Estimation of the domain of attraction for polynomial systems via LMI's

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Abstract

Investigation of the stability properties of stationary points of nonlinear systems lies at the heart of modern control engineering. In this contribution we will show how modern results of real algebraic geometry, a branch of pure mathematics, will be used to compute subsets of the region of attraction of asymptotically stable stationary points of polynomial systems. This computation will be done in a numerically stable and efficient way by reformulating the problem as a linear matrix inequality (LMI). For this refomulation new results from real algebraic geometry will be used. The results presented in this contribution show very clearly that a multidisciplinary approach to nonlinear control systems leads to new insight and new powerful conditions. Some conclusions and an outlook will finish this contribution.

Keywords: Control Theory, Polynomial Systems, Stability Theory.

1 Introduction

The class of systems investigated in this paper is given by the following state differential equation

$$\dot{x} = f(x)\ ,\ x(0) = x^0\ , \tag{1}$$

where $x \in R^n$ represents the state vector of the system and the function $f(x)$ is a polynomial function of the state vector. We make the following two assumptions

- $f(0) = 0$, i.e., the state vector $x = 0$ is a stationary point of the system,
- $A = \frac{\partial f}{\partial x}(0)$ is a Hurwitz's matrix, i.e., it has all eigenvalues in the left open half plane of the complex plane.

Thus, the stationary point $x = 0$ is asymptotically stable and the problem which will be investigated in this paper is to estimate the domain of attraction of $x = 0$. The main tool in achieving this goal is the use of an appropriate Lyapunov function. In the following we will use the quadratic Lyapunov function $V(x) = x^T Qx$ where Q is a positive definite symmetrical $n \times n$ matrix. We assume that the matrix $P = A^T Q + QA$ is negative definite, thus $V(x)$ is a valid Lyapunov function for the linearized system. The set

$$\Omega_c = \{x \mid V(x) \leq c\}\ ,\ c > 0 \tag{2}$$

is contained in the unknown domain of attraction if the inequality

$$\dot{V}(x) = f(x)^T Qx + x^T Qf(x) < 0 \tag{3}$$

is valid for all $x \in \Omega_c$, $x \neq 0$ [1],[4],[5]. The problem is to maximize c because the corresponding set Ω_c is the largest subset of the domain of attraction which can be guaranteed with the chosen Lyapunov function. In the literature several different methods have been proposed in order to achieve this goal [1],[10],[11]. In this paper we will use modern results of real algebraic geometry and will extensively use the polynomial nature of f, V, and $\dot{V}$ in order to maximize c. The main optimization will be formulated as a generalized eigenvalue optimization under linear matrix inequality constraints. For this type of optimization problem very powerful implementations of numerical methods are available, e.g. the MATLAB Toolbox [2]. Two examples will conclude the paper.

2 Representation Theorems

In this section the basic facts from real algebraic geometry [6], [9] which are needed in the following are given. We follow the notation of [3]. Let $K = S(p_1, \ldots, p_m)$ be a compact basic closed semi-algebraic subset of R^n, i.e., $p_1, \ldots, p_m$ are polynomials and

$$S(p_1, \ldots, p_m) = \{x \in R^n \mid p_i(x) \geq 0,\ 1 \leq i \leq m\}. \tag{4}$$

In [3] it is proved that if $m = 2$ every polynomial p which is strictly positive on K, i.e., $p(x) > 0$ for all

$x \in K$, has a representation of the form

$$p(x) = q_0(x) + q_1(x)p_1(x) + q_2(x)p_2(x) \tag{5}$$

where q_0, q_1, and q_2 are polynomials which are sums of squares of polynomials, i.e., there exist three integers n_0, n_1, and n_2, and polynomials $q_{0i}(x)$, $i = 1, \ldots n_0$, $q_{1i}(x)$, $i = 1, \ldots n_1$, and $q_{2i}(x)$, $i = 1, \ldots n_2$, such that the following holds

