An LMI-based approach for characterizing the solution set of polynomial systems

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Abstract

This paper considers the problem of solving certain classes of polynomial systems. This is a well known problem in control system analysis and design. A novel approach is developed as a possible alternative to the commonly employed algebraic geometry and homotopy methods. The first result of the paper shows that the solution set of the polynomial system belongs to the kernel of a symmetric matrix. Such a matrix is obtained via the solution of a suitable Linear Matrix Inequality (LMI) involving the maximization of the minimum eigenvalue of an affine family of symmetric matrices. The second result concerns the computation of the solutions from the kernel of the obtained matrix. In particular, it is shown that the solutions can be recovered quite easily if the dimension of the kernel is smaller than the degree of the polynomial system. Finally, some application examples are illustrated to show the features of the approach and to make a brief comparison with the algebraic geometry techniques.

1 Introduction

Polynomial system solving is a key problem in a large number of system analysis and control problems. It is well known that, while Euler-Newton iterative methods work quite satisfactorily when local solutions are of interest, no efficient method is available for computing all the solutions, except for cases when some information on the solution structure is known a-priori, as for the celebrated algebraic Riccati equation.

For general polynomial systems, algebraic geometry and homotopy methods have been often used in several problems. To name but a few, algebraic geometry methods have been employed for output feedback stabilization [1], equilibria location of nonlinear systems [2], multidimensional system analysis [3], robustness analysis of control systems [4, 5], while homotopy methods have been introduced for solving modified algebraic Riccati equations related to the synthesis of robust low order compensators [6, 7]. Algebraic geometry methods, which are based on elimination theory and Tarksi's decision theorem [8, 9, 10], suffer the problem of spurious solutions in the resultant scalar polynomial, thus leading to a significant inflation degree and making them effective for low dimensional systems only. On the other hand, homotopy methods are based on continuation techniques and are able to provide with probability one all the solutions of small polynomial systems as fixed points of suitable nonlinear maps [11].

In this paper a different approach for determining the solution set of polynomial systems is introduced. This approach is based on some properties of homogeneous forms [12] and powerful convex optimization techniques for optimization problems in the form of Linear Matrix Inequalities (LMIs) [13, 14, 15]. More specifically, it is first shown how to characterize the solution set by means of an affinely parameterized quadratic homogeneous form in suitable transformed variables. In particular, it turns out that, for those parameter vectors corresponding to a positive definite form, the solutions of the polynomial system belong to the kernel of the symmetric matrix associated to the form. It is successively shown that the parameter vector corresponding to the kernel with the smallest dimension can be obtained via the solution of an LMI consisting in the maximization of the minimum eigenvalue of a symmetric matrix. Finally, it is shown how to compute the solution set from the obtained kernel. In particular, it turns out that the relationship between the degree of the polynomial system and the dimension of the smallest kernel is important for an easy computation of the solutions. Also, some examples illustrating the main features of the approach and providing a preliminary comparison with the polynomial resultants method are reported.

The paper is organized as follows. In Section 2 the polynomial system problem is formulated and an equivalent representation of the solution set by means of an affinely parameterized quadratic homogeneous form is introduced. Section 3 is devoted to the problem of computing the matrix of such form with the smallest kernel, via a suitable LMI. Section 4 shows how the solutions of the polynomial system can be computed from the obtained kernel. Section 5 provides some illustrative examples. Some concluding comments end the paper in Section 6. Proofs of the results in the paper are reported in [16].

Notation. E_n: $n \times n$ identity matrix; $A > 0$ ($A \geq 0$): positive definite (semidefinite) matrix; A': transpose of A; $A_{i,j}$: entry (i,j) of A; $\mathrm{Ker}(A)$: kernel of A; $\mathrm{Im}(A)$: linear space generated by the columns of A; $\emptyset$: empty set; $\operatorname*{Im}_{x} f(x)$: image of the function $f(x)$; $\lambda_M(A)$ ($\lambda_m(A)$): maximum (minimum) real eigenvalue of A; $\partial\mathcal{A}$: boundary of the set $\mathcal{A}$.

