Analysis, Eigenstructure Assignment and H_2 Multi-Channel Synthesis with Enhanced LMI Characterizations

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Abstract

This paper describes a new framework for the analysis and synthesis of control systems, which constitutes a genuine continuous-time extension of results that are only available in discrete time. In contrast to earlier results the proposed methods involve a specific transformation on the Lyapunov variables and a reciprocal variant of the Projection Lemma, in addition to the classical linearizing transformations on the controller data. For a wide range of problems including robust analysis and synthesis, multi-channel H_2 state- and output-feedback syntheses, the approach leads to potentially less conservative LMI characterizations. This comes from the fact that the technical restriction of using a single Lyapunov function is to some extent ruled out in this new approach. Moreover, the approach offers new potentials for problems that cannot be handled using earlier techniques. As an instance, the eigenstructure assignment problem blended with Lyapunov-type constraints is given a simple and tractable formulation.

1 Introduction

In recent times, LMI techniques have come to be essential tools for the analysis and synthesis of control systems, and more specifically in the area of robust control. This is due to three main factors. First of all, LMI techniques offer the advantage of operational simplicity in regard to classical approaches which necessitates the more cumbersome material of Riccati equations and the like. A small number of concepts and basic principles are sufficient to develop tools which can then be used in practice. Secondly, they render accessible a vast panorama of control problems including robustness analysis, nominal H_∞, H_2 and robust control syntheses, multi-objective synthesis and Linear Parameter-Varying synthesis. Some of them cannot be handled in the classical setting. Thirdly, these techniques are effective numerical tools exploiting a solidly based branch of convex programming with efficient softwares attached to the theoretical body.

A closer look to the literature in the recent years reveals that most LMI control methods make use of Lyapunov variables and possibly additional variables, often called scalings or multipliers, which in some sense translate how ideal behaviors are altered by uncertainties or perturbations. These methods are constantly refined in two directions:

- more and more sophisticated classes of multipliers are developed to take maximum advantage of the information available on the nature and structure of the uncertainties [4, 16, 14]
- for similar reasons, single Lyapunov functions are replaced with a "multiple" Lyapunov function [11, 3, 10] in order to end up with more accurate tools and reduce the degree of conservatism inherent to robust control problems.

This paper is a contribution in the second direction and accounts for the fact that a major drawback of most existing LMI formulations is that the Lyapunov function used for checking system performances is itself involved in controller variables. This leads to unnecessary restrictions on the set of solutions and limits the practical appeal of these methods. This weakness is apparent in, for instance, robust state-feedback control of polytopic systems where a common Lyapunov function is used for all vertices of the polytope but also in multi-channel H_2 state- and output-feedback synthesis problems where one has to use a single Lyapunov function for all performance constraints. Although multiple or parameter-dependent Lyapunov functions have been used successfully in analysis problems [11, 10], this remains an issue for synthesis problems because of the intricate interrelations between plant and controller data on one side and the Lyapunov variables on the other side.

A significant breakthrough towards the (partial) elimination of this weakness is the work in [7, 6]. In this work, the authors propose new LMI representations in which the interrelations discussed before are to some degree bypassed through the use of an auxiliary slack variable. The consequence of these developments is a reduction of conservatism in a broad class of problems including robust control, multi-objective control, decentralized control etc. The work in [7, 6] exploits some natural properties of discrete-time LMI characterizations which enjoy some sort of factorized structure. Analogous LMI formulations for continuous-time systems remain open and challenging and they constitute the motivation of our developments in this paper. Namely, we

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introduce general techniques and tools which have important consequences in the context of analysis and synthesis with LMIs. Key ideas consist in a reciprocal variant of the Projection Lemma which permits to recast most usual LMI characterizations as augmented LMI representations. These new representations involve (slack) variables which provide additional flexibility in a wide range of problems. Topics under consideration include classical LMI problems such as analysis with parameter-dependent Lyapunov functions, robust control of polytopic systems, multi-objective/channel state- and output-feedback synthesis, but also some new problems such as eigenstructure assignment or pole placement combined with Lyapunov-type constraints. It is worth mentioning that the latter problems are not tractable within the usual LMI setting but can be given simple and practical formulations with the proposed tools and techniques.

2 Instrumental tools

Throughout the paper, we shall make an intensive use of the *Projection Lemma* [9]. It is omitted here for brevity.

Also, we provide a variant of a reciprocal version of the Projection Lemma that will facilitate subsequent derivations.

Lemma 2.1 (Reciprocal Projection Lemma) *Let P be any given positive definite matrix. The following statements are equivalent:*

$$(i) \quad : \Psi + S + S^T < 0\,. \tag{1}$$

(ii) : *the LMI problem*

$$\begin{bmatrix} \Psi + P - (W + W^T) & S^T + W^T \\ S + W & -P \end{bmatrix} < 0 \tag{2}$$

is feasible with respect to W.

Proof: It suffices to compute the projection conditions according to the Projection Lemma [9] with respect to the general variable W. ■

As one can see, the Reciprocal Projection Lemma produces a dilated characterization in a space of augmented dimensions both in terms of constraints and variables. The conditions (1) and (2) are strictly equivalent, but the slack variable W provides additional flexibility in a broad class of problems. The remainder of the paper is an examination of its consequences.

3 Stability problems

In this section, we introduce new alternative characterizations of the fundamental Lyapunov stability Theorem for linear systems. All forms, are of course equivalent but are more or less practical when analysis or synthesis aspects come into play. The results below also constitute the core of the development in the subsequent sections. It introduces a new transformation on the Lyapunov variables which helps to reduce the degree of conservatism in some delicate problems. This will appear more in light for robust synthesis and multi-objective synthesis problems.

Theorem 3.1 (Stability theorems) *The following matrix inequality conditions (i)-(v), with symmetric matrix variables X and Y and general matrix variables W and V, are equivalent.*

$$(i) \quad : \quad A \text{ is Hurwitz } \ (\operatorname{Re}\lambda_i(A) < 0)\,. \tag{3}$$

$$(ii) \quad : \quad \begin{bmatrix} A^T X + XA & 0 \\ 0 & -X \end{bmatrix} < 0\,. \tag{4}$$

$$(iii) \quad : \quad \begin{bmatrix} Y - (W + W^T) & AY + W^T \\ YA^T + W & -Y \end{bmatrix} < 0\,. \tag{5}$$

$$(iv) \quad : \quad \begin{bmatrix} -(V + V^T) & V^T A + X & V^T \\ A^T V + X & -X & 0 \\ V & 0 & -X \end{bmatrix} < 0\,. \tag{6}$$

$$(v) \quad : \quad \begin{bmatrix} -(V + V^T) & V^T A^T + X & V^T \\ AV + X & -X & 0 \\ V & 0 & -X \end{bmatrix} < 0 \tag{7}$$

Proof: Note that the equivalence between (i) and (ii) is the standard Lyapunov Theorem for continuous-time linear systems. The equivalence between (iii) and (iv) is obtained by performing the congruence transformation

$$\begin{bmatrix} V & 0 \\ 0 & X \end{bmatrix} \text{ with } V := W^{-1},\ X := Y^{-1}$$

in (5), which yields

$$\begin{bmatrix} V^T X^{-1} V - (V + V^T) & V^T A + X \\ A^T V + X & -X \end{bmatrix} < 0\,.$$

Then, a Schur complement operation with respect to the term $V^T X^{-1} V$ leads to (6). Thus, we shall only prove that (iv) and (v) reduce to (ii). Considering (iv), it can be written in the alternative form

$$\begin{bmatrix} 0 & X & 0 \\ X & -X & 0 \\ 0 & 0 & -X \end{bmatrix} + \begin{bmatrix} -I \\ A^T \\ I \end{bmatrix} V \begin{bmatrix} I & 0 & 0 \end{bmatrix} + (*) < 0\,.$$

Since V is a general matrix, it can be eliminated using the Projection Lemma [9]. The projection conditions simplifies to $X > 0$ and

$$\begin{bmatrix} A^T X + XA - X & X \\ X & -X \end{bmatrix} < 0\,,$$

which is exactly the $(1,1)$-term in (ii) by a Schur complement argument.

A fairly short proof of the equivalence between (ii) and (iv) can also be obtained by invoking Lemma 2.1. Indeed, the Lyapunov inequality (ii) is equivalent to

$$AY + YA^T < 0, \text{ with } Y := X^{-1}.$$

The use of Lemma 2.1 with $\Psi := 0$, and $S = AY$ yields

$$\begin{bmatrix} -(W+W^T)+P & AY+W^T \\ YA^T+W & -P \end{bmatrix}$$

or equivalently with $X := Y^{-1}$,

$$\begin{bmatrix} -(W+W^T)+P & A+W^TX \\ A^T+XW & -XPX \end{bmatrix} < 0 \qquad (8)$$

By the congruence transformation

$$\begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix}, \text{ with } V := W^{-1},$$

and a Schur complement operation with respect to the term V^TPV, the inequality (8) in turn becomes

$$\begin{bmatrix} -(V+V^T) & V^TA+X & V^T \\ A^TV+X & -XPX & 0 \\ V & 0 & -P^{-1} \end{bmatrix} < 0,$$

which linearizes to (6) with the special choice $P := X^{-1}$.

Finally, (v) is the dual of (iv) in the transformation $A \to A^T$ and can be shown to be equivalent to (ii) by similar arguments. ■

Remark 3.2 There are a few points to have in mind to understand these alternative forms and their usefulness. In (iii), the Lyapunov terms AY and YA^T are separated by means of an intermediate (slack) variable W. Furthermore and more importantly, the classical product terms A^TX and XA fully disappear in (iv) and (v), and this will offer new potentials both for analysis and synthesis. Similar ideas have been presented earlier in [7, 6] for the discrete-time case.

The LMI condition (6) is significantly more costly than its original form (3) because of the additional general matrix variable V ($V \in \mathbf{R}^{n\times n}$ where $n = \dim A$). We shall see however that this extra computational overhead is more than offset by a reduction of conservatism in delicate problems.

4 H_2 performance multi-channel problems

There are important extensions of the stability results in Section 3 to performance specifications.

For future use, we introduce the system governed by

$$\begin{aligned} \dot{x} &= Ax + Bw \\ z &= Cx + Dw, \end{aligned} \qquad (9)$$

where w is the exogenous input and z is the controlled output.

4.1 H_2 performance

The following results for H_2 performance parallels those for the stability Theorem 3.1. We are considering the system described in (9) with the strict properness assumption, i.e. $D = 0$.

Theorem 4.1 (H_2 performance) *The following statements, involving symmetric matrix variables X, Z and general matrix variables V_1 and V_2 are equivalent.*

$$(i) \; : \; A \text{ is stable and } \|C(sI-A)^{-1}B\|_2 < \gamma. \qquad (10)$$

$$(ii) \; : \; \begin{bmatrix} A^TX+XA & XB \\ B^TX & -\gamma I \end{bmatrix} < 0, \quad \begin{bmatrix} X & C^T \\ C & Z \end{bmatrix} > 0, \; \mathrm{Tr}\, Z < 1. \qquad (11)$$

$$(iii) \; : \; \begin{bmatrix} -(V_1+V_1^T) & V_1^TA+X & V_1^TB & V_1^T \\ A^TV_1+X & -X & 0 & 0 \\ B^TV_1 & 0 & -\gamma I & 0 \\ V_1 & 0 & 0 & -X \end{bmatrix} < 0, \quad \begin{bmatrix} X & C^T \\ C & Z \end{bmatrix} > 0, \; \mathrm{Tr}\, Z < 1. \qquad (12)$$

$$(iv) \; : \; \begin{bmatrix} -(V_1+V_1^T) & V_1^TA^T+X & V_1^TC^T & V_1^T \\ AV_1+X & -X & 0 & 0 \\ CV_1 & 0 & -\gamma I & 0 \\ V_1 & 0 & 0 & -X \end{bmatrix} < 0, \quad \begin{bmatrix} X & B \\ B^T & Z \end{bmatrix} > 0, \; \mathrm{Tr}\, Z < 1. \qquad (13)$$

Proof: The equivalence between (i) and (ii) is a standard result. See [18, 17]. The first LMI in (12) is easily inferred from the following equivalent form of the first LMI in (ii)

$$YA^T + AY + \gamma^{-1}BB^T < 0, \quad Y := X^{-1},$$

and the application of Lemma 2.1 with $S := AY$, $\Psi := \gamma^{-1}BB^T$, $P = X^{-1}$ and $V_1 = W^{-1}$.

