraints in terms of the relevant variables $\mathcal{X}$, Q, R and S. However, (2) must be satisfied for all the elements in $\boldsymbol{\Delta}$. If $\boldsymbol{\Delta}$ consists of finite number of elements, the numerical verification of (2) is possible. On the other hand, if $\boldsymbol{\Delta}$ consists of infinite number of elements, the verification may be impossible in general. However, if $\boldsymbol{\Delta}$ is given by a polytopic set and $R < 0$ is additionally assumed, we can verify (2), since it can be reduced to the constraints on the vertices of $\boldsymbol{\Delta}$. As is seen so far, some additional assumptions are necessary in order to verify (2) numerically. Nevertheless, we will not deal with those assumptions and will leave R be mere symmetric, since we mainly focus on (3); the convexity of (2) in terms of Q, R and S suffices to the purpose of this paper.

In Proposition 2, the requirement for Q is only the symmetry. However, if $0 \in \boldsymbol{\Delta}$ holds, we can assume $Q > 0$ without loss of generality. In the sequel, we assume $0 \in \boldsymbol{\Delta}$ and thus $Q > 0$. When $Q > 0$ holds, we define a template matrix $\Phi^{(i)}$ for given matrices $\Theta_{\mathcal{Y}}^{(i)}$, $\Theta_{\mathcal{A}}^{(i)}$, $\Theta_{\mathcal{B}}^{(i)}$, $\Theta_{\mathcal{C}}^{(i)}$ and $\Theta_{\mathcal{D}}^{(i)}$ as follows:

$$\Phi^{(i)} = \begin{bmatrix} \Phi_{11}^{(i)} & {\Phi_{21}^{(i)}}^T & {\Phi_{31}^{(i)}}^T \\ \Phi_{21}^{(i)} & \Phi_{22}^{(i)} & {\Phi_{32}^{(i)}}^T \\ \Phi_{31}^{(i)} & \Phi_{32}^{(i)} & \Phi_{33}^{(i)} \end{bmatrix} \quad (q = 0),$$

where

$$\begin{aligned} \Phi_{11}^{(i)} &= s(\Theta_{\mathcal{A}}^{(i)} + {\Theta_{\mathcal{A}}^{(i)}}^T) + r\Theta_{\mathcal{Y}}, \\ \Phi_{22}^{(i)} &= R - SQ^{-1}S^T, \\ \Phi_{33}^{(i)} &= -Q^{-1}, \qquad \Phi_{21}^{(i)} = s\Theta_{\mathcal{C}}^{(i)}, \\ \Phi_{31}^{(i)} &= {\Theta_{\mathcal{B}}^{(i)}}^T, \qquad \Phi_{32}^{(i)} = {\Theta_{\mathcal{D}}^{(i)}}^T. \end{aligned}$$

We omit the definition of $\Phi^{(i)}$ for the case of $q > 0$, due to the brevity of the space; it can be defined similarly. Template matrix $\Phi^{(i)}$ makes it possible to describe (3) simply.

Proposition 3 *Suppose that $\mathcal{X} > 0$ and $Q > 0$ hold. Then, (3) is equivalent to $\Phi^{(1)} < 0$, where*

$$\Theta_{\mathcal{Y}}^{(1)} = \mathcal{X}, \quad \begin{bmatrix} \Theta_{\mathcal{A}}^{(1)} & \Theta_{\mathcal{B}}^{(1)} \\ \Theta_{\mathcal{C}}^{(1)} & \Theta_{\mathcal{D}}^{(1)} \end{bmatrix} = \begin{bmatrix} \mathcal{A}\mathcal{X} & \mathcal{B} \\ \mathcal{C}\mathcal{X} & \mathcal{D} \end{bmatrix}. \quad (4)$$

3 Problem Formulation

The robust stabilization problem dealt with in this paper is based on the robust stability analysis in Section 2. Since the robust stability condition in Proposition 2 can be cast as a constantly scaled H_∞ norm condition, the synthesis problem can be regarded as a constantly scaled H_∞ control problem.