2 Problem formulation and preliminaries

Let us define the system of polynomial equations S_x as follows

$$S_x : \left\{ \begin{array}{rcl} p_1(x) & = & 0 \\ & \vdots & \\ p_k(x) & = & 0 \end{array} \right. \tag{1}$$

where $x = (x_1, \ldots, x_n)' \in \mathbb{R}^n$ and $p_i(\cdot) : \mathbb{R}^n \to \mathbb{R}$, $1 \leq i \leq k$, is a polynomial in x of degree less or equal to m. The solution set $\mathcal{X}$ of system S_x is defined as

$$\mathcal{X} = \{x \in \mathbb{R}^n : \ p_i(x) = 0, \ \ 1 \leq i \leq k\}.$$

Throughout the paper, we assume that $\mathcal{X}$ contains a finite number of solutions.
In the following, we provide a useful alternative representation of the solution set $\mathcal{X}$. Let us write any polynomial $p_i(x)$ as

$$p_i(x) = \sum_{j=0}^{m} h_{i,j}(x)$$

where $h_{i,j}(x)$ are homogeneous forms in x of degree j. Let us introduce an auxiliary variable x_{n+1} and let $y \in \mathbb{R}^{n+1}$ be defined as

$$y = (x', x_{n+1})'. \tag{2}$$

Consider the related polynomials

$$q_i(y) = \sum_{j=0}^{m} h_{i,j}(x) x_{n+1}^{m-j}, \quad i = 1, \cdots, k.$$

By construction, $q_i(y)$ is a homogeneous form in y of degree m. Moreover, the polynomial system

$$S_y : \left\{ \begin{array}{rcl} q_1(y) & = & 0 \\ & \vdots & \\ q_k(y) & = & 0 \end{array} \right. \tag{3}$$

is equivalent to system S_x if y_{n+1} is set to 1. In fact, the following result holds.

Lemma 1 *Let $\mathcal{Y}$ be the set of nontrivial solutions of system S_y*

$$\mathcal{Y} = \left\{y \in \mathbb{R}^{n+1}, y \neq 0 : \ q_i(y) = 0, \ \ 1 \leq i \leq k\right\}.$$

Then,

$$\mathcal{Y} = \{y : \ y = a(x', 1)' \ \ \forall a \in \mathbb{R}, a \neq 0, \ \forall x \in \mathcal{X}\}. \ \blacksquare$$

From Lemma 1, it is straightforward to verify that

$$\mathcal{X} = \{x = (y_1, \ldots, y_n)' \in \mathbb{R}^n : \ y \in \mathcal{Y} \ \text{ and } \ y_{n+1} = 1\}. \tag{4}$$

Now, let us define the homogeneous form $q(y)$ as follows

$$q(y) = \sum_{i=1}^{k} q_i^2(y). \tag{5}$$

It is obvious that, for $y \neq 0$

$$q(y) = 0 \iff y \in \mathcal{Y}. \tag{6}$$

Before we characterize the set $\mathcal{Y}$ according to (5)-(6), let us introduce some useful notation regarding homogeneous forms. Let $x^{\{m\}}$ denote the base vector for a homogeneous form of degree m, in $x \in \mathbb{R}^n$. For example, if $n = 2$ and $m = 3$ one has $x^{\{3\}} = [x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3]'$. The dimension of vector $x^{\{m\}}$ is denoted by $\sigma(n,m)$ and is given by

$$\sigma(n,m) = \left(\begin{array}{c} n+m-1 \\ n-1 \end{array} \right).$$

Hence, if $y^{\{m\}}$ is the base vector for the homogeneous form $q(y)$ in (5), there exists a symmetric matrix $M_0 \in \mathbb{R}^{d \times d}$, being $d = \sigma(n+1, m)$, such that

$$q(y) = y^{\{m\}\prime} M_0 y^{\{m\}}.$$

It turns out that M_0 is not unique. Indeed, let us define the set $\mathcal{L}$ as follows

$$\mathcal{L} = \left\{ L = L' \in \mathbb{R}^{d \times d} : \ y^{\{m\}\prime} L y^{\{m\}} = 0 \ \ \forall y \in \mathbb{R}^{n+1} \right\}.$$

It can be verified that $\mathcal{L}$ enjoys the following property.

Lemma 2 *The set $\mathcal{L}$ is a linear space of dimension d_α, where*

$$d_\alpha = \frac{d(d+1)}{2} - \sigma(n+1, 2m). \qquad \blacksquare$$

Therefore, $\mathcal{L}$ admits a linear parameterization. Thus, we can define the map $L(\cdot) : \mathbb{R}^{d_\alpha} \to \mathbb{R}^{d \times d}$ as follows (see [12] for details):

1. $L(\alpha)$ is a linear function of α;
2. $\operatorname*{Im}_{\alpha \in \mathbb{R}^{d_\alpha}} L(\alpha) = \mathcal{L}$.