Finally, (iv) is the dual equivalent of (iii) in the transformation $(A,B,C) \to (A^T, C^T, B^T)$. ■

4.2 Analysis and synthesis for uncertain polytopic systems

There are many potential applications and combinations of Theorems 3.1 and 4.1. We shall discuss some examples to illustrate the improvements over more classical characterizations. We must keep in mind however that the reasoning below has a general nature.

Consider an uncertain polytopic system described as

$$\dot{x} = A(\alpha)x := \sum_{i=1}^{N} \alpha_i A_i x, \qquad \alpha_i \geq 0,\ \sum_{i=1}^{N} \alpha_i = 1 \qquad (14)$$

i.e.,

$$A(\alpha) \in \text{co}\,\{A_1, \ldots, A_N\}\,. \qquad (15)$$

By virtue of the properties of convex combinations, and using the characterization (6), the uncertain system (14) is stable whenever there exist matrices $\{X_i\}_{i=1,\ldots,N}$ and V such that

$$\begin{bmatrix} -(V+V)^T & V^T A_i + X_i & V^T \\ A_i^T V + X_i & -X_i & 0 \\ V & 0 & -X_i \end{bmatrix} < 0 \qquad i = 1, \ldots, N. \qquad (16)$$

Note that the LMI condition (16) establishes robust stability of the polytopic system (14) through the use of a parameter-dependent Lyapunov functions. More specifically, it is readily verified by invoking the necessary part of the Projection Lemma [9] that

$$V(x, \alpha) := x^T (\sum_{i}^{N} \alpha_i X_i) x$$

is a Lyapunov function for the polytopic system. This generally results in dramatic improvements as compared to the customary quadratic stability analysis which uses the same Lyapunov matrix for all vertices of the polytope, $X_i := X$, $i = 1, \ldots, N$. See [10, 8] for comparison results for affine systems and [2, 19] for polytopic systems.

As an immediate byproduct, the approach applies with the same ease to the synthesis of state-feedback controllers ($u = Kx$) for polytopic uncertain systems. It then suffices to use the linearizing transformation

$$R := KV \qquad K = RV^{-1}\,. \qquad (17)$$

to end up with LMI characterizations of the solutions.

5 Mixed eigenstructure assignment with Lyapunov-type constraints

In this section, we deal with another interesting application of the proposed techniques. We consider the problem of exact assignment of the closed-loop eigenvalues to prescribed locations of the complex plane while enforcing Lyapunov-type constraints such as those discussed in previous sections. This problem should not be confused with the clustering technique in [18] which cannot handle the precise location of individual eigenvalues. Pole placement is a very classical method for control design and can be very efficient in the hands of practitioners. There are however few tractable extensions which allow combination with other objectives such as quadratic performance. Hence, one has to resort to some general optimization softwares to locally determine solutions that achieve some compromise between pole placement and some other objective. Hereafter, we describe an alternative approach which requires only solving LMIs, and thus is highly practical. We should also stress out that the developments below are not possible with the customary LMI characterizations, and this again reemphasizes the role played by the new representations and tools described above.

Consider the open-loop plant governed by

$$\begin{array}{rcl} \dot{x} & = & Ax + B_1 w + B_2 u, \qquad B_2 \in \mathbf{R}^{n\times m} \\ z & = & C_1 x + D_{11} w + D_{12} u\,. \end{array} \qquad (18)$$

We are seeking a state-feedback control $u = Kx$ such that

- the closed-loop spectra satisfies

  $$\text{spec}\ (A + B_2 K) = \{\lambda_1, \ldots, \lambda_n\}$$

  where the λ_i's denote desired closed-loop eigenvalues.

- a family of Lyapunov-type specifications (Sections 3 and 4) are met in closed-loop.

The assignment of the set of eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ by state-feedback is quite a simple problem [15, 12] and can be carried out in two steps:

1- compute for $i = 1, \ldots, n$ bases

$$\begin{bmatrix} M_i \\ N_i \end{bmatrix} \text{ of the nullspace of } \ [\, A - \lambda_i I \quad B_2 \,] \qquad (19)$$

with conformable partitioning.

2- for arbitrary vectors ξ_i of dimension m, compute the state-feedback controller

$$K = RV^{-1}, \quad \text{where } R = [\, N_1, \ldots, N_n \,] \operatorname*{diag}_{i=1}^{n} \xi_i, \qquad (20)$$

$$V = [\, M_1, \ldots, M_n \,] \operatorname*{diag}_{i=1}^{n} \xi_i\,. \qquad (21)$$

Note that for this procedure to be valid, the non-real eigenvalues must occur in conjugate pairs. Also, for a complex-conjugate eigenvalue pair $(\lambda_j, \lambda_{j+1})$ an analogous nullspace computation in (19) is possible. See for instance [1] and references therein. Summing up, the generic form of the state-feedback gain which assigns the closed-loop spectra as desired is

$$K := RV^{-1}, \text{ where } \ R = N\,\Xi, \qquad V = M\,\Xi\,, \qquad (22)$$

where

$$\begin{array}{rcl} N & := & [\ldots, N_i, \ldots, N_j, N_{j+1}, \ldots]\,, \\ M & := & [\ldots, M_i, \ldots, M_j, M_{j+1}, \ldots]\,, \\ \Xi & := & \text{diag}(\ldots, \xi_i, \ldots, \xi_j, \xi_{j+1}, \ldots), \end{array} \qquad (23)$$

with the subscripts i and j, $j+1$ for real and complex-conjugate eigenvalues, respectively. Note that the matrix Ξ gathers all degrees of freedom. They can in turn be used to shape the right eigenvectors $v_i := M_i \xi_i$ of the closed-loop matrix $A + B_2 K$ and thereby to achieve some modal distribution or internal decoupling.

By virtue of the identity of the controller formulas in (17) and (22), it becomes fairly easy to blend the eigenstructure assignment method with any of the Lyapunov-type specifications encountered so far. This is obtained by picking up as V and R matrices in the Lyapunov-type constraints those of (22). As an illustration, equipped with the LMI characterization (13), eigenstructure assignment with an H_2 performance constraint and the standard assumption $D_{11} = 0$, can be recast as an LMI program in the variables Ξ, X and Z:

$$\begin{bmatrix} -(M\Xi + \Xi^T M^T) & * & * & * \\ AM\Xi + B_2 N\Xi + X & -X & * & * \\ C_1 M\Xi + D_{12} N\Xi & 0 & -\gamma I & * \\ M\Xi & 0 & 0 & -X \end{bmatrix} < 0,$$

$$\begin{bmatrix} X & B_1 \\ B_1^T & Z \end{bmatrix} > 0, \ \mathrm{Tr}\, Z < 1\,.$$

When solved, the controller gain is easily determined with the help of the formulas (22) and (23).

The success of this procedure hinges on the availability of degrees of freedom in excess with respect to a pure pole placement. These degrees of freedom, in matrix Ξ, are related to the dimensions of the nullspaces in (19). For single input systems, these dimensions drop to 1 and 2 respectively, so that there is no longer freedom for other performance constraints. The approach is therefore only valid for multivariable systems. One way to overpass this difficulty is to allow dynamic state-feedback controllers. This is realized with the augmentation scheme

$$(A, B, C) \rightarrow \left(\begin{bmatrix} A & 0 \\ 0 & 0_{k\times k} \end{bmatrix}, \begin{bmatrix} 0 & B_2 \\ I_k & 0 \end{bmatrix}, \begin{bmatrix} 0 & I_k \\ I & 0 \end{bmatrix} \right).$$