$$q_k(x) = \sum_{i=1}^{n_k} (q_{ki}(x))^2 \ , \ k = 0, 1, 2 \ . \tag{6}$$

In [8] an algorithm for testing if a given polynomial is a sum of squares is given. The basic tool is the Gramian matrix associated to a polynomial. In order to introduce the Gramian matrix some notations are needed. We denote a multiindex by $\alpha = (\alpha_1, \ldots, \alpha_n)$ and define $\mid \alpha \mid = \sum_{i=1}^n \alpha_i$, and $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$. Let $p(x) = \sum_{|\alpha| \le 2k} p_\alpha x^\alpha$ be the corresponding representation of a polynomial of degree $2k$ where the coefficients are given by p_α. Let the vector $x^{[k]}$ consist of all monomials x^α with $\mid \alpha \mid \le k$ then the polynomial $p(x)$ of degree $2k$ has a representation

$$p(x) = {x^{[k]}}^T G x^{[k]} \tag{7}$$

as quadratic form with a real symmetrical matrix G called Gramian matrix for p. The Gramian matrix is not unique because monomials of degree less than $2k$ can have different representations as products of monomials of degree less than k. For the polynomial $p(x) = 1 + 2 * x^2 + x^4$ in one variable we have $k = 2$ and

$$x^{[2]} = \begin{pmatrix} 1 & x & x^2 \end{pmatrix}^T \tag{8}$$

and all possible Gramian matrices are given by

$$G = \begin{pmatrix} 1 & 0 & 1-\lambda \\ 0 & 2\lambda & 0 \\ 1-\lambda & 0 & 1 \end{pmatrix} \tag{9}$$

where a real parameter λ has been introduced in order to parametrize the set of possible Gramian matrices. In this case the non-uniqueness can be traced back to the following different representations $1 \times x^2 = x \times x$ of the monomial x^2. A fundamental theorem in real algebraic geometry states that a polynomial is a sum of squares iff it has at least one positive semidefinite Gramian matrix [7], [8]. In our above example the Gramian matrix is positive semidefinite if $0 \le \lambda \le 2$ is fulfilled. Thus, the polynomial in the example is a sum of squares. In general the possible Gramian matrices of a multivariable polynomial are linearly parametrized by a finite number of λ. With respect to this representation the task to find a positive definite Gramian matrix is a linear matrix inequality (LMI) which is readliy solved using available numerical methods [2]. Thus, the problem to decide if a polynomial $p(x)$ is strictly positive on K for $m = 2$ is reduced to the problem to decide if a representation of the form (5) exists which is formulated as an LMI problem by making an ansatz for q_1, and q_2 and computing q_0 by comparison of coefficients. Unfortunately due to deep mathematical reasons [9], the degrees of q_0, q_1, and q_2 can not be computed a priori from the degrees of p, p_1, and p_2. Thus, if the ansatz for q_1, and q_2 was not sufficient no conclusion can be drawn. Nevertheless, we will see in the next section how these results can be used in order to estimate the domain of attraction and to optimize c.

3 Formulation of the LMI problem

In this section we will show how the results of the previous section can be applied to the estimation of the domain of attraction. The direct approach of taking $\Omega_c = K$ and $p(x) = -\dot{V}(x)$ will not work because the repesentation (5) is only valid if p is strictly positive on K but $-\dot{V}(x)$ is not strictly positive on Ω_c because it takes the value 0 at $x = 0$. Thus, $x = 0$ has to be excluded and K has to be defined accordingly. In order to do this we divide Ω_c into two sets, namely,

$$\begin{aligned} \Omega_c &= \{x \mid V(x) \le c\} \ , \\ &= \{x \mid V(x) \le c_0\} \ \cup \{x \mid c_0 \le V(x) \le c\} \end{aligned} \tag{10}$$

where $c_0 < c$ is assumed. For the first set $\Omega_{c_0} = \{x \mid V(x) \le c_0\}$ we now give a procedure in order to compute c_0 in such a way that $\dot{V}(x)$ is negative for all $x \in \Omega_{c_0}$ and $x \ne 0$. Let the degree of $\dot{V}(x)$ be l, then we have the representation