Moreover, let us introduce the affine function $M(\cdot) : \mathbb{R}^{d_\alpha} \to \mathbb{R}^{d \times d}$, defined as $M(\alpha) = M_0 + L(\alpha)$. Then, we have that the Square Matrix Representation (SMR) of the homogeneous form $q(y)$, and hence of the system S_y, is given by

$$q(y) = y^{\{m\}\prime} M(\alpha) y^{\{m\}} = 0, \ \ \forall \alpha \in \mathbb{R}^{d_\alpha}. \tag{7}$$

In the next sections, we will exploit the SMR to introduce a new approach to the characterization of the solution set $\mathcal{Y}$, based on convex optimization.

3 Characterization of the solution set $\mathcal{Y}$

From (6) and (7) it follows that, in order to find the solution set of the equation system S_y, we have to perform the following tasks:

1. for a given α, compute the set
$$\mathcal{V}(\alpha) = \left\{ v \in \mathbb{R}^d : v' M(\alpha) v = 0 \right\}; \tag{8}$$
2. calculate the solution set $\mathcal{Y}$ as
$$\mathcal{Y} = \left\{ y \in \mathbb{R}^n, y \neq 0 : \; y^{\{m\}} \in \mathcal{V}(\alpha) \right\}. \tag{9}$$

It is well known that $\mathcal{V}(\alpha)$ may be a very intricate set depending on the eigenstructure of $M(\alpha)$. Hence, in order to simplify the characterization of $\mathcal{Y}$, we have to choose an appropriate matrix $M(\alpha)$. Said another way, we can exploit the degree of freedom in the choice of α, to obtain a useful form for $\mathcal{V}(\alpha)$ in (8).

Consider the set $\mathcal{A} = \left\{ \alpha \in \mathbb{R}^{d_\alpha} : M(\alpha) \geq 0 \right\}$. We are ready to state the following result.

Lemma 3 *Let $\alpha \in \mathcal{A}$ and $y \in \mathcal{Y}$. Then, $y^{\{m\}} \in$ $\mathrm{Ker}(M(\alpha))$.* ■

Lemma 3 means that, for any $\alpha \in \mathcal{A}$, $\mathcal{V}(\alpha)$ is a linear space, specifically the kernel of $M(\alpha)$. Observe that the dimension of $\mathcal{V}(\alpha)$ is not constant in $\mathcal{A}$. Hence, we can greatly simplify the characterization of $\mathcal{Y}$ according to Lemma 3, by selecting $M(\alpha)$ so that $\alpha \in \mathcal{A}$ and the dimension of $\mathrm{Ker}(M(\alpha))$ is as small as possible. Indeed, let us define the sets

$$\mathcal{A}_i = \{ \alpha \in \mathcal{A} : \; \dim[\mathrm{Ker}(M(\alpha))] = i \}$$

with the assumption $\dim(\emptyset) = -1$. We have the following result.

Lemma 4 *Let i, j be such that $i_m \leq i < j \leq i_M$ where*

$$\begin{aligned} i_m &= \min_{\alpha \in \mathcal{A}} \dim[\mathrm{Ker}(M(\alpha))], \\ i_M &= \max_{\alpha \in \mathcal{A}} \dim[\mathrm{Ker}(M(\alpha))]. \end{aligned}$$

Then, $\dim(\mathcal{A}_i) > \dim(\mathcal{A}_j)$ *and* $\mathcal{A}_j \subseteq \partial \mathcal{A}_i$. ■

Lemmas 3 and 4 suggest a strategy for the selection of a proper α. Due to Lemma 3, we have that α must belong to set $\mathcal{A}$. Moreover, since as already mentioned the characterization of $\mathcal{Y}$ is simplified if the dimension of $\mathrm{Ker}(M(\alpha))$ is small, the best choice would be to pick an α from $\mathcal{A}_{i_m}$, according to Lemma 4. Let us point out that such a vector is given by the following optimization problem

$$\alpha^* = \arg\max_{\alpha \in \mathcal{A}} \mathrm{rank}(M(\alpha)). \tag{10}$$

Notice that in general α^* is not unique. The next result is useful for tackling problem (10).

