Estimation of the domain of attraction for polynomial systems using multidimensional grids

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Abstract

Investigation of the stability properties of stationary points of nonlinear systems lies at the heart of modern control engineering. In this contribution we will show how the theorem of Ehlich and Zeller is used to compute subsets of the domain of attraction of asymptotically stable stationary points of polynomial systems. The theorem of Ehlich and Zeller is a tool to bound the values of a polynomial over an interval using the values of the polynomial on a finite grid in the interval. We will present the generalizations of this theorem to multivariable polynomials and to trigonometric polynomials. A bisection strategy will be presented which allows the guaranteed computation of a subset of the domain of attraction. An instructive example will be presented and some conclusions and an outlook will finish this contribution.

Keywords: Control Theory, Polynomial Systems, Stability Theory.

1 Introduction

The class of systems investigated in this paper is given by the following state differential equation

$$\dot{x} = f(x) \ , \ x(0) = x^0 \ , \tag{1}$$

where $x \in R^n$ represents the state vector of the system and the function $f(x)$ is a polynomial function of the state vector. We make the following two assumptions

- $f(0) = 0$, i.e., the state vector $x = 0$ is a stationary point of the system,
- $A = \frac{\partial f}{\partial x}(0)$ is a Hurwitz's matrix, i.e., it has all eigenvalues in the left open half plane of the complex plane.

Thus, the stationary point $x = 0$ is asymptotically stable and the problem which will be investigated in this paper is to estimate the domain of attraction of $x = 0$. The main tool in achieving this goal is the use of an appropriate Lyapunov function. In the following we will use the quadratic Lyapunov function $V(x) = x^T Q x$ where Q is a positive definite symmetrical $n \times n$ matrix. We assume that the matrix $P = A^T Q + QA$ is negative definite, thus $V(x)$ is a valid Lyapunov function for the linearized system. The set

$$\Omega_c = \{x \mid V(x) \leq c\} \ , \ c > 0 \tag{2}$$

is contained in the unknown domain of attraction if the inequality

$$\dot{V}(x) = f(x)^T Q x + x^T Q f(x) < 0 \tag{3}$$

is valid for all $x \in \Omega_c$, $x \neq 0$ [1],[4],[5],[6]. The problem is to maximize c because then the corresponding set Ω_c is the largest subset of the domain of attraction which can be guaranteed with the chosen Lyapunov function. In the literature several different methods have been proposed in order to achieve this goal [1],[8],[9]. In this paper we will give an improved version of the method presented in [8]. In oder to do this we will first formulate the theorem of Ehlich and Zeller and the corresponding generalizations.

2 Theorem of Ehlich and Zeller

This section will closely follow the corresponding section in [8]. In the following; $J = [a, b]$ denotes a nonempty compact real interval with $J \subset R$. We define the set of Chebychev points in J for a given natural number $N > 0$ by $X(N, J) := \{x_1, x_2, ..., x_N\}$, where $x_i := \frac{a+b}{2} + \frac{b-a}{2} \cos\left(\frac{(2i-1)\pi}{2N}\right)$ for $i = 1, ..., N$. For a function h defined on a set I we define the norm $||h||^I := \max_{x \in I} |h(x)|$ which is the usual maximum norm. Let P_n be the set of polynomials in one variable with $degP \leq n$. Then the following inequality

$$||P||^J \leq C\left(\frac{n}{N}\right) \ ||P||^{X(N,J)} \tag{4}$$

with

$$N > n \text{ and } C(q) := \left[\cos(\frac{q}{2}\pi)\right]^{-1} \text{ for } 0 \leq q \leq 1 \tag{5}$$

is valid for every $P \in P_n$ and every interval J. Inequality (4) is remarkable because the norm $\|P\|^{X(N,J)}$ on the right hand side of (4) depends on the values of P at the Chebychev points only. This result was given by Ehlich and Zeller in [2]. Using (4); the following inequalities

$$P^J_{min} \geq \frac{1}{2}\left\{ \left(C\left(\frac{n}{N}\right)+1\right) P^{X(N,J)}_{min} - \left(C\left(\frac{n}{N}\right)-1\right) P^{X(N,J)}_{max}\right\}, \qquad (6)$$

$$P^J_{max} \leq \frac{1}{2}\left\{ \left(C\left(\frac{n}{N}\right)+1\right) P^{X(N,J)}_{max} - \left(C\left(\frac{n}{N}\right)-1\right) P^{X(N,J)}_{min}\right\} \qquad (7)$$

which are valid for every $P \in P_n$ and $N > n$ are given by Gärtel in [3] where $P^I_{min} := \min_{x\in I} P(x)$ and $P^I_{max} := \max_{x\in I} P(x)$ are the maximum and minimum of P in the set I, respectively. For trigonometric polynomials and for rational functions similar inequalities are given by Gärtel. We will give these results at the end of this section.