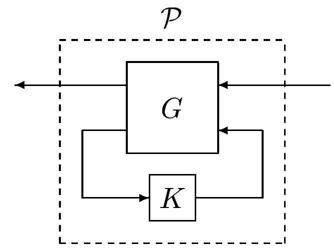

Figure 2: Closed-loop system

The synthesis problem is defined for a generalized plant (see Fig. 2) given by

$$G(\varsigma) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \quad (5)$$

where $A \in \mathbf{R}^{n\times n}$, $B_j \in \mathbf{R}^{n\times m_j}$, $C_i \in \mathbf{R}^{p_i\times n}$ and $D_{ij} \in \mathbf{R}^{p_i\times m_j}$ $(i, j = 1, 2)$ are given matrices. We here assume $D_{22} = 0$. Note that we can assume without loss of generality that $[\, C_2 \; D_{21} \,]$ has full row rank. Using suitable output and coordinate transformations, we can assume further that $[\, C_2 \; D_{21} \,]$ has the following form [2]:

$$[\, C_2 \quad D_{21} \,] = \left[\begin{array}{c:c} E & H \\ J & 0 \end{array}\right], \quad \begin{array}{l} J = [\, 0 \quad I_{p_2-\kappa} \,], \\ \kappa = \mathrm{rank}(D_{21}) \end{array} \quad (6)$$

where $H \in \mathbf{R}^{\kappa\times m_1}$ has full row rank, if $\kappa > 0$ holds. The form in (6) can be seen typically in synthesis of servo control systems [2].

The synthesis problem is to find a controller $K(\varsigma)$ such that the closed-loop system composed by $\boldsymbol{\Delta}$ and $\mathcal{P}(\varsigma)$ is robustly stable, where $\mathcal{P}(\varsigma)$ is another closed-loop system constructed by $G(\varsigma)$ and $K(\varsigma)$ as depicted in Fig. 2. According to (6), we assume a state space realization of $K(\varsigma)$ as follows:

$$K(\varsigma) = \left[\begin{array}{c|cc} A_k & B_{k1} & B_{k2} \\ \hline C_k & D_{k1} & D_{k2} \end{array}\right]. \quad (7)$$

Then, the state-space realization (1) of $\mathcal{P}(\varsigma)$ is given by

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} = \left[\begin{array}{cc:c} A & 0 & B_1 \\ 0 & 0 & 0 \\ \hdashline C_1 & 0 & D_{11} \end{array}\right] + \left[\begin{array}{cc} 0 & B_2 \\ I & 0 \\ \hdashline 0 & D_{21} \end{array}\right] \begin{bmatrix} A_k & B_{k1} & B_{k2} \\ C_k & D_{k1} & D_{k2} \end{bmatrix} \left[\begin{array}{cc:c} 0 & I & 0 \\ E & 0 & H \\ J & 0 & 0 \end{array}\right]. (8)$$

For the above systems, the precise formulation of the problem is given as follows:

Problem 1 (Constantly scaled H_∞ control problem) *Let $G(\varsigma)$ be a generalized plant given by (5) and (6), and $\boldsymbol{\Delta} \subseteq \mathbf{R}^{m_1 \times p_1}$ be a given set such that $0 \in \boldsymbol{\Delta}$ holds. Then, find a controller $K(\varsigma)$ in (7) such that there exist $0 \in \mathcal{X} \in \mathbf{R}^{n_{c\ell} \times n_{c\ell}}$, $0 < Q \in \mathbf{R}^{m_1 \times m_1}$, $R = R^T \in \mathbf{R}^{p_1 \times p_1}$ and $S \in \mathbf{R}^{p_1 \times m_1}$ satisfying (2) and (3) for $\boldsymbol{\Delta}$ and matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ defined by (8), where $n_{c\ell}$ is the size of $\mathcal{A}$.*

A solvability condition for the constantly scaled H_∞ control problem can be given by using the standard LMI solutions [6, 10] on the H_∞ control problem. Taking account of the form in (6) and the results in [2], a more detailed result can be given.