Theorem 1 *Let λ^* be defined by the following convex optimization problem*

$$\lambda^* = \max_{\alpha \in \mathbb{R}^{d_\alpha}} \lambda_m(M(\alpha)). \tag{11}$$

Then, $\lambda^* \geq 0$. Moreover:

1. *if $\mathcal{Y} \neq \emptyset$, then $\lambda^* = 0$;*
2. *if $\lambda^* > 0$, then $\mathcal{Y} = \emptyset$.* ■

Suppose $\mathcal{Y} \neq \emptyset$ and let us start the maximization procedure (11) from a vector $\alpha_0 \notin \mathcal{A}$. Since $\lambda^* = 0$, we have that the maximum is achieved for an $\alpha \in \mathcal{A}$. Moreover, from Lemma 4 one can argue that (except for degenerate cases) the optimizing α in (11) will belong to $\mathcal{A}_{i_m}$. Hence, it is a natural choice to select α^* as the α solving the convex optimization problem (11). Notice that problem (11) can be written as follows

$$\begin{array}{rc} \lambda^* = & \displaystyle\max_{t \in \mathbb{R}, \alpha \in \mathbb{R}^{d_\alpha}} t \\ \text{subject to} & M(\alpha) - tE_d \geq 0, \end{array} \tag{12}$$

which is a maximization of a linear function constrained by a Linear Matrix Inequality (LMI). This can be solved very efficiently from a computational point of view [13, 14, 15].

Remark 1 Notice that the role played by the LMI optimization problem (12) is twofold: first, it provides an α^* such that the matrix $M(\alpha^*)$ is positive semidefinite (which is guaranteed by construction of problem (11)); second, it selects an α^* such that the dimension of $\mathrm{Ker}(M(\alpha^*))$ is minimal. A small dimension of $\mathrm{Ker}(M(\alpha^*))$ will turn out to be useful in the computation of the solutions, as it will be shown in the sequel.

4 Computation of the solution set

Let us see how to compute the solution set of the equation system S_y from (8)-(9), with $\mathcal{V}(\alpha) = \mathcal{V}(\alpha^*)$. In order to highlight the connection with the solutions of S_x, stressed in (4), we use the notation $y = (x_1, \ldots, x_n, x_{n+1})'$, introduced in (2). Recall from (4) that we are interested in the solutions satisfying $x_{n+1} = 1$.

Let $\mathcal{K}^*$ be the kernel of $M(\alpha^*)$ and i^* the dimension of $\mathcal{K}^*$. We have to consider two cases separately

- I. $i^* \leq m+1$;
- II. $i^* > m+1$.

CASE **I:** $i^* \leq m+1$

In this case, it is always possible to find the solutions of S_y by computing the roots of a monovariable polynomial of degree not greater than $m+1$. In fact, let us write

$$\mathcal{K}^* = \mathrm{Im}(V) \tag{13}$$

where $V = [v_1, \ldots, v_{i^*}] \in \mathbb{R}^{d \times i^*}$ and v_i, $1 \leq i \leq i^*$, are linearly independent. Any $v \in \mathcal{K}^*$ can be written as

$$v = \sum_{i=1}^{i^*} \mu_i v_i \tag{14}$$

where $\mu \in \mathbb{R}^{i^*}$ is a suitable parameter. Now, recall from Lemma 3 that each solution y of the equation system

S_y must satisfy $y^{\{m\}} \in \mathcal{K}^*$, which means that $y^{\{m\}}$ can be expressed as v in (14). Let us consider the rows $r_1, \ldots, r_{i^*}$ of V corresponding to the monomials x_{n+1}^m, $x_j x_{n+1}^{m-1}$, …, $x_j^{i^*-1} x_{n+1}^{m+1-i^*}$ in $y^{\{m\}}$, for a chosen j such that $1 \leq j \leq n$. By means of pivot operations, it is generally possible to obtain a new base for $\mathcal{K}^*$, such that the rows $\tilde{r}_i$, $1 \leq i \leq i^*$, of the new matrix $\tilde{V}$ satisfy

$$\begin{pmatrix} \tilde{r}_{i^*} \\ \vdots \\ \tilde{r}_1 \end{pmatrix} = E_{i^*} \tag{15}$$

(cases in which it is not possible to reach the form (15) will be treated later). In this new base, every $v \in \mathcal{K}^*$ can be written as a linear combination of the new vectors $\tilde{v}_i$, and consequently, weