Global Stabilization and Peaking Reduction for Nonlinear Uncertain Systems with Unmodeled Actuator Dynamics [1]

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Abstract: The problem of the global stabilization of a class of nonlinear uncertain plants is considered in this paper. The analysis includes the uncertain actuator dynamics, and the global convergence of the system state to zero is ensured by a second order sliding mode control technique. The proposed approach is chattering-free, and it also reduces the peaking of the state that is typically shown during the transient phase by variable structure systems and other robust control schemes. Simulation examples highlight the good features of the proposed control law.

Keywords: Uncertain systems, Nonlinear systems, Second order sliding modes.

1 Introduction

Variable Structure Systems (VSS) in which the control is able to constrain the trajectories of an uncertain system on a pre-specified manifold (the sliding manifold) by "brutal force" seem to be the most obvious and heuristic way to withstand the uncertainty which is always present in real plants. In such systems the control immediately reacts to any deviation of the system, steering it back to the manifold by means of a sufficiently energetic control effort. A motion on the sliding manifold will be characterized by the desired dynamical performances of the controlled system [5, 11, 12].

Higher order sliding mode control (HOSMC) generalizes the basic sliding mode idea, by using a discontinuous control signal acting on the higher time derivatives of the sliding quantity. Preserving the main advantages of the original approach, with respect to robustness and simplicity of implementation, they can deal with high relative degree systems and/or remove the chattering effect, and even guarantee higher accuracy in presence of plant and/or control devices imperfections [8, 9, 3].

HOSM's can appear naturally when a fast dynamic actuator is present between the relay and the controlled process [9]. Moreover, in presence of slow actuator dynamics, the relay control may traduce in limit cycles or even instability.

The main drawback of second order sliding mode algorithms SOSMC schemes appeared up to now is tied to the lack of global stability results, while FOSMC is naturally global. All the published SOSMC algorithms present local convergence properties (semi-global stability). This is a consequence of the fact that the uncertain terms appearing in the nonlinear differential equation modeling the system are assumed to be bounded by known positive functions satisfying strict growth conditions [2, 3, 4, 8].

A second drawback of SOSMC is that, similarly to other robust control techniques , peaking may occur during the transients [1, 7].

In this paper we propose a modified version of the sub-optimal SOSMC algorithm [2, 3, 4], which features global convergence properties for a large class of nonlinear uncertain systems, and, at the same time, counteracts the magnitude of the peaking phenomenon. In particular we consider the case of actuator dynamics heavily affecting the system uncertainties, while the actual control input is neither manipulable nor measurable.

The paper is organized as follows: in Section 2 an introductory example is provided to evidence the problems linked to the presence of the actuator dynamics. In section 3 the control problem is formulated, while in Section 4 the convergence prop-

[1] This work has been partially supported by the MURST Project "Navigation, Guidance and Control of Unmanned Vehicles for Submarine Activities"

erty of the proposed control algorithm is proved. In Section 5 some simulation results are shown. Some final comments close the paper.

2 An introductory example

Consider the system

$$\begin{aligned} \dot{x} &= ax^2 + u \qquad |a| \leq 4 \\ \tau\dot{u} &= -u + v \end{aligned} \tag{1}$$

where x is the state, u is the system input, v is the actuator input and a, τ are unknown constants. The quadratic growth of the nonlinear dynamics advises that the finite time escape to infinity of the state may occur if the controller is not able to promptly counteract the system explosion.

With $x(0) = 2$, $\tau = 0$ (no actuator device) and $a = 2$ the relay control

$$v = -(k^2 + 4x^2)\text{sign}(x) \tag{2}$$

with $k^2 = 1$ steers x to zero, while with $\tau = 0.2$ the delayed control action leads to a slower transient characterized by state peaking. Instability occurs if $\tau = 0.3$, and x diverges in finite time (fig. 1). FOSMC recovers the stability if the k^2 parameter is increased up to 3 (fig. 2).