Lemma 4 *Let $G(\varsigma)$ be a generalized plant given by (5) and (6), $\boldsymbol{\Delta} \subseteq \mathbf{R}^{m_1 \times p_1}$ be a given set such that $0 \in \boldsymbol{\Delta}$ holds. Then, the constantly scaled H_∞ control problem is solvable, iff there exist $0 < X \in \mathbf{R}^{n \times n}$, $0 < Y \in \mathbf{R}^{\nu \times \nu}$, $Z \in \mathbf{R}^{\nu \times (n-\nu)}$, $\mathbb{A} \in \mathbf{R}^{\nu \times n}$, $\mathbb{B} \in \mathbf{R}^{\nu \times \kappa}$, $\mathbb{C} \in \mathbf{R}^{m_2 \times n}$, $\mathbb{D} \in \mathbf{R}^{m_2 \times \kappa}$, $0 < Q \in \mathbf{R}^{m_1 \times m_1}$, $R = R^T \in \mathbf{R}^{p_1 \times p_1}$ and $S \in \mathbf{R}^{p_1 \times m_1}$ such that (2), $\Theta_{\mathcal{Y}}^{(2)} > 0$ and $\Phi^{(2)} < 0$ hold, where*

$$
\begin{aligned}
\Theta_{\mathcal{A}}^{(2)} &= \begin{bmatrix} AX + B_2\mathbb{C} & (A + B_2\mathbb{D}E)N^T \\ \mathbb{A} & (\hat{Y}A + \mathbb{B}E)N^T \end{bmatrix}, \\
\Theta_{\mathcal{B}}^{(2)} &= \begin{bmatrix} B_1 + B_2\mathbb{D}H \\ \hat{Y}B_1 + \mathbb{B}H \end{bmatrix}, \\
\Theta_{\mathcal{C}}^{(2)} &= \begin{bmatrix} C_1X + D_{12}\mathbb{C} & (C_1 + D_{12}\mathbb{D}E)N^T \end{bmatrix}, \\
\Theta_{\mathcal{D}}^{(2)} &= D_{11} + D_{12}\mathbb{D}H, \\
\Theta_{\mathcal{Y}}^{(2)} &= \begin{bmatrix} X & N^T \\ N & Y \end{bmatrix}, \qquad \hat{Y} = \begin{bmatrix} Y & Z \end{bmatrix}, \\
N &= \begin{bmatrix} I_\nu & 0 \end{bmatrix}, \qquad \nu = n - (p_2 - \kappa).
\end{aligned}
$$

Lemma 4 provides the necessary and sufficient condition for the constantly scaled H_∞ control problem solvable. Moreover, if the problem is solvable, a ν-th order controller exists [2]. The existence of the reduced order controller is the consequence of the fact that $p_2 - \kappa$ number of the state variable are exactly available due to the form in (6).

Lemma 4 is, in general, a non-convex condition in terms of the relevant variables. However, if the generalized plant has the special structure such as the SF and the FI problems, the synthesis problem can be reduced to convex problems [3, 14]. Moreover, it has been shown in [1] that a larger class of problems can be reduced to convex problems. In [1], the class is characterized by additional assumptions for $G(\varsigma)$. Similarly to [1], we consider the following assumption for $G(\varsigma)$:

Assumption 1 (Pseudo full information condition) *One of the following conditions holds for the generalized plant $G(\varsigma)$ given by (5) and (6):*

- *$\nu = 0$, i.e. $J = I_n$, holds.*
- *$\nu > 0$, $N\Omega N^T \in \mathbf{A}(\nu)$ and $N\Gamma = 0$ hold,[1] where*

$$\Omega = A - B_1 H^\dagger E, \quad \Gamma = B_1(I - H^\dagger H).$$

In the sequel, we call Assumption 1 the pseudo full information (PFI) condition due to the following reason: Suppose that the first condition holds. Then, all the state variables of $G(\varsigma)$ are exactly available, since $J = I_n$ holds. In this case, additional condition $\kappa = 0$, i.e. $D_{21} = 0$, gives the state feedback problem, while another condition $\kappa = m_1$, i.e. $D_{21}^T D_{21} > 0$ gives the full information problem. It follows that the SF and the FI problems are the special cases of Assumption 1. On the other hand, in the case of $\nu > 0$, we can define a subsystem $G_{\rm sub}(\varsigma)$ as follows:

$$G_{\rm sub}(\varsigma) = \left[\begin{array}{c|c} NAN^T & NB_1 \\ \hline EN^T & H \end{array}\right].$$

Note that $G_{\rm sub}(\varsigma)$ corresponds to the ν number of state variables which are not exactly available. Then, Assumption 1 ensures that the (pseudo) inverse system $G_{\rm sub}^\dagger(\varsigma)$ is stable, where

$$G_{\rm sub}^\dagger(\varsigma) = \left[\begin{array}{c|c} N\Omega N^T & -NB_1H^\dagger \\ \hline H^\dagger EN^T & H^\dagger \end{array}\right].$$

This condition may not look related to the SF and the FI problem. However, it will be clarified that the both conditions in Assumption 1 indeed play the same role in deriving the main results of this paper.

