IRREDUCIBILITY CONDITIONS FOR NONLINEAR INPUT-OUTPUT DIFFERENCE EQUATIONS

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Abstract

The purpose of this paper is to present a new necessary and sufficient condition for irreducibility of nonlinear input-output (i/o) difference equation which extends directly the corresponding condition for the linear case. The condition is presented in terms of the common left factors of two polynomials describing the behavior of the system; the basic difference is that unlike the linear case the polynomials related to the nonlinear system belong to a *non-commutative* polynomial ring. This condition provides a bases for finding the minimal (irreducible) equivalent representation of the i/o equation which is a suitable starting point for constructing a minimal state space representation.

1 Introduction

This paper is devoted to the realization problem of the nonlinear input/output (i/o) difference equation in the state space form. In the linear case, it is well-known that a realization of a polynomial i/o equation

$$p(\delta)y = q(\delta)u$$

will be minimal (i. e. both controllable and observable) iff the i/o equation is irreducible i. e. iff the polynomials $p(\delta)$ and $q(\delta)$ have no common factors except constants. So, to obtain a minimal realization, a first task would be to reduce the i/o equation if the original equation is not irreducible. The reduction problem of the nonlinear i/o equations was studied in [1] and [2] for continuous-time and discrete-time case, respectively. In both papers the practical criterion for evaluating irreducibility was given in terms of certain subspaces of one-forms $\mathcal{H}_k$, defined by the system equations.

The purpose of this paper is to present a new necessary and sufficient condition for irreducibility of nonlinear i/o difference equation which mimics closely (extends directly) the irreducibility condition for the linear case. We show that the nonlinear system behavior can be also represented by a polynomial equation over the difference field of the nonlinear system and, a new condition is presented in terms of the common left factors of the two polynomials. The basic difference is that the polynomials related to the nonlinear system belong to a non-commutative polynomial ring. This condition provides the bases for alternative procedure for examining irreducibility of nonlinear i/o difference equations, and for finding the irreducible representation. Note that though the difference field related to the nonlinear system induces a non-commutative polynomial ring, the celebrated Euclidean algorithm for finding the greatest common left factor of two polynomials is still applicable in our case since the ring considered satisfies the Ore conditions.

Our results were inspired by the results of the paper [3] which gives the controllability condition of nonlinear i/o differential equation in terms of the common factors of the polynomials. Note that though the main results look like very similar both for the continuous-time and the discrete-time case there are substantial differences. The main difference stems from the different properties of the derivative and shift operators, which define the different multiplication rules between the derivative (or shift) operator and an element of the field and gives rise to different non-commutative polynomial rings.

2 The polynomial description of the nonlinear i/o equation

Consider a discrete-time single-input single-output nonlinear system Σ described by a high order input-output (i/o) difference equation relating the input u, the output y and a finite number of their time shifts

$$\varphi(y(t),\ldots,y(t+n),u(t),\ldots,u(t+n-1))=0 \quad (1)$$

where $u \in I\!R$, $y \in \mathcal{Y} \subset I\!R$, and φ is a real analytic function defined on $\mathcal{Y}^{n+1} \times I\!R^n$. This equation provides a natural generalization of the autoregressive moving-average representation which is very popular in linear systems theory when φ is a linear function.

In order to be able to construct the difference field related to the system, we assume that the following assumptions hold for system (1).

A1 Equation (1) can be solved uniquely (at least locally) for $y(t+n)$:

$$y(t+n)=\phi(y(t),\ldots,y(t+n-1),u(t),\ldots,u(t+n-1)) \quad (2)$$

Note that this can always be guaranteed about a regular point where $(y(t),\ldots,y(t+n),u(t),\ldots,u(t+n-1))$ is such that $\partial\varphi(\cdot)/\partial y(t+n)\neq 0$.

A2 $\varphi(0,...,0)=0$. We make this assumption because the mathematical tools we are going to employ require that instead of working with the equations themselves we work with their differentials and will not able to distinguish between the systems with $\varphi(\cdot)=0$ and $\varphi(\cdot)+const=0$. So both systems would be recognized as the same system (i.e. equivalent). This should be avoided.

A3 $\partial\varphi(\cdot)/\partial(y(t),u(t))$ is different from zero. In the case of shift-invariant systems we may assume this without loss of generality since if this is not the case, we may always apply the backward shift operator a sufficient number of times so that this condition is satisfied; and we still have the same system.

A4 The radical (i. e. perfect) difference ideal I_φ generated by equation (1) us prime. The latter means that φ cannot be decomposed (factored out) into simpler equations. In this way we avoid equations like $y(t+2)y(t)+u(t)[u(t)-y(t)-y(t+2)]=0$ which may be considered as a composition of two different systems: $y(t+2)=u(t)$ and $y(t)=u(t)$.

Let $\mathcal{K}$ denote the field of meromorphic functions in a finite number of the variables $\{y(0),\ldots,y(n-1),u(t),\ t\geq 0\}$. The forward-shift operator $\delta:\mathcal{K}\to\mathcal{K}$ is defined by shifting the arguments of the function and replacing $y(t+n)$ by the right hand side of equation (2). Under assumption A3 δ is one-to-one and the pair $(\mathcal{K},\delta)$ is a difference field. In [4] an explicit construction of the inversive closure $(\mathcal{K}^*,\delta^*)$ of $(\mathcal{K},\delta)$ is given. Hereinafter, we assume that the inversive closure $(\mathcal{K}^*,\delta^*)$ is given, and use the same symbol to denote the difference field $(\mathcal{K},\delta)$ and its inversive closure.

Over the field $\mathcal{K}$ one can define a difference vector space, $\mathcal{E}:=\mathrm{span}_{\mathcal{K}}\{\mathrm{d}\varphi \mid \varphi\in\mathcal{K}\}$. The operator $\delta:\mathcal{K}\to\mathcal{K}$ induces a forward-shift operator $\delta:\mathcal{E}\to\mathcal{E}$ by

$$\sum_i a_i \mathrm{d}\varphi_i \mapsto \sum_i (\delta a_i)\mathrm{d}(\delta\varphi_i), \quad a_i,\varphi_i\in\mathcal{K}.$$

Instead of working with equation (2), we can work with its differential

$$\begin{aligned} dy(t+n) - \sum_{i=1}^{n-1}\frac{\partial\phi}{\partial y(t+i)}dy(t+i) \\ -\sum_{j=1}^{n-1}\frac{\partial\phi}{\partial u(t+j)}du(t+j)=0 \end{aligned} \quad (3)$$

Since $dy(t+i)=\delta^i dy(t)$, $du(t+j)=\delta^j du(t)$, we can rewrite (3) as

$$p(\delta)dy(t)=q(\delta)du(t) \quad (4)$$

where

$$p(\delta)=\delta^n-\sum_{i=1}^{n-1}\frac{\partial\phi}{\partial y(t+i)}\delta^i$$

$$q(\delta)=\sum_{j=1}^{n-1}\frac{\partial\phi}{\partial u(t+j)}\delta^j$$

and $\partial\phi/\partial y(t+j)\in\mathcal{K}$, $\partial\phi/\partial u(t+j)\in\mathcal{K}$. Equation (4) presents the nonlinear system behavior in terms of the polynomials $\{p(\delta),q(\delta)\}$ over the difference field $\mathcal{K}$.

The difference field $\mathcal{K}$ and the shift operator δ induce a polynomial ring, denoted by $\mathcal{K}[\delta]$. A polynomial $p(\delta)\in\mathcal{K}[\delta]$ is written as

$$p(\delta)=p_n\delta^n+p_{n-1}\delta^{n-1}+\ldots+p_1\delta+p_0. \quad (5)$$

The degree of the polynomial in (5) is n if $p_n\neq 0$, and the polynomial $p(\delta)$ is called monic if $p_n=1$.

While all the other algebraic operations in the ring satisfy the operations in the field (of meromorphic functions) $\mathcal{K}$, the multiplication between the shift operator δ and an element $\varphi\in\mathcal{K}$ can be defined by the following rule

$$\delta\varphi=\delta\varphi\delta$$

so for example

$$(p\delta^n)(q\delta^m)=p\delta^n(q)\delta^{n+m}.$$

If the multiplication is defined in the above way, the non-commutative ring $\mathcal{K}[\delta]$ is called the *twisted polynomial ring* twisted by δ, and it is proved to satisfy the Ore condition, i. e. to be the Ore ring [5]. If the non-commutative ring satisfies the Ore condition, one can construct the division ring of fractions, a process exactly like that of constructing the field of rational numbers from the ring of integers.

In the difference field $\mathcal{K}$, there is no non-zero *zero-element* in the sense that if $\varphi_1, \varphi_2 \in \mathcal{K}$ with $\varphi_1 \neq 0$, $\varphi_2 \neq 0$ then $\varphi_1\varphi_2 \neq 0$. It follows that for three polynomials $p(\delta), p_1(\delta), p_2(\delta) \in \mathcal{K}[\delta]$, with $\deg p_1(\delta) = d_1 > 0$ and $\deg p_2(\delta) = d_2 > 0$, such that $p(\delta) = p_1(\delta)p_2(\delta)$, the degree of $p(\delta)$ satisfies

$$\deg p(\delta) = \deg p_1(\delta) + \deg p_2(\delta) = d_1 + d_2 \qquad (6)$$

where $p_1(\delta)$ is called the *left divisor* of $p(\delta)$ and $p(\delta)$ is called left divisible by $p_1(\delta)$.

If for $p_1(\delta)$, $p_2(\delta) \in \mathcal{K}[\delta]$ such that $p_c(\delta)$ is a left divisor of $p_1(\delta) - p_2(\delta)$, then $p_c(\delta)$ is called the *common left factor* of $p_1(\delta)$ and $p_2(\delta)$; $p_c(\delta)$ is called the *greatest common left factor* of $p_1(\delta)$ and $p_2(\delta)$ if $\deg p_c(\delta)$ is the greatest of all common left factors of $p_1(\delta) - p_2(\delta)$.

The celebrated Euclidean Algorithm is applicable for finding the greatest common left divisor of two polynomials. The Euclidean algorithm is based on the fact that given two polynomials, $p_1(\delta)$ and $p_2(\delta)$ with $\deg p_1(\delta) > \deg p_2(\delta)$, there exists a unique (right) quotient polynomial $\gamma_1(\delta)$ and a unique remainder polynomial $p_3(\delta)$ such that

$$p_1(\delta) = p_2(\delta)\gamma_1(\delta) + p_3(\delta), \quad \deg p_3(\delta) < \deg p_2(\delta).$$

By successive use of the above polynomial division formula we can write

$$\begin{aligned} p_2(\delta) &= p_3(\delta)\gamma_2(\delta) + p_4(\delta) \\ &\vdots \\ p_{k-2}(\delta) &= p_{k-1}(\delta)\gamma_{k-2}(\delta) + p_k(\delta) \\ p_{k-1}(\delta) &= p_k(\delta)\gamma_{k-1}(\delta). \end{aligned}$$

The algorithm stops in a finite number of steps when the remainder $p_{k+1}(\delta) = 0$ and then the greatest common left divisor (*gcld*) of $p_1(\delta)$ and $p_2(\delta)$ is $p_k(\delta)$. The *gcld* is only unique up to a constant, but it can be made unique by requiring it to be monic.

Two polynomials $\tilde{p}_1(\delta)$ and $\tilde{p}_2(\delta)$ are obtained such that

$$\begin{aligned} p_1(\delta) &= p_k(\delta)\tilde{p}_1(\delta) \\ p_2(\delta) &= p_k(\delta)\tilde{p}_2(\delta). \end{aligned}$$

If $\deg p_k(\delta) = 0$, then the polynomials $p_1(\delta)$ and $p_2(\delta)$ have no common left divisor and are called coprime (relatively prime).

3 Irreducibility of the i/o equation

The relative degree r of a one-form $\omega \in \mathcal{E}$ is defined to be the least integer such that $\delta^r\omega \notin \mathrm{span}_{\mathcal{K}}\{dy(0), \ldots, dy(n-1), du(0), \ldots, du(n-1)\}$. If such an integer does not exist, set $r = \infty$.

A sequence of subspaces $\{\mathcal{H}_k\}$ of $\mathcal{E}$ is defined for $k \geq 1$ by

$$\begin{aligned} \mathcal{H}_1 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y(0), \ldots, \mathrm{d}y(n-1), \\ &\qquad \mathrm{d}u(0), \ldots, \mathrm{d}u(n-1)\}, \\ \mathcal{H}_{k+1} &= \{\omega \in \mathcal{H}_k \mid \Delta\omega \in \mathcal{H}_k\}. \end{aligned} \qquad (7)$$

It is clear that the sequence (7) is decreasing. Denote by k^* the least integer such that

$$\mathcal{H}_1 \supset \cdots \supset \mathcal{H}_{k^*} \supset \mathcal{H}_{k^*+1} = \mathcal{H}_{k^*+2} = \cdots =: \mathcal{H}_\infty \qquad (8)$$

The subspace $\mathcal{H}_k$ contains the one-forms whose relative degree is equal to k or higher than k.

Definition 1 *A function φ_r in $\mathcal{K}$ is said to be an autonomous element for a system Σ of the form (1) if there exists and integer μ and a non-zero meromorphic function F so that*

$$F(\varphi_r, \delta\varphi_r, \ldots, \delta^\mu\varphi_r) = 0. \qquad (9)$$

Definition 2 *The system (1) is said to be irreducible (forward accessible) if there does not exist any non-zero autonomous element in $\mathcal{K}$.*

If the system is not irreducible, its behavior can be expressed as

$$\varphi = F(\varphi_r, \delta\varphi_r, \ldots, \delta^\mu\varphi_r) = 0$$

where $\varphi_r = \varphi_r(y(t), \ldots, y(t+r), u(t), \ldots, u(t+r-1))$. and $r < n$.

In [2] a practical criterion for evaluating irreducibility was given in terms of the $\mathcal{H}_k$ subspaces.

Theorem 1 *The system (2) is irreducible if and only if $\mathcal{H}_\infty = \{0\}$.*

Below we will give an alternative irreducibility condition.

Theorem 2 *The nonlinear system (2) is irreducible (forward accessible) in the sense of Definition 2 if and only if the polynomials $p(\delta)$ and $q(\delta)$ in (4) have no common left divisors.*

Proof. *Necessity.* Suppose that the nonlinear system (2) is not irreducible. According to Definition 2, there exist functions $\varphi_r \in \mathcal{K}$ and $F \in \mathcal{K}$ such that (9) holds.

We can differentiate the functions φ_r and F in (5) and use $d\delta^i y = \delta^i dy$, $d\delta^j u = \delta^j du$ and $d\delta^m F = \delta^m dF$ to obtain

$$
\begin{aligned}
d\varphi_r &= \sum_{i=0}^{k} \frac{\partial \varphi_r}{\partial y(t+i)} dy(t+i) \\
&+ \sum_{j=0}^{k-1} \frac{\partial \varphi_r}{\partial u(t+j)} du(t+j) \\
&= \tilde{p}(\delta) dy - \tilde{q}(\delta) du \\
dF &= \sum_{k=0}^{\mu} \frac{\partial F}{\partial \delta^k \varphi_r} d\delta^k \varphi_r = \rho(\delta) d\varphi_r \\
&= \rho(\delta)[\tilde{p}(\delta) dy - \tilde{q}(\delta) du] = 0.
\end{aligned} \tag{10}
$$

Since $\{dy(t), \ldots, dy(t+n-1), du(t), \ldots, du(t+n-1)\}$ is a set of independent vectors in the difference vector space, we can match the equations (10) and (4) to obtain

$$p(\delta) = \rho(\delta)\tilde{p}(\delta), \quad q(\delta) = \rho(\delta)\tilde{q}(\delta).$$

It further follows from (6) that

$$\deg p = \deg \rho + \deg \tilde{p}, \quad \deg q = \deg \rho + \deg \tilde{q}.$$

In (9), $\mu > 0$, then $\deg \rho > 0$. Hence the polynomials $p(\delta)$, $q(\delta)$ have a common left factor $\rho(\delta)$.

Sufficiency. Suppose that the polynomials $p(\delta)$, $q(\delta)$ with $\deg p(\delta) = n$ and $\deg q(\delta) \leq n-1$ have a common left factor $\rho(\delta)$, with $\deg \rho(\delta) > 0$, such that the equation (4) can be written as

$$p(\delta)dy(t) - q(\delta)du(t) = \rho(\delta)[\tilde{p}(\delta)dy(t) - \tilde{q}(\delta)du(t)] = 0$$

We obtain

$$\rho(\delta)\omega = 0.$$

Using the results of [2], one can show that $\tilde{p}(\delta)dy(t) - \tilde{q}(\delta)du(t)$ is exact one-form (or can be made exact by multiplying the integrating factor), so one can write

$$\tilde{p}(\delta)dy(t) - \tilde{q}(\delta)du(t) = d\varphi_r$$

and obtain

$$\rho(\delta)d\varphi_r = 0$$

which will imply the existence of F such that (9) holds. Hence the system is not irreducible.

4 Examples

Example 3.3 Consider the system described by the input-output difference equation

$$\varphi_0 = y(t+1) - y(t)u(t) = 0 \tag{11}$$

which can alternatively written as

$$[\delta - u(t)]dy(t) = y(t)du(t).$$

Since $\deg q(\delta) = \deg y(t) = 0$, the polynomials $\delta - u(t)$ and $y(t)$ have no common left factor and the system is irreducible.