The application of the SOSMC algorithm [2, 3]

$$v = -(k^2 + 4x^2)\text{sign}\,(x - \beta x_M) \tag{3}$$

where $k^2 = 3$, $\beta = \frac{1}{2}$ and x_M is the last singular value (local minimum, local maximum or horizontal flex) of x, considerably reduces the peaking phenomenon (Fig. 2) . In this paper it is formalized the intuitive idea of real-time updating the β parameter to improve the convergence properties [4]. A significant enlargement of the controlled class of systems and a further reduction of, both, the peaking phenomenon and of the convergence time are attained (see Fig. 2).

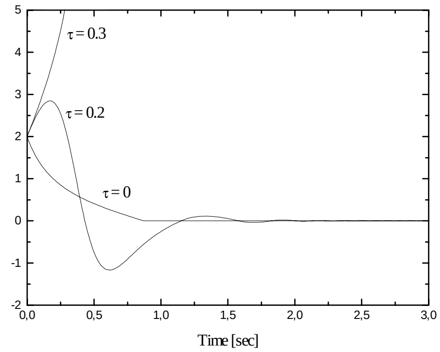

Figure 1: Relay control, $k^2 = 1$. The state behaviour with different values of τ.

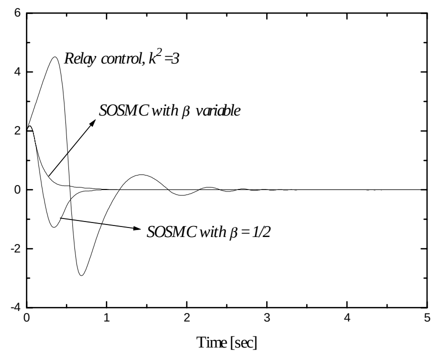

Figure 2: The x behaviour. Comparison between the relay control and the SOSMC.

3 Problem formulation

Consider a single–input system represented in normal form with a first-order actuator dynamics

$$x^{(n)} = f(\mathbf{x}, u) \tag{4}$$

$$\dot{u} = h(u) + v \tag{5}$$

where $\mathbf{x} = [x, \dot{x}, \ldots, x^{(n-1)}] \in R^n$ is the measurable state vector, $u \in R$ is the plant input, $v \in R$ is the actuator input and $f(\mathbf{x}, u)$, $h(u)$ are sufficiently smooth uncertain functions. Assume that any solution of (4) is well defined for all $t > 0$, provided u is bounded and continuous.

It is well known that if the system output is defined by a proper linear combination of the state variables

$$y_1(\mathbf{x}) = \mathbf{cx} \tag{6}$$

with $\mathbf{c} = [c_1, c_2, \ldots c_{n-1}, 1]$, and c_i, $(i = 1, 2, \ldots, n-1)$, real positive constants such that the polynomial $P(z) = z^{n-1} + \sum_{i=1}^{n-1} c_i z^{i-1}$ is Hurwitz, then the origin of the state space is a globally asymptotically stable (GAS) equilibrium point for its zero dynamics [6].

Indeed, it results

$$\begin{cases} \dot{\hat{\mathbf{x}}}(t) &= \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{b}}y_1(t) \\ x_n(t) &= -\hat{\mathbf{c}}\hat{\mathbf{x}} + y_1(t) \end{cases} \tag{7}$$

$$\begin{cases} \dot{y}_1(t) & = y_2(t) \\ \dot{y}_2(t) & = \varphi(\mathbf{x}, u) + g(\mathbf{x}, u)v \end{cases} \tag{8}$$

where $\hat{\mathbf{x}}=[x_1, x_2, \ldots, x_{n-1}]^T$, $\hat{\mathbf{c}}=[c_1, c_2, \ldots, c_{n-1}]$, $\hat{\mathbf{A}}$ is a $(n-1) \times (n-1)$–matrix in companion form with the last row coinciding with vector $-\hat{\mathbf{c}}$, $\hat{\mathbf{b}} = [0, \ldots, 0, 1]^T \in R^{n-1}$, and

$$\begin{aligned} \varphi(\mathbf{x}, u) &= \textstyle\sum_{i=1}^{n-1} \frac{\partial f}{\partial x_i} x_{i+1} + \sum_{i=1}^{n-2} c_i x_{i+2} \\ &\quad + \left(\frac{\partial f}{\partial x_n} + c_{n-1}\right) f(\mathbf{x}, u) + \frac{\partial f}{\partial u} h(u) \\ g(\mathbf{x}, u) &= \frac{\partial f}{\partial u}(\mathbf{x}, u) \end{aligned} \tag{9}$$

If a feedback actuator control $v(\mathbf{x}, t)$ steers $y_1(\mathbf{x})$ to zero in finite time, the system state converges asymptotically to zero, and the control task is achieved. For tracking purposes, it could be defined a similar output $y_1(\mathbf{e})$ in the error space.