The PFI condition can be easily verified for $G(\varsigma)$ given by (5) and (6). However, even if the PFI condition does not hold for a given realization of $G(\varsigma)$, there may exist a coordinate transformations for $G(\varsigma)$ such that the PFI condition holds. In other words, the PFI condition depends on coordinate transformations. Fortunately, the existence of such a coordinate transformation can be reduced to an LMI problem and it can be easily confirmed.

4 Main results

4.1 Solvability condition

Assuming the PFI condition for the generalized plant, two solvability conditions, which are main results of this paper, will be shown for the constantly scaled H_∞ control problem. Those solvability conditions involve fewer number of variables

[1] In the case of $\kappa = 0$, $NAN^T \in \mathbf{A}(\nu)$ and $NB_1 = 0$ are assumed

than Lemma 4. The first condition serves to derive an explicit formula of a possible controller. In addition, the controller leads to a simple realization of the corresponding closed-loop system. On the other hand, the second condition serves to derive a *convex* sufficient condition so that the constantly scaled H_∞ control problem is solvable.

Theorem 5 *Let $\boldsymbol{\Delta} \subseteq \mathbf{R}^{m_1 \times p_1}$ be a given set and $G(\varsigma)$ be a generalized plant given by (5) and (6). Suppose that $0 \in \boldsymbol{\Delta}$ and the PFI condition for $G(\varsigma)$ hold. Then, the constantly scaled H_∞ control problem is solvable, iff there exist $0 < X \in \mathbf{R}^{n\times n}$, $\mathbb{C} \in \mathbf{R}^{m_2 \times n}$, $\mathbb{D} \in \mathbf{R}^{m_2\times(p_2-\kappa)}$, $0 < Q \in \mathbf{R}^{m_1\times m_1}$, $R = R^T \in \mathbf{R}^{p_1 \times p_1}$ and $S \in \mathbf{R}^{p_1\times m_1}$ such that (2) and $\Phi^{(3)} < 0$ hold, where $\Phi^{(3)} = X$,*

$$\begin{bmatrix} \Theta^{(3)}_{\mathcal{A}} & \Theta^{(3)}_{\mathcal{B}} \\ \Theta^{(3)}_{\mathcal{C}} & \Theta^{(3)}_{\mathcal{D}} \end{bmatrix} = \begin{bmatrix} AX + B_2\mathbb{C} & B_1 + B_2\mathbb{D}H \\ C_1X + D_{12}\mathbb{C} & D_{11} + D_{12}\mathbb{D}H \end{bmatrix}.$$

Furthermore, if there exist X, $\mathbb{C}$, $\mathbb{D}$, Q, R and S such that the above conditions hold, a possible controller $K(\varsigma)$ is given by (7) with the following matrices:

$$\begin{bmatrix} A_k & B_{k2} \\ C_k & D_{k2} \end{bmatrix} = \begin{bmatrix} N(\Omega + B_2\hat{F}) \\ \hat{F} \end{bmatrix} \begin{bmatrix} N \\ J \end{bmatrix}^T, \qquad \begin{bmatrix} B_{k1} \\ D_{k1} \end{bmatrix} = \begin{bmatrix} N(B_1H^\dagger + B_2\mathbb{D}) \\ \mathbb{D} \end{bmatrix}, \tag{9}$$

where $\hat{F} = \mathbb{C}X^{-1} - \mathbb{D}E$.

Theorem 5 provides the necessary and sufficient condition for the constantly scaled H_∞ control problem solvable. Moreover, Theorem 5 gives the explicit formula (9) of the possible controller. In spite of the fact that the order of the generalized plant is n, the controller in (9) has the ν-th order. The existence of the reduced order controller is the consequence of the results in [2]. A similar result has been given in [1] for the continuous systems. However, formulae of controllers have not been shown in [1]. Note also that Theorem 5 provides the generalized results unifying results for continuous and discrete time systems.

The solvability condition in Theorem 5 is still non-convex in terms of the relevant variables. Hence, the numerical verification of the solvability requires much computational effort [7, 16, 17]. However, it may be easier than Lemma 4, since Y, Z, $\mathbb{A}$ and $\mathbb{B}$ are not involved in Theorem 5. This simplification is owing to the fact that an arbitrarily large Y exists under the PFI condition. It implies that, if the problem with the PFI condition is solvable, possible Y in Lemma 4 can be extremely large. In that case, numerical tests may not conclude the problem as solvable, since the extremely large Y leads to ill-conditions for numerical manipulations. A similar difficulty has been pointed out also in [18]. On the other hand, the solvability condition in Theorem 5 leads to better conditions for numerical verifications, since Y does not appear.

It is easy to confirm the fact that (2) and $\Phi^{(3)} < 0$ hold, iff the constantly scaled H_∞ control problem for the following generalized plant is solvable:

$$G_{\mathrm{mod}}(\varsigma) = \left[\begin{array}{c|c:c} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hdashline 0 & H & 0 \\ I & 0 & 0 \end{array}\right]. \tag{10}$$

Note that the solvability for $G_{\mathrm{mod}}(\varsigma)$ is given by the existence of a static feedback $[\,\mathbb{D}\ F\,]$, since all the state variables of $G_{\mathrm{mod}}(\varsigma)$ are exactly available [2]. It follows that the solvability condition for $G(\varsigma)$ is equivalent to that for $G_{\mathrm{mod}}(\varsigma)$, provided the PFI condition is assumed for $G(\varsigma)$. Note that $G_{\mathrm{mod}}(\varsigma)$ satisfies the PFI condition.

In addition to the solvability condition, the closed-loop system for $G(\varsigma)$ and $K(\varsigma)$ is equivalent to that for $G_{\mathrm{mod}}(\varsigma)$ and $[\,\mathbb{D}\ F\,]$.

Proposition 6 *Let $G(\varsigma)$ be a generalized plant given by (5) and (6), and $K(\varsigma)$ be a controller given by (7) and (9). Suppose that the PFI condition holds for $G(\varsigma)$. Then, the closed-loop system $\mathcal{P}(\varsigma)$ defined by (1) and (8) is equivalent to the following system:*

$$\mathcal{P}_{\mathrm{mod}}(\varsigma) = \left[\begin{array}{c|c} A + B_2F & B_1 + B_2\mathbb{D}H \\ \hline C_1 + D_{12}F & D_{11} + D_{12}\mathbb{D}H \end{array}\right] \tag{11}$$

where $F = \mathbb{C}X^{-1}$.

$\mathcal{P}_{\mathrm{mod}}(\varsigma)$ is the closed-loop system given by $G_{\mathrm{mod}}(\varsigma)$ and $[\,\mathbb{D}\ F\,]$. Concerning the existence of a static feedback, the LMI synthesis results in [6, 10] often lead to a result; a control problem can be solved by a static feedback, if the solvability condition can be described in terms of only X, i.e. without Y. Even though Theorem 5 is stated in terms of only X, there does not necessarily exist a static feedback realizing $\mathcal{P}_{\mathrm{mod}}(\varsigma)$ for $G(\varsigma)$ in the case of $\nu > 0$. This gap is caused by the difference in ways of eliminating Y. A static feedback exists, if there exists small Y, i.e. sufficiently small so that an assign $Y = NX^{-1}N^T$ is possible. On the other hand, Theorem 5 eliminates Y (and Z) by allowing it sufficiently large.

4.2 Convex sufficient condition

In general, Theorem 5 is a non-convex problem. However, the problem can be reduced to a convex problem, if $G(\varsigma)$ corresponds to the SF or the FI problem. Moreover, the problem can be reduced to a convex problem for a larger class of problems, if $G(\varsigma)$ satisfies the PFI condition with $\kappa = 0$, i.e. $D_{21} = 0$, or $\kappa = m_1$, i.e. $D_{21}^T D_{21} > 0$ [1]. However, the case $0 < \kappa < m_1$ has not been discussed. In this section, we deal with the latter case and give a convex sufficient condition by explicitly taking account of the null space of H.