$$\dot{V}(x) = x^T P x + \sum_{3 \le |\alpha| \le l} \dot{v}_\alpha x^\alpha \tag{11}$$

where $\dot{v}_\alpha$ are the coefficients of the terms of degree higher than or equal to 3 and all quadratic terms are written as a quadratic form with the negative definite symmetric real matrix P. Now we bound $\dot{V}$ from above on the set $\{x \mid V(x) = r^2\}$ and use $V(x) = x^T Q x$. We compute

$$\max_{x^T Q x = r^2} x^T P x \ = \ -\mu r^2 \ , \ \mu > 0 \ , \tag{12}$$

$$\max_{x^T Q x = r^2} x_i \ = \ r \left(e_i^T Q^{-1} e_i\right)^{\frac{1}{2}} \ , \ 1 \le i \le n \tag{13}$$

where $-\mu$ is the largest generalized eigenvalue of the generalized eigenvalue problem $\lambda Q v = P v$ and e_i is the unit vector in the i-th direction in R^n. Now we introduce the following vector

$$z = \begin{pmatrix} \left(e_1^T Q^{-1} e_1\right)^{\frac{1}{2}} \\ \left(e_2^T Q^{-1} e_2\right)^{\frac{1}{2}} \\ \vdots \\ \left(e_{n-1}^T Q^{-1} e_{n-1}\right)^{\frac{1}{2}} \\ \left(e_n^T Q^{-1} e_n\right)^{\frac{1}{2}} \end{pmatrix} \tag{14}$$

and have immediately the following bound

$$\max_{x^TQx=r^2} x^\alpha \leq r^{|\alpha|} z^\alpha \ . \tag{15}$$

Using (12) and (15) we compute from (11) the following bound

$$\begin{aligned}\max_{x^TQx=r^2} \dot{V}(x) &\leq -\mu r^2 + \sum_{3\leq|\alpha|\leq l} |\ \dot{v}_\alpha\ |\ z^\alpha r^{|\alpha|} \\ &= r^2\ h(r). \end{aligned} \tag{16}$$

Using the bound (16) it is easliy verified that $\dot{V}(x)$ is negative definite in the sets Ω_{r^2} for all r which are smaller than the unique real positive root $r^\star$ of the polynomial

$$h(r) = -\mu + \sum_{k=1}^{l-2} \left(\sum_{|\alpha|=k+2} |\ \dot{v}_\alpha\ |\ z^\alpha \right) r^k \tag{17}$$

Thus, if we choose $c_0 \leq r^{\star 2}$ the first set in the decomposition (10) has the desired property and in order to maximize c we have to investigate the polynomial $\dot{V}(x)$ on the set

$$\begin{aligned} K_c &= \{x \mid c_0 \leq V(x) \leq c\} \\ &= S(V(x)-c_0\ ,\ c-V(x)) \end{aligned} \tag{18}$$

where it has to be strictly negative. It is obvious that the set (18) is a closed basic semialgebraic set which is defined by two polynomial inequalities and therefore the representation

$$\begin{aligned} -\dot{V}(x) &= q_0(x) + q_1(x)(V(x)-c_0) + \\ & \quad q_2(x)(c-V(x)) \end{aligned} \tag{19}$$

with polynomials $q_0(x)$, $q_1(x)$, and $q_2(x)$ which are sums of squares has to hold if $\dot{V}(x)$ is negative definite in Ω_c. As already mentioned in the last section the first step now is to make an ansatz for q_1 and q_2 as polynomials with undetermined coefficients of even degree $2k_1$, and $2k_2$, respectively. The corresponding Gramian matrices are denoted by G_1, and G_2, respectively. These matrices depend linearly on the coefficients of the corresponding polynomials and linearly on the corresponding λ, thus, the conditions that q_1, and q_2 are sums of squares are given by the LMI's