The inequalities (4),(6),(7) are valid for polynomials in one variable; they are extended to polynomials of several variables using the following replacements. The interval J is replaced by

$$\hat{J} = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \qquad (8)$$

which represents a hyperrectangle. For the degree of P with respect to the i-th variable x_i we introduce the abbreviation n_i and the set of Chebychev points in $\hat{J}$ is given by

$$X(\hat{N}, \hat{J}) := X(N_1, [a_1, b_1]) \times \cdots \times X(N_n, [a_n, b_n]) \quad (9)$$

where N_i is the number of Chebychev points in the interval $[a_i, b_i]$. Then the inequalites

$$P^{\hat{J}}_{min} \geq \frac{1}{2}\left\{(K+1)P^{X(\hat{N},\hat{J})}_{min} - (K-1)P^{X(\hat{N},\hat{J})}_{max}\right\} (10)$$

$$P^{\hat{J}}_{max} \leq \frac{1}{2}\left\{(K+1)P^{X(\hat{N},\hat{J})}_{max} - (K-1)P^{X(\hat{N},\hat{J})}_{min}\right\} (11)$$

with

$$K = \prod_{i=1}^{n} C\left(\frac{n_i}{N_i}\right) \qquad (12)$$

under the conditions $N_i > n_i,\ i = 1, ..., n$, are valid. The given inequalities are now extended to trigonometric polynomials. A trigonometric polynomial of degree n in the variable φ is defined as

$$P(\varphi) := a + \sum_{k=1}^{n} (\alpha_k \sin(k\varphi) + \beta_k \cos(k\varphi)) \qquad (13)$$

which is just a finite Fourier series. Let P be a trigonometric polynomial with $degP \leq n$. Then the following inequality

$$\|P\|^{[0,2\pi]} \leq C\left(\frac{n}{N}\right)\|P\|^{\Psi(N)} \qquad (14)$$

with

$$\Psi(N) = \left\{\frac{(i-1)\pi}{N}, i = 1, \ldots, 2N\right\} \qquad (15)$$

and $N > n$ is valid. The inequalities (6,7) are generalized to

$$P^{[0,2\pi]}_{min} \geq \frac{1}{2}\left\{(K+1)P^{\Psi(N)}_{min} - (K-1)P^{\Psi(N)}_{max}\right\}, (16)$$

$$P^{[0,2\pi]}_{max} \leq \frac{1}{2}\left\{(K+1)P^{\Psi(N)}_{max} - (K-1)P^{\Psi(N)}_{min}\right\} (17)$$

with $K = C(\frac{n}{N})$. The generalization to several variables leads to (10),(11) with the replacements $\hat{J} = [0, 2\pi]^n$ and $X(\hat{N}, \hat{J}) = \Psi(N_1) \times \Psi(N_2) \times \cdots \times \Psi(N_n) = \Psi(\hat{N})$ and with K given by (12). Exactly the same results hold if the function under investigation is a polynomial in some of the variables and a trigonometric polynomial in the remaining variables. That means if $P(x_1, x_2, \ldots, x_n)$ is a trigonometric polynomial in the variables $x_1, \ldots, x_r$ and a polynomial in the variables $x_{r+1}, \ldots, x_n$ with the corresponding degrees $n_1, \ldots n_n$, the bounds (10),(11) are valid with $\hat{J} = [0, 2\pi]^r \times [a_{r+1}, b_{r+1}] \times \cdots [a_n, b_n]$, $X(\hat{N}, \hat{J}) = \Psi(N_1) \times \cdots \times \Psi(N_r) \times X(N_{r+1}, [a_{r+1}, b_{r+1}]) \times \cdots \times X(N_n, [a_n, b_n])$ and K given by (12). We apply these bounds in the next sections to compute upper bounds on polynomials and trigonometric polynomials in several variables based on values at a finite number of points.