Theorem 7 *Let $\boldsymbol{\Delta} \subseteq \mathbf{R}^{m_1 \times p_1}$ be a given set and $G(\varsigma)$ be a generalized plant given by (5) and (6). Suppose that $0 \in \boldsymbol{\Delta}$ and the PFI condition hold. Moreover, suppose that $0 < \kappa < m_1$ holds. Then, the constantly scaled H_∞ control problem is solvable, if there exist $0 < X \in \mathbf{R}^{n \times n}$, $\mathbb{C} \in \mathbf{R}^{m_2 \times \nu}$, $\mathbb{D} \in \mathbf{R}^{m_2 \times r}$, $0 < \bar{Q} \in \mathbf{R}^{m_1 \times m_1}$, $R = R^T \in \mathbf{R}^{p_1 \times p_1}$ and $\bar{S} \in \mathbf{R}^{p_1 \times m_1}$ such that the following inequalities hold:*

$$\begin{bmatrix} W \\ \Delta^T \end{bmatrix}^T \Pi \begin{bmatrix} W \\ \Delta^T \end{bmatrix} \geq 0 \quad \forall \Delta \in \boldsymbol{\Delta},$$

$$\begin{bmatrix} B_2 \\ D_{12} \end{bmatrix}^\perp \left\{ \begin{bmatrix} A & I \\ C_1 & 0 \end{bmatrix} \begin{bmatrix} qX & sX \\ sX & rX \end{bmatrix} \begin{bmatrix} A & I \\ C_1 & 0 \end{bmatrix}^T + \begin{bmatrix} \bar{B}_1 & 0 \\ \bar{D}_{11} & I \end{bmatrix} \Pi \begin{bmatrix} \bar{B}_1 & 0 \\ \bar{D}_{11} & I \end{bmatrix}^T \right\} \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix}^{\perp T} < 0,$$

$$\begin{bmatrix} \bar{D}_{112}^T \\ I \end{bmatrix}^T \Pi_2 \begin{bmatrix} \bar{D}_{112}^T \\ I \end{bmatrix} < 0 \quad (q = 0) \qquad (12)$$

where

$$\Pi = \begin{bmatrix} \bar{Q} & \bar{S}^T \\ \bar{S} & R \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} \bar{Q}_2 & \bar{S}_2^T \\ \bar{S}_2 & R \end{bmatrix},$$

$$\bar{Q} = \begin{bmatrix} \bar{Q}_1 & \bar{Q}_3^T \\ \bar{Q}_3 & \bar{Q}_2 \end{bmatrix}, \quad \bar{S} = [\, \bar{S}_1 \quad \bar{S}_2 \,], \quad W = \begin{bmatrix} H \\ (H^T)^\perp \end{bmatrix},$$

$$\begin{bmatrix} \bar{B}_1 \\ \bar{D}_{11} \end{bmatrix} = \begin{bmatrix} \bar{B}_{11} & \bar{B}_{12} \\ \bar{D}_{111} & \bar{D}_{112} \end{bmatrix} = \begin{bmatrix} B_1 \\ D_{11} \end{bmatrix} W^T.$$

Again, (12) for the case of $q > 0$ is omitted for the sake of the space.

Theorem 7 reveals that, even if neither $D_{21} = 0$ nor $D_{21}^T D_{21} > 0$ holds, the convex condition can be derived, although it is the sufficient condition. For the constantly scaled H_∞ control problem, it is easy to obtain a convex sufficient condition. In fact, if Q, R and S are fixed so that (2) holds, (3) can be reduced to an H_∞ control norm constraint. In that case, the synthesis problem is reduced to a pure H_∞ control problem which is a convex problem. However, since all the scaling matrices are fixed in the pure H_∞ control problem, the resultant condition tends to be highly conservative. On the other hand, the sufficient condition in Theorem 7 is less conservative than the pure H_∞ control problem with the fixed scaling matrices, since scaling matrices $\bar{Q}$, R and $\bar{S}$ are still involved.

The solvability condition for the SF and the FI cases can be regarded as the extreme cases of (12). Indeed, the SF problem corresponds to the case that $\bar{Q}_2$ in (12) is $\bar{Q}$, while the FI problem is corresponding to the case that $\bar{Q}_2$ does not appear. In other words, the sufficient condition in Theorem 7 gives necessity in the extreme cases. Note that the conservatism of Theorem 7 depends on the rank of D_{21}.

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