In this case, sufficient freedom can be gained to tackle multi-constrained problems. Another possibility of practical interest, is to perform a *partial* pole placement. Hence, only a subset of column vectors of V and R have to satisfy the subspace inclusions defined in (19). The remaining freedom can then be used to meet additional Lyapunov-type constraints. The relationships (22) become,

$$K := RV^{-1}, \text{ where } R = N\,\Xi_N, \qquad V = M\,\Xi_M\,,$$

where

$$\begin{aligned} N &:= [N_1, \ldots, N_l, \overbrace{I, I, \ldots, I}^{n-l \ \text{times}}], \\ M &:= [M_1, \ldots, M_l, \overbrace{I, I, \ldots, I}^{n-l \ \text{times}}], \\ \Xi_M &:= \mathrm{diag}(\xi_1, \ldots, \xi_l, \widetilde{v}_1, \ldots, \widetilde{v}_{n-l}), \\ \Xi_N &:= \mathrm{diag}(\xi_1, \ldots, \xi_l, \widetilde{r}_1, \ldots, \widetilde{r}_{n-l}). \end{aligned}$$

The vectors ξ_i's, $i = 1, \ldots, l$ correspond to the placement of l ($l < n$) poles and reduce to a scalar for single input systems. The vectors $\widetilde{v}_k$ and $\widetilde{r}_k$ are not spent in the pole placement problem and thus are useful to meet independent Lyapunov-type constraints. Finally, note that LMI characterizations for partial pole placement and Lyapunov-type constraints will involve the variables X_j for each constraint and Ξ_N, Ξ_M.

6 H_2 Multi-channel output-feedback synthesis

In this section, we take advantage of the characterizations derived in Sections 3 and 4 to provide a new more accurate method for output-feedback synthesis with multi-objective/channel constraints. The reader is referred to [6] for the discrete-time version of this work. Also, since the proof has strong similarities with the results in [18, 13], we shall put a special emphasis on new aspects. The general setup of the problem is as follows. Consider a plant

$$P(s) \quad \left\{ \begin{array}{lcl} \dot{x} & = & Ax + B_1 w + B_2 u, \quad A \in \mathbf{R}^{n\times n} \\ z & = & C_1 x + D_{11} w + D_{12} u \\ y & = & C_2 x + D_{21} w \end{array} \right. \tag{24}$$

where

- $u \in \mathbf{R}^{m_2}$ is the vector of control inputs,
- $w \in \mathbf{R}^{m_1}$ is a vector of exogenous inputs,
- $y \in \mathbf{R}^{p_2}$ is the measurement vector
- $z \in \mathbf{R}^{p_1}$ is a vector of controlled variables.

Let $T(s)$ denote the closed-loop transfer functions from w to z for some dynamic output-feedback control law $u = K(s)y$. Our goal is to compute a full-order output-feedback controller

$$K(s) \quad \left\{ \begin{array}{lcl} \dot{x}_K & = & A_K x_K + B_K y, \qquad A_K \in \mathbf{R}^{n\times n} \\ u & = & C_K x_K + D_K y \end{array} \right. \tag{25}$$

which meets a family of input-output specifications such as those discussed in Sections 3 and 4. For future use, we introduce the following the state-space data of the closed-loop system described by the augmented dynamics $(\mathcal{A}\ \mathcal{B}_1; \mathcal{C}_1\ \mathcal{D}_{11}) :=$

$$\left[\begin{array}{cc|c} A & 0 & B_1 \\ 0 & 0 & 0 \\ \hline C_1 & 0 & D_{11} \end{array} \right] + \left[\begin{array}{cc} 0 & B_2 \\ I & 0 \\ \hline 0 & D_{12} \end{array} \right] \mathcal{K} \left[\begin{array}{cc|c} 0 & I & 0 \\ C_2 & 0 & D_{21} \end{array} \right], \tag{26}$$

where

$$\mathcal{K} := \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}.$$

With Each specification/channel is associated an LMI constraint of the form encountered in Theorems 3.1 and 4.1. See also [18, 13] for a richer catalog. The desired characterization for output-feedback synthesis with multi-objective/channel specifications can be derived in three steps:

1- introduce a different Lyapunov variable X_j for each specification/channel,

2- introduce a variable V common to all channels and specifications,

3- perform adequate congruence transformations and use linearizing changes of variables to end up with LMI synthesis conditions.