3 Enclosure for the Optimization Problem

In this section we will show how the results of the previous section can be applied to the estimation of the domain of attraction. The basic idea of our algorithm will be a generalized bisection. In order to start we need an initial approximation to the domain of attraction in form of a set Ω_{c_0}. Our goal now is to compute a value c_0 such that $\dot{V}(x)$ is negative definite in Ω_{c_0}. Let the degree of $\dot{V}(x)$ be l, then we have the representation

$$\dot{V}(x) = x^T P x + \sum_{3\leq|\alpha|\leq l} \dot{v}_\alpha x^\alpha \qquad (18)$$

where $\dot{v}_\alpha$ are the coefficients of the terms of degree higher than or equal to 3 and all quadratic terms are written as a quadratic form with the negative definite symmetric real matrix P. Now we bound $\dot{V}$ from above on the set $\{x \mid V(x) = r^2\}$ and use $V(x) = x^T Q x$. We compute

$$\max_{x^TQx=r^2} x^T P x = -\mu r^2 \ , \ \mu > 0 \ , \tag{19}$$

$$\max_{x^TQx=r^2} x_i = r \left(e_i^T Q^{-1} e_i\right)^{\frac{1}{2}} \ , \ i = 1, \ldots n \tag{20}$$

where $-\mu$ is the largest generalized eigenvalue of the generalized eigenvalue problem $\lambda Q v = P v$ and e_i is the unit vector in the i-th direction in R^n. Now we introduce the following vector

$$z = \begin{pmatrix} \left(e_1^T Q^{-1} e_1\right)^{\frac{1}{2}} \\ \left(e_2^T Q^{-1} e_2\right)^{\frac{1}{2}} \\ \vdots \\ \left(e_{n-1}^T Q^{-1} e_{n-1}\right)^{\frac{1}{2}} \\ \left(e_n^T Q^{-1} e_n\right)^{\frac{1}{2}} \end{pmatrix} \tag{21}$$

and have immediately the following bound

$$\max_{x^TQx=r^2} x^\alpha \le r^{|\alpha|} z^\alpha \ . \tag{22}$$

Using (19) and (22) we compute from (18) the following bound

$$\max_{x^TQx=r^2} \dot V(x) \le -\mu r^2 + \sum_{3\le|\alpha|\le l} \mid \dot v_\alpha \mid z^\alpha r^{|\alpha|} =$$

$$= r^2 \left(-\mu + \sum_{k=1}^{l-2} \left(\sum_{|\alpha|=k} \mid \dot v_\alpha \mid z^\alpha \right) r^k \right) \ . \tag{23}$$

Using the bound (23) it is easliy verified that $\dot V(x)$ is negative definite in the sets Ω_{r^2} for all r which are smaller than the unique real positive root $r^\star$ of the polynomial

$$h(r) = -\mu + \sum_{k=1}^{l-2} \left(\sum_{|\alpha|=k} \mid \dot v_\alpha \mid z^\alpha \right) r^k \tag{24}$$

Thus, we choose $c_0 = {r^\star}^2$, and $r_0 = r^\star$ and have found our initial approximation which will be improved now in a systematic way. Now we will use the results of the previous section to compute upper and lower bounds for the optimal c denoted by c_{opt} which guarantees the maximal subset of the domain of attraction which could be computed with the chosen Lyapunov function. First we compute the Cholesky decomposition $Q = L^T L$ of the matrix Q where L is a regular upper triangular matrix [7]. Using this L we change variables according to $y = Lx$ which leads to the representations

$$\tilde V(y) = V(L^{-1}y) = y^T y \tag{25}$$

and

$$\dot{\tilde V}(y) = \dot V(L^{-1}y) \tag{26}$$

respectively. The crucial fact is that the set Ω_c is transformed into the set

$$\tilde\Omega_c = \{z | z^T z < c\} \tag{27}$$

which is the open interior of a hypersphere. Thus, we have to look for the largest hypersphere such that $\dot{\tilde V}(y)$ is negative in the interior of this sphere except at the origin where it is zero. Now the idea of the method described in detail below is to increase r_0 thus; we choose $r = 2r_0$ and we can guarantee that $\dot{\tilde V}(x)$ is negative definite in the larger sphere $\{y \,|\, ||y|| \le r\}$ if it is negative in $\Delta = \{y \,|\, r_0^2 \le y^T y \le r^2\}$ which is the region between the two hyperspheres with radius r_0 and r, respectively. To verify negativity in Δ the n-dimensional polar coordinates $\rho, \alpha_1, \ldots, \alpha_{n-1}$ are introduced by