$$G_1 > 0 \quad , \quad G_2 > 0 \tag{20}$$

where the coefficients of q_1 and q_2 and the λ are the decision variables. The polynomial q_0 is computed from (19) by comparison of coefficients and the degrees of q_1 and q_2 have to be choosen such that the degree of q_0 which is given by $\max(l, 2k_1+2, 2k_2+2)$ is even because otherwise q_0 could not be a sum of squares. This is the first condition which has to be fulfilled by the ansatz. It is clear that the Gramian matrix for q_0 depends linearly on the coefficients of the ansatz polynomials and linear on the λ which have to be introduced in forming the Gramian for q_0. The dependence on c is also linearly but only through products with the coefficients of q_2. Thus, we have the following representation for the Gramian of q_0

$$G_0 = G_{00} + c\,G_{01} \tag{21}$$

where G_{00} and G_{01} depend linearly on the coefficients of the ansatz polynomials and the λ. The optimization problem is given by

$$\max c \tag{22}$$

subject to the constraints

$$G_0 > 0 \quad , \quad G_1 > 0 \quad , \quad G_2 > 0 \tag{23}$$

and is known as a generalized eigenvalue problem under LMI constraints [2]. This optimization problem is not well suited because nearly all numerical algorithms require that G_{01} has to be positive definite at the optima but this is impossible in our situation because G_{01} is that part of the Gramian of q_0 which is generated only by the polynomial q_2 without multiplication with $V(x)$ and thus can at best be positive semidefinite. In order to circumvent this problem we introduce a real positive constant $c^\star$ which fulfills the inequality $c_0 \leq c^\star \leq r^{\star 2}$; this is possible due to the choice of c_0 and $r^\star$. Now we know that $\dot{V}(x)$ is strictly negative in $K_{c^\star}$ and if with our ansatz a representation of the form (19) exists in $K_{c^\star}$ the LMI conditions given by

$$\begin{aligned} G_0 = G_{00} + c^\star G_{01} &> 0\ , \\ G_1 &> 0\ , \\ G_2 &> 0 \end{aligned} \tag{24}$$

have a feasible solution. If the LMI conditions (24) have no feasible solution the degrees of the ansatz polynomials have to be increased. In the following we assume that the LMI conditions (24) have a feasible solution. In this case our optimization problem can be refomulated as follows

$$\eta^\star = \min \eta \tag{25}$$

subject to the constraints

$$\eta G_0 + G_{01} > 0\ , \tag{26}$$

$$G_0 > 0\ , \tag{27}$$

$$G_1 > 0\ , \tag{28}$$

$$G_2 > 0 \tag{29}$$

and the optimal value c_{opt} for c is given by

$$c_{opt} = c^\star + \eta^{\star -1}\ . \tag{30}$$

In this form the generalized eigenvalue optimization can be solved routinely with available software, e.g. the MATLAB Toolbox [2]. Therefore the goal of this section is achieved, namely, the computation of c_{opt} has been reduced to an LMI optimization which can be solved in a reliable way.

4 Examples

In this section we will give two examples in order to illustrate the presented results. The state space differential equations for the first example are given by

$$\begin{aligned} \dot{x}_1 &= -x_1 \ , && (31) \\ \dot{x}_2 &= -x_2 + x_1^2 x_2 && (32) \end{aligned}$$

and the Lyapunov function is chosen as

$$V(x) = x_1^2 + x_2^2 \qquad (33)$$

the time derivative of the Lyapunov function along the trajectories of (31), and (32) is given by

$$\dot{V}(x) = -2x_1^2 - 2x_2^2 + 2x_1^2 x_2^2 \ . \qquad (34)$$

The polynomial $h(r)$ is given by

$$h(r) = -2 + 2r^2 \qquad (35)$$

and we compute $r^\star = 1$ and choose $c_0 = c^\star = 1$. For the polynomials $q_1(x)$ and $q_2(x)$ we make the following ansatz