In accordance with the partition of $\mathcal{A}$ in (26), we introduce a partition of V and of its inverse $W := V^{-1}$ in the form

$$V := \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}, \qquad W := \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}.$$

By the strict nature of the LMI constraints involved and a perturbation argument, there is no loss of generality in assuming that V_{21} and W_{21} are invertible. See for instance [5] for a detailed justification. We then introduce the notation

$$\Pi_V := \begin{bmatrix} V_{11} & I \\ V_{21} & 0 \end{bmatrix}, \qquad \Pi_W := \begin{bmatrix} I & W_{11} \\ 0 & W_{21} \end{bmatrix}.$$

Π_V and Π_W are invertible matrices by the assumptions on V_{21}, W_{21} and H_{21}, G_{21}. We can easily infer the identities

$$W\Pi_V = \Pi_W, \quad V\Pi_W = \Pi_V .$$

Systematic congruence transformations in Π_W lead for each specification/channel to matrix inequalities which solely involves the terms

$$\begin{array}{ll} \Pi_W^T V^T \mathcal{A} \Pi_W = \Pi_V^T \mathcal{A} \Pi_W, & \mathcal{B}_1^T V \Pi_W = \mathcal{B}_1^T \Pi_V, \quad \mathcal{C}_1 \Pi_W, \\ \Pi_W^T V \Pi_W = \Pi_W^T \Pi_V, & \Pi_W^T X_j \Pi_W , \end{array} \tag{27}$$

possibly involving the selection matrices L_i and R_i, respectively. Explicit computation and inspection of these terms reveal that by invertibility of V_{21} and W_{21}, one can perform the following linearizing changes of variables:

$$\begin{aligned}
\widehat{\boldsymbol{A}}_K &:= V_{11}^T A W_{11} + V_{21}^T A_K W_{21} + V_{11}^T B_2 C_K W_{21} + \\
&\quad\; V_{21}^T B_K C_2 W_{11} + V_{11}^T B_2 D_K C_2 W_{11} && (28)\\
\widehat{\boldsymbol{B}}_K &:= V_{21}^T B_K + V_{11}^T B_2 D_K && (29)\\
\widehat{\boldsymbol{C}}_K &:= C_K W_{21} + D_K C_2 W_{11} && (30)\\
\widehat{\boldsymbol{D}}_K &:= D_K , && (31)\\
\widehat{\boldsymbol{X}}_j &:= \Pi_W^T X_j \Pi_W && (32)\\
\boldsymbol{U} &:= V_{11}^T W_{11} + V_{21}^T W_{21} . && (33)
\end{aligned}$$

The LMI terms in (27) then become affine in the introduced bold variables. As an instance, the first term can be written:

$$\Pi_V^T \mathcal{A} \Pi_W = \begin{bmatrix} V_{11}^T A + \widehat{\boldsymbol{B}}_K C_2 & \widehat{\boldsymbol{A}}_K \\ A + B_2 \widehat{\boldsymbol{D}}_K C_2 & A W_{11} + B_2 \widehat{\boldsymbol{C}}_K \end{bmatrix}.$$

Consequently, sufficient existence conditions for the multi-objective/channel output-feedback synthesis problem can be recast as an LMI program in the variables $\widehat{\boldsymbol{A}}_K$, $\widehat{\boldsymbol{B}}_K$, $\widehat{\boldsymbol{C}}_K$, $\widehat{\boldsymbol{D}}_K$ and V_{11}, W_{11}, $\boldsymbol{U}$ and $\widehat{\boldsymbol{X}}_j$. In stark contrast with earlier results, a different Lyapunov function is employed for each specification/channel. Hence far better results can generally be expected. When a solution to the LMIs has been found, the sought controller is easily derived from the following simple scheme:

- compute a factorization $V_{21}^T W_{21}$ of $\boldsymbol{U} - V_{11}^T W_{11}$ and deduce (invertible) V_{21} and W_{21}. Note that this is always possible by perturbation if necessary.
- compute the controller data A_K, B_K, C_K and D_K by reversing the formulas in (28)-(31).

A more conservative approach is to utilize symmetric V and W matrices, in which case the variable $\boldsymbol{U}$ reduces to the constant identity. See the full version of the paper for a complete description of the LMI characterizations for H_2 constraints.

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