Because of the uncertain structure of the system dynamics, a robust control technique must be used to fulfill the control objective. The considered system output y_1 can be considered as a sliding quantity whose vanishing defines a linear sliding manifold in the state space. As the relative degree between the output and the manipulable actuator input is two, a second order sliding mode control technique is well suited.

Under sensible smoothness assumptions on the uncertain dynamics, a mapping between the $\mathbf{x}-u$ and $y_1 - y_2$ spaces exists such that the following correspondence occurs

$$\begin{cases} \|\mathbf{x}\| \le X \\ |u| \le U \end{cases} \Longleftrightarrow \begin{cases} |y_1| \le Y_1 \\ |y_2| \le Y_2 \end{cases} \tag{10}$$

where X, U, Y_1 and Y_2 are positive constants. More specifically, we assume that the uncertainties satisfy the following bounds

$$|\varphi(\mathbf{x}, t, u)| \le \Phi\left[\|\mathbf{x}(t)\|, |u|\right] \le \overline{\Phi}(\|\mathbf{x}(t)\|, |y_2|) \tag{11}$$

$$|h(u)| \le M + N|u| \tag{12}$$

$$0 < G_1 \le g(\mathbf{x}, u) \le G_2 \tag{13}$$

where G_1, G_2, M, N are known positive constants and $\Phi(\cdot)$, $\overline{\Phi}(\cdot)$ are known positive convex functions without any particular assumption regarding their growth.

4 Main Result

The SOSMC problem consists in constraining in finite time system (4),(5) onto the second order sliding set (2-sliding set) $y_1 = y_2 = 0$, with its uncertain dynamics (7)–(9) satisfying inequalities (11)-(13). Our proposal is explained as follows.

A region $\mathcal{R}_k$ in the state space, containing the 2-sliding set, within which constant bounds of the uncertainties can be found, is defined.

The control must be able to fulfill the following tasks:

1. drive the system trajectories in finite time within the region $\mathcal{R}_k$
2. constrain the system motion within this region for the entire control time interval.
3. guarantee the finite-time reaching of the 2-sliding set y_1=y_2=0

To this end, consider the rectangular region in the $y_1 O y_2$ plane

$$\mathcal{R}_k \equiv \Big\{ (y_1, y_2) \in R^2 : |y_1| \le |y_{1_{M_k}}| , |y_2| \le \eta\sqrt{|y_{1_{M_k}}|} \Big\} \tag{14}$$

where $y_{1_{M_k}}$ is the generic k–th singular value (that is local minima, local maxima or horizontal flexes) of y_1, and η is an arbitrarily-chosen positive constant.

Into this region, by (11), a constant upper bound of the modulus of the uncertain drift term $|\varphi(\mathbf{x}, u)|$ in (8) is given by

$$\Phi_k^* = \overline{\Phi}\left(\|\mathbf{x}\|_{M_k}, \eta\sqrt{|y_{1_{M_k}}|}\right) \tag{15}$$

where $\|\mathbf{x}\|_{M_k}$ is an upper bound of $\|\mathbf{x}\|$ into the region $\mathcal{R}_k$. In fact, due to the bounded-input-bounded-state (BIBS) property of the linear subsystem in (7) the following condition can be derived [3]

$$\|\mathbf{x}(t)\| \le \|\mathbf{x}\|_{M_k} = Q_x \|\mathbf{x}(T_{M_k})\| + Q_y |y_{1_{M_k}}| \quad \forall (y_1, y_2) \in \mathcal{R}_k \tag{16}$$

where Q_x and Q_y are properly defined constants.

The control strategy is designed so that the controlled plant is proved to reach, and not to leave, the region $\mathcal{R}_k$. After that, in order to guarantee the finite-time reaching of the sliding set, a sequence of singular values is generated, featuring the following contraction property

$$|y_{1_{M_{k+1}}}| \leq \gamma^{2k} |y_{1_{M_1}}| \qquad \gamma^2 < 1 \qquad k = 1, 2, \ldots \tag{17}$$

so that that both y_1 and y_2 are steered to zero.

The possibility of achieving this two-fold task strictly depends on the assumptions made on the uncertain plant dynamics. In previous works the problem has been solved by assuming linear growth of the term $\varphi(\mathbf{x}, u)$ in (8) with respect to the control u, which could be much restrictive when considering systems with nonlinear dependence on the control. Here this assumption is relaxed, and the class of plants (4)–(13) is considered.

The key idea to achieve the peaking reduction is the direct counteraction of the growth of $|y_2|$ during the transient, as a direct correspondence occurs between the sliding quantity derivative and peaking.

The following Theorem solves the considered problem

Theorem 1: *Consider system* (4)-(6) *with its uncertain input-output dynamics* (8) *satisfying* (11)-(13) *and with completely available state. Assume that* $u(0) = 0$, *then the control law*

$$v(t) = -\frac{1}{G_1} \left[\Phi \left[\|\mathbf{x}\|, \overline{u}(t) \right] + \chi \right] \operatorname{sign} \left[y_1(t) - y_1(0) \right] \tag{18}$$

$$\begin{array}{l} \overline{u}(0) = 0 \\ \dot{\overline{u}}(t) = M + N|\overline{u}(t)| + |v(t)| \end{array} \tag{19}$$

where $\chi > 0$, *ensures the reaching of the condition* $y_2 = 0$ *at time instant* $t = T_{M_1}$. *From this time instant on, the control law*

$$\begin{array}{l} v(t) = -V_{M_k} \operatorname{sign} \left[y_1(t) - \beta_k y_{1_{M_k}} \right] \\ \qquad T_{M_k} < t \leq T_{M_{k+1}} \qquad k = 1, 2, \ldots \end{array} \tag{20}$$

guarantees the finite-time vanishing of the system output y_1 *and of its derivative, provided that the controller parameters are chosen according to*

$$V_{M_k} = \frac{\alpha}{G_1} \left[\Phi_k^* + \frac{1}{3} \eta^2 \right] \qquad \alpha > 1 \tag{21}$$

$$\beta_k = max \left\{ \frac{1}{2}, 1 - \frac{\eta^2}{2[\Phi_k^* + G_2 V_{M_k}]} \right\} \tag{22}$$

where T_{M_k} *are the time instants at which* $\dot{y}_1(t) = y_2(t)$ *is zero,* $y_{1_{M_k}} = y_1(T_{M_k})$, Φ_k^* *is defined as in* (15)-(16) *and* Q_x, Q_y *are properly defined constants* (see [3]). *As a consequence, the state* $\mathbf{x}(t)$ *is globally exponentially stabilized.*

Proof. See the appendix.

5 A Simulative Example

The sufficient conditions stated in Theorem 1 are derived on the basis of a worst-case analysis, thus resulting in being very conservative. Computer simulations show that the control effort may be set to smaller values then (21), thus obtaining higher smoothness of the resulting control law and higher accuracy in the practical realization of the control scheme, or even to constant values, reducing the on-line computational effort. Moreover, the perfect knowledge of the time instants at which y_2 is zero would require the availability of an infinite-bandwidth peak detector, which is an ideal device. In real applications, the use of an approximate peak detector must be considered, and, as a result, the real sliding motion occurs in a neighborhood of the sliding set.

As an example, consider system (4)-(5) with $n = 2$ and

$$\left\{ \begin{array}{l} \mathbf{f}(\mathbf{x}, t, u) = 3sin(3t) + x_1 + x_2 + 2cos(x_1) \\ \qquad\qquad + \left(1 + \frac{1}{2} sin(x_1)\right) u \\ \dot{u} = v \end{array} \right. \tag{23}$$

The state $\mathbf{x}$ is fully measurable, and the initial conditions are set to $\mathbf{x}(0) = [1 \;,\; 1]^T$. The actuator dynamics is represented by an integrator, which may be also designedly introduced for chattering elimination purposes (anti-chattering procedure [3]).

The SOSMC control with $\beta = 1/2$ is first applied. Then, it is compared with the new approach here proposed for on-line updating the β parameter. Great benefits can be evidenced as for state peaking (Fig. 3) and control activity (Fig. 4).

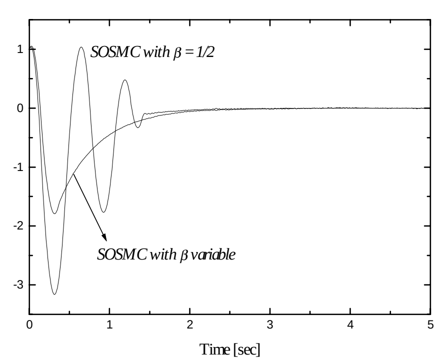

Figure 3: The state x_2.

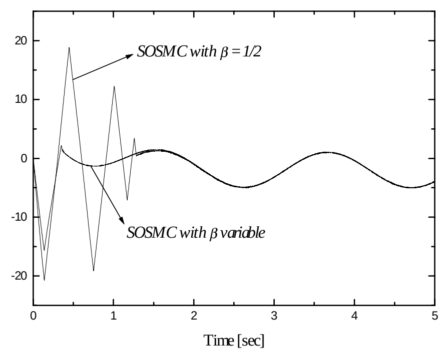

Figure 4: The actual control.

6 Conclusions

The problem of the global stabilization for a class of nonlinear uncertain systems expressed in normal form, encompassing non zero input stable (ZIS) and non bounded input bounded state (BIBS) stable systems, is considered in this paper. A feasible solution, based on second order sliding modes, is proposed. With respect to standard first order SMC, it results to be endowed by important properties of theoretical and practical relevance, like the chattering elimination, the robustness against unmodeled actuator dynamics and the reduction of the peaking phenomenon. The proposed algorithm is easy to implement and therefore suitable to be used in practice.

APPENDIX

Proof of Theorem 1

Initialization: $0 \leq t \leq T_{M_1}$

The initialization phase is designed such that the axis $y_2(t) = 0$ is reached in a finite time T_{M_1}. The term $\overline{u}$ upper bounds the modulus of $u(t)$ at any $t \geq 0$. It is not difficult to verify that the condition $y_2 \dot{y}_2 < 0$ is guaranteed. This implies that, at $t = T_{M_1} \leq \frac{y_2(t_1)}{\chi}$, the first singular value of $y_1(t)$, ${y_1}_{M_1}$, is reached, and its value is available for control purposes.

General case $T_{M_k} < t \leq T_{M_{k+1}}$ $(k \geq 1)$

Let ${y_1}_{M_k}$ be the k–th singular value of $y_1(t)$, T_{M_k} the corresponding time instant and T_{c_k} the time instant subsequent to T_{M_k} at which a control switching occurs. Let us define the region $\mathcal{R}_k$ in the $y_1 O y_2$ plane according to (14).

Suppose, without lost of generality, that the actual value of the $k - th$ singular value is positive. Analogous considerations are still valid if ${y_1}_{M_k} < 0$.

The proof of the finite time convergence to the origin of the state plane can be split into the following steps.

Contraction property:

Within the time interval $T_{M_k} < t < T_{c_k}$ the control is given by $v = -V_{M_k}$, and the system behaviour can be analyzed as follows:

$\dot{y}_2$ is less than zero, and the system dynamics satisfies the following differential inequalities

$$\begin{aligned} \dot{y}_2(t) &\leq -\left[G_1 V_{M_k} - \Phi_k^*\right] \\ \dot{y}_2(t) &\geq -\left[G_2 V_{M_K} + \Phi_k^*\right] \end{aligned} \tag{24}$$

Which can be graphically intended in terms of parabolic limit curves for the state trajectory (Fig. 5).

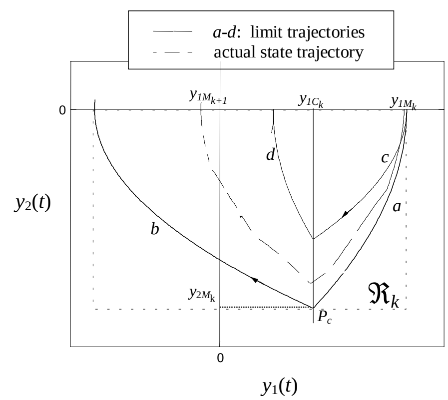

Figure 5: The 2-sliding limit trajectories

All the possible system trajectories evolve within the region $\mathcal{R}_k$, which is evidenced in figure. For contraction purposes, the worst-case trajectory is that to which corresponds the faster decrease of y_2 (line a in fig. 5), and is represented by

$$y_1 = {y_1}_{M_k} - \frac{1}{2} \frac{y_2^2}{\Phi_k^* + G_2 V_{M_k}} \tag{25}$$

The switching time instant $t_{c_k} : y_1(t_{c_k}) = \beta_k {y_1}_{M_k}$ is defined according to (22) such that

$$|y_2(t_{c_k})| \quad \leq \quad \eta \sqrt{|{y_1}_{M_k}|} \tag{26}$$

in order not to leave the region $\mathcal{R}_k$.

From $t = t_{c_k}$ on, the control commutes to $v = V_{M_k}$. As a result, $|y_2|$ starts to decrease.

The differential inequalities that describe the system behaviour are modified as follows

$$\begin{array}{l} \dot{y}_2(t) \geq G_1 V_{M_k} - \Phi_k^* \\ \dot{y}_2(t) \leq G_2 V_{M_k} + \Phi_k^* \end{array} \qquad (27)$$

which correspond to the limiting arcs b and d in Fig. 5.

To ensure the contraction condition (17), the worst case trajectory b, corresponding to the slower decrease of $|y_2|$, must be considered.

Such a limit line is described by

$$y_1 = y_1(t_{c_k}) + \frac{1}{2}\frac{y_2^2 - y_2^2(t_{c_k})}{G_1 V_{M_k} - \Phi_k^*} \qquad (28)$$

$y_{1_{M_{k+1}}}$ can be easily evaluated by substituting $y_2 = 0$ into (28), and the contraction condition (17) is equivalent to

$$\frac{1}{2}\frac{\eta^2}{G_1 V_{M_k} - \Phi_k^*} < 1 + \beta_k \qquad (29)$$

which, by (21) and (22), is true by assumption. Therefore, the region $\mathcal{R}_k$ is not left during the considered time interval $t \in (t_{M_k}, t_{M_{k+1}}]$.

By (24)-(28)

$$\frac{y_{1_{M_{k+1}}}}{y_{1_{M_k}}} \in \left[-\left(\beta_k + \frac{\eta^2}{2(G_1 V_{M_k} - \Phi_k^*)}\right); \frac{[\beta_k G_2 - (1-\beta_k) G_1] V_{M_k} + \Phi_k^*}{G_1 V_{M_k} + \Phi_k^*} \right] \qquad (30)$$

Thus, it is possible to define

$$\gamma_k^2 = \max \left\{ \left(\beta_k + \frac{\eta^2}{2(G_1 V_{M_k} - \Phi_k^*)}\right); \frac{[\beta_k G_2 - (1-\beta_k) G_1] V_{M_k} + \Phi_k^*}{G_1 V_{M_k} + \Phi_k^*} \right\} \qquad (31)$$

and, correspondingly, $\gamma^2 = \max_k \{\gamma_k^2\}$, so that

$$\left\{ \begin{array}{l} |y_{1_{M_{k+1}}}| \leq \gamma^{2k} |y_{1_{M_1}}| \\ |y_{2_{M_k}}| \leq \eta \gamma^{k-1} \sqrt{|y_{1_{M_1}}|} \end{array} \right. \qquad (32)$$

holds.

Finite time convergence:

Starting from $(y_{1_{M_k}}, 0)$ at time instant $t = T_{M_k}$, and considering the limit trajectories $a - b$ and $c - d$ in Fig. 5, it can be shown by algebraic computations that the $a - b$ trajectory is slower than the $c - d$ one. As for the reaching-time T_r, a constant Γ can be found such that

$$T_r \leq t_{M_1} + \Gamma \frac{\sqrt{|y_{1_{M_1}}|}}{1-\gamma} \qquad (33)$$

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