$$\begin{aligned} q_1(x) &= b_{00} + b_{10}x_1 + b_{01}x_2 + \\ &\quad b_{20}x_1^2 + b_{11}x_1x_2 + b_{02}x_2^2 \ , && (36) \\ q_2(x) &= c_{00} + c_{10}x_1 + c_{01}x_2 + \\ &\quad c_{20}x_1^2 + c_{11}x_1x_2 + c_{02}x_2^2 && (37) \end{aligned}$$

using the vector

$$x^{[1]} = \begin{pmatrix} 1 & x_1 & x_2 \end{pmatrix}^T \qquad (38)$$

we compute the corresponding Gramian matrices as

$$G_1 = \begin{pmatrix} b_{00} & \frac{b_{10}}{2} & \frac{b_{01}}{2} \\ \frac{b_{10}}{2} & b_{20} & \frac{b_{11}}{2} \\ \frac{b_{01}}{2} & \frac{b_{11}}{2} & b_{02} \end{pmatrix} \ , \qquad (39)$$

$$G_2 = \begin{pmatrix} c_{00} & \frac{c_{10}}{2} & \frac{c_{01}}{2} \\ \frac{c_{10}}{2} & c_{20} & \frac{c_{11}}{2} \\ \frac{c_{01}}{2} & \frac{c_{11}}{2} & c_{02} \end{pmatrix} \qquad (40)$$

and observe that no λ have to be introduced. The polynomial $q_0(x)$ is of degree four in two variables and in order to compute the Gramian matrix the vector

$$x^{[2]} = \begin{pmatrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{pmatrix}^T \qquad (41)$$

is used and the 6×6 Gramian matrix is given by

$$G_0 = \begin{pmatrix} \beta^c_{00} & \frac{1}{2}\beta^c_{10} & \frac{1}{2}\beta^c_{01} & \frac{1}{2}\lambda_2 & \lambda_3 & \frac{1}{2}\lambda_1 \\ \frac{1}{2}\beta^c_{10} & \psi_1 & \frac{1}{2}\beta^c_{11} - \lambda_3 & -\frac{1}{2}\beta_{10} & \lambda_4 & -\frac{1}{2}\beta_{10} - \lambda_5 \\ \frac{1}{2}\beta^c_{01} & \frac{1}{2}\beta^c_{11} - \lambda_3 & \psi_2 & -\frac{1}{2}\beta_{01} - \lambda_4 & \lambda_5 & -\frac{1}{2}\beta_{01} \\ \frac{1}{2}\lambda_2 & -\frac{1}{2}\beta_{10} & -\frac{1}{2}\beta_{01} - \lambda_4 & -\beta_{20} & -\frac{1}{2}\beta_{11} & \frac{1}{2}\lambda_6 \\ \lambda_3 & \lambda_4 & \lambda_5 & -\frac{1}{2}\beta_{11} & \psi_3 & -\frac{1}{2}\beta_{11} \\ \frac{1}{2}\lambda_1 & -\frac{1}{2}\beta_{10} - \lambda_5 & -\frac{1}{2}\beta_{01} & \frac{1}{2}\lambda_6 & -\frac{1}{2}\beta_{11} & -\beta_{02} \end{pmatrix} \qquad (42)$$

where the abbreviations

$$\begin{aligned} \beta_{ij} &:= b_{ij} - c_{ij} \ , && (43) \\ \beta^c_{ij} &:= b_{ij} - c \cdot c_{ij} \ , && (44) \\ \psi_1 &:= 2 - \beta_{00} + \beta^c_{20} - \lambda_2 \ , && (45) \\ \psi_2 &:= 2 - \beta_{00} + \beta^c_{02} - \lambda_1 \ , && (46) \\ \psi_3 &:= -2 - \beta_{20} - \beta_{02} - \lambda_6 \ , && (47) \end{aligned}$$

and six λ have been introduced. The vector of the 18 LMI decision variables is given by

$$\begin{aligned} &(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, b_{00}, b_{10}, b_{01}, b_{20} \\ &\quad b_{11}, b_{02}, c_{00}, c_{10}, c_{01}, c_{20}, c_{11}, c_{02}) && (48) \end{aligned}$$