$$\begin{aligned} y_1 &= \rho\cos(\alpha_1)\cdots\cos(\alpha_{n-3})\cos(\alpha_{n-2})\cos(\alpha_{n-1}) \ , \\ y_2 &= \rho\cos(\alpha_1)\cdots\cos(\alpha_{n-3})\cos(\alpha_{n-2})\sin(\alpha_{n-1}) \ , \\ y_3 &= \rho\cos(\alpha_1)\cdots\cos(\alpha_{n-3})\sin(\alpha_{n-2}) \ , \\ &\vdots \\ y_n &= \rho\sin(\alpha_1) \ . \end{aligned} \tag{28}$$

Using these coordinates the set Δ is the image of $[r_0, r]\times[0, 2\pi]\times[0,\pi]\times\cdots\times[0,\pi]$ which means that $\dot{\tilde V}(y)$ is a polynomial in ρ and a trigonometric polynomial in $(\alpha_1, \ldots, \alpha_{n-1})$. Now an upper bound for $\dot{\tilde V}(y)$ in Δ is computed using the results of the previous section. If, during the computation of $\dot{\tilde V}(y)$ at the lattice points, a positive value is found the whole computation is interrupted and the radius $\tilde r$ of this point is an upper bound for the radius of the sphere lying entirely in $\dot{\tilde V}(y) < 0$. If the upper bound of $\dot{\tilde V}(y)$ in Δ is negative we set $r_0 = r$, choose again $r = 2r_0$ and repeat the process. If the upper bound is positive but all function values on the lattice are negative, the number of points in the lattice is increased and the calculation is repeated until $\dot{\tilde V}(y) < 0$ for all $y \in \Delta$ is guaranteed or a point in Δ with a positive value for $\dot{\tilde V}(y)$ is found. If the region of attraction is not the whole of R^n after a finite number of steps, an upper bound for the radius is found. When upper and lower bound for the radius are determined, both bounds are improved by bisection. The algorithm which has been described in [8] is given by the following steps:

1. Compute r_0 with $\dot{\tilde V}(x) < 0$ for all x with $x^T x \le r_0^2$ and $x \ne 0$; this has been described at the beginning of this section. Set $r_l = r_0, r_u = \infty$ and go to step 2.

2. If $r_u - r_l$ is less than a given tolerance ϵ go to step 6, else go to step 3.

3. If $r_u = \infty$, set $r = 2r_l$, else set $r = \frac{1}{2}(r_l + r_u)$. Go to step 4.

4. Compute $\dot{\tilde V}(x)$ at the lattice points in $[r_l, r] \times [0, 2\pi] \times \cdots \times [0, \pi]$ introduced in section 2. Go to step 5.

5. If $\dot{\tilde{V}}(x)$ is positive at a lattice point set r_u to the corresponding radius and go to step 2. If $\dot{\tilde{V}}(x)$ can be ensured to be negative in Δ by the bounds of section 2 set $r_l = r$ and go to step 2. If the bound is positive but all function values of $\dot{\tilde{V}}(x)$ at the lattice points are negative increase the number of lattice points and repeat step 4.

6. Stop.

This algorithm converges in a finite number of steps to bounds r_l and r_u for the optimal radius which satisfy $r_u - r_l < \epsilon$ for any given $\epsilon > 0$ which measures the accuracy of the computed solution. After the algorithm stops it is guaranteed that $\{y \,|\, y^T y \le r_l^2\}$ is completely inside the region with $\dot{\tilde{V}}(y) < 0$ and the sphere $\{y \,|\, y^T y \le r_u^2\}$ contains at least one point with $\dot{\tilde{V}}(x) > 0$. Thus, the set $\Omega = \{x \,|\, V(x) \le r_l^2\}$ is a subset of the region of attraction for the stationary solution $x = 0$ of (1) which is guaranteed with certainty. This is the largest subset of the region of attraction which can be guaranteed with the chosen Lyapunov function.

One drawback of the algorithm given above is that the bisection is carried out for r only. Thus, the angles have to be considered over their entire range while the region with respect to the radius decreases in size. This problem has been circumvented in a recent implementation of the algorithm which also carries out bisection steps in the angles. In order to be able to compute bounds in this case using the results of section 2 the trigonometric functions have to be replaced by rational functions via

$$\cos(\alpha_i) = \frac{1-t_i^2}{1+t_i^2}\ ,\ \sin(\alpha_i) = \frac{2t_i}{1+t_i^2}\ ,\ i = 1, \ldots, n-1 \tag{29}$$

because the results of section 2 are valid only for the interval $[0, 2\pi]$ and therefore are not usefull when bisection is applied. The new real parameters t_i range over the whole real line. The points at infinity are taken into account using the transformation $t_i = \frac{1}{s_i}$ for all values outside of the real interval $[-1, 1]$. Using this transformation into the real parameters t_i the function $\dot{\tilde{V}}(x)$ is given by a multivariable rational function with strictly positive denominator. Thus the sign can be decided from the multivariable numerator polynomial. In the modified algorithm a complete bisection with respect to the t_i is carried out in the steps 4 and 5 of the presented algorithm. This leads to a drastic improvement with respect to the consumed computer time. The example which will be presented in the next section has been computed using this modified new algorithm.

4 Example

The following example will illustrate that the proposed method for a three dimensional problem. The state space differential equations are given by

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 x_3^2\ , && (30)\\ \dot{x}_2 &= -x_2 + x_1 x_2\ , && (31)\\ \dot{x}_3 &= -x_3 && (32) \end{aligned}$$

and we choose the Lyapunov function

$$V(x) = x_1^2 + x_2^2 + x_3^2\ . \tag{33}$$

The time derivative of the Lyapunov function along the trajectories is given by

$$\dot{V}(x) = -2x_1^2 - 2x_2^2 - 2x_3^2 + 2x_1x_2x_3^2 + 2x_1x_2^2 \tag{34}$$

and the polynomial $h(r)$ is computed as

$$h(r) = -2 + 2r + 2r^2 \tag{35}$$

which has the unique positive real root $r^\star = \frac{1}{2}(\sqrt{5} - 1)$. With $r_0 = r^\star$ using the MATLAB Implementation of our algorithm we have computed

$$\begin{aligned} r_l &= 2.1875...\ , && (36)\\ r_u &= 2.2188... && (37) \end{aligned}$$

which result in the values

$$\begin{aligned} c_l &= 4.78515...\ , && (38)\\ c_u &= 4.92307... && (39) \end{aligned}$$

leading to the enclosing interval $[c_l, c_u]$ for c_{opt}. For this example the exact value for c_{opt} is given by 4.9187584... which has been computed with the help of a computer algebra system. The width of the enclosing interval depends on the chosen ϵ in the algorithm (Figure 1).

5 Conclusions and Outlook

In this paper asymptotically stable stationary points of polynomial systems have been investigated. A new algorithm for the computation of a subset of the region of attraction based on a quadratic Lyapunov function has been presented. With this algorithm the solution of the corresponding global optimization problem can be computed to any given accuracy. An example presented in this paper illustrates the results which can be achieved with this new algorithm. The method can be extended to systems where the right hand sides of the state space differential equations are continuously differentiable because these systems can be approximated arbitrarily well by a polynomial system. This extension will be investigated in future work. The use of interval arithmetic in oder to improve the bounding step in the algorithm will be the topic of future research also.

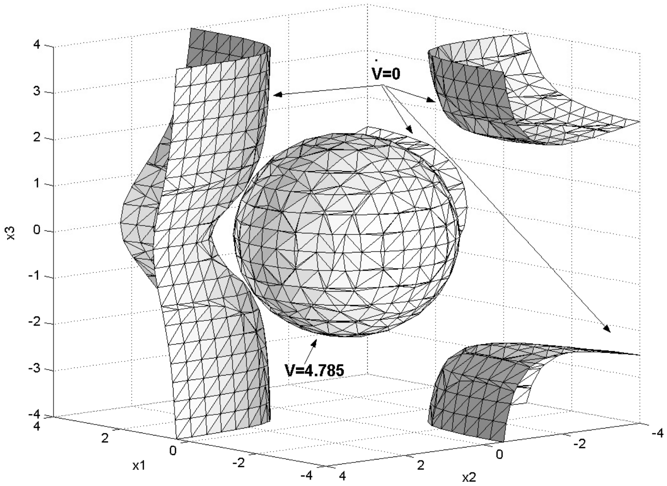

Figure 1: domain of attraction

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