Example 3.4 Consider the system described by the input-output difference equation

$$
\begin{aligned}
&\delta\varphi_0 - \varphi_0 = y(t+2) - y(t+1)u(t+1) \\
&+y(t)u(t) - y(t+1) = 0
\end{aligned} \tag{12}
$$

which can be alternatively written as

$$[\delta^2 - (1+u(t+1))\delta + u(t)]dy(t) = [y(t+1)\delta - y(t)]du(t)$$

Applying Euclidean algorithm one can find that under the assumption $y(t) \neq 0$

$$p(\delta) = q(\delta)\left[\frac{1}{y(t)}\delta - \frac{u(t)}{y(t)}\right]$$

which defines the system with

$$\tilde{p}(\delta) = \left[\frac{1}{y(t)}\delta - \frac{u(t)}{y(t)}\right]$$
$$\tilde{q}(\delta) = 1$$

and so, the reduced system is

$$d[y(t+1) - y(t)u(t)] = 0.$$

Hence, the system (12) is not irreducible.

Example 3.5 Consider the input-output difference equation

$$
\begin{aligned}
&\delta\varphi_0 + y(t)\varphi_0 = y(t+2) - y(t+1)u(t+1) \\
&+y(t)y(t+1) - y^2(t)u(t) = 0.
\end{aligned} \tag{13}
$$

which can be alternatively written as

$$
\begin{aligned}
&[\delta^2 + (y(t) - u(t+1))\delta + (y(t+1) \\
&-2y(t)u(t))]dy(t) = [y(t+1)\delta - y^2(t)]du(t)
\end{aligned}
$$

Applying Euclidean algorithm one can find that $p(\delta)$ and $q(\delta)$ have no common left factors since

$$p(\delta) = q(\delta)\left[\frac{1}{y(t)}\delta - \frac{u(t)}{y(t)}\right] + [y(t+1) - y(t)u(t)].$$

Hence, the system (13) is irreducible.

Example 3.6 Consider the systemdescribed by the input-output difference equation

$$
\begin{aligned}
&\delta\varphi_0 + u(t+1)\varphi_0 \\
&= y(t+2) - y(t)u(t)u(t+1) = 0.
\end{aligned} \tag{14}
$$

which can be alternatively written as

$$[\delta^2 - u(t)u(t+1)]dy(t) = [y(t)u(t)\delta + y(t)u(t+1)]du(t)$$

Applying Euclidean algorithm one can find that

$$p(\delta) = q(\delta)\left[\frac{u(t)}{y(t+1)}\delta - \frac{u(t)}{y(t)}\right]$$

which defines

$$\tilde{p}(\delta) = \frac{u(t)}{y(t+1)}\delta - \frac{u(t)}{y(t)}$$
$$\tilde{q}(\delta) = 1$$

and so, the reduced system is

$$d\left[\frac{y(t+1)}{y(t)u(t)}\right] = 0.$$

5 Symbolic implementation using Mathematica

The reduction procedure that bases on the $\mathcal{H}_k$ subspaces has been implemented in *Mathematica* and tested on numerous examples [6]. Though it is straightforward to test irreducibility for the examples of the form (2) of medium complexity, one encounters the main difficulties in finding the irreducible form. The latter includes, in general, finding the integrating factor and integration the (integrable in principle) differential form. Note that in *Mathematica* almost no facilities are available for integrating the differential forms. Another difficulty is related to the computation of the backward shift which reduces to finding the solution of the system of nonlinear equations.

A reduction procedure that bases on polynomial approach has been also implemented in *Mathematica* and seem to be somewhat easier to apply. A main difference is that within a polynomial approach one can complete the reduction in one step whereas this is not so within the subspace approach. Moreover, the polynomial approach requires less computations and therefore the expressions do not become so huge. The problems related to finding the backward shift and the integrating factor, still exist. Backward shift is necessary in order to shift back the coefficients of the polynomials in a proper way when applying the Euclidean algorithm and sometimes, like in Example 3.4, one has to multiply

$$\tilde{p}(\delta)\mathrm{d}y(t) - \tilde{q}(\delta)\mathrm{d}u(t)$$

by an integrationg factor to make it an exact one-form.

6 Conclusion

This paper presents a new condition for irreducibility of nonlinear i/o difference equation which is an important subproblem in the realization of the i/o equation in the classical state-space form. The condition is formulated in terms of the common left factor of the system polynomials which are defined over the difference field and belong to a non-commutative polynomial ring. On the bases of the above condition an effective procedure is suggested to examine the system irreducibility and, if necessary, to transform the system into the irreducible form. The proposed condition and procedure are consistent with those for the linear system.

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