and the matrices G_{00} and G_{01} can be extracted from G_0. The MATLAB LMI Control Toolbox has been used to solve the corresponding optimization problem using the routine *gevp*. The computed optimal value for c using the default stopping criteria of *gevp* is given by

$$c_{opt} = 3.9987825... \qquad (49)$$

and the computed values of the LMI decision variables lead to the following polynomials

$$\begin{aligned} q_0(x) &= \frac{1}{2}(x_1^2 - x_2^2)^2 \ , && (50) \\ q_1(x) &= 0 \ , && (51) \\ q_2(x) &= \frac{1}{2}(x_1^2 + x_2^2) && (52) \end{aligned}$$

in the representation theorem. After some easy calculations the following representation is found

$$\begin{aligned} -\dot{V}(x) &= 2x_1^2 + 2x_2^2 - 2x_1^2x_2^2 \ , \\ &= \frac{1}{2}(x_1^2 - x_2^2)^2 + \\ &\quad (4 - x_1^2 - x_2^2) \cdot (\frac{1}{2}(x_1^2 + x_2^2)) && (53) \end{aligned}$$

from which the optimal value of $c_{opt} = 4$ can be read of directly. Thus, a very good approximation has been computed with the help of the MATLAB Toolbox and the discrepancy between (49) and $c_{opt} = 4$ is due to the numerical stopping criterion. This example shows very clearly that the presented LMI formulation of the computation of the domain of attraction leads to reliable results because the underlying optimization problem is convex and is solved with reliable numerical methods.

The second example will illustrate that the proposed method is also applicable to higher dimensional problems. The state space differential equations are given by

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 x_3^2 \ , && (54) \\ \dot{x}_2 &= -x_2 + x_1 x_2 \ , && (55) \\ \dot{x}_3 &= -x_3 && (56) \end{aligned}$$

and we choose the Lyapunov function

$$V(x) = x_1^2 + x_2^2 + x_3^2 \ . \qquad (57)$$

The time derivative of the Lyapunov function along the trajectories is given by

$$\dot{V}(x) = -2x_1^2 - 2x_2^2 - 2x_3^2 + 2x_1 x_2 x_3^2 + 2x_1 x_2^2 \ . \quad (58)$$

and the polynomial $h(r)$ is computed as

$$h(r) = -2 + 2r + 2r^2 \qquad (59)$$

and the unique positive real root is $r^\star = \frac{1}{2}(\sqrt{5} - 1)$. We choose $c_0 = c^\star = 0.25$. For the polynomials q_1 and q_2 an ansatz as quadratic functions is made and the corresponding Gramian matrices G_1 and G_2 are real symmetric 4×4 matrices. The polynomial q_0 is a polynomial of degree four in three variables and the corresponding Gramian matrix is a real symmetric 10×10 matrix with 20 λ parameters. The optimization problem is solved easily with the help of *gevp* from [2] and the computed optimal value is given by

$$c_{opt} = 4.9123337... \qquad (60)$$

which is very close to the exact optimal value 4.9187584... computed with the help of a computer algebra system. The difference can be traced back to the numerical stopping criteria in *gevp*. The consumed computation time using the presented approach is approximately $40sec$ on a Pentium III computer whereas the computer algebra computation took approximately $1h$ on the same computer.

5 Conclusions and Outlook

In this paper the important problem of computing an approximation to the domain of attraction for polynomial systems has been adressed. Using a new representation theorem for strictly positive polynomials on compact basic semialgebraic sets the necessary optimization is formulated as a generalized eigenvalue optimization under LMI constraints. In two examples the usefullness of the presented approach is shown. With a very simple ansatz excellent results can be obtained for the guaranteed subset of the domain of attraction. Future work will concentrate on the incorporation of the controller design into the optimization and on the optimization of the Lyapunov function. It is expected that this will result in bilinear matrix inequalities which have to be solved using a branch and bound method. Another direction of future reseach is the use of higher order Lyapunov functions which for example naturally arise in the application of Zubov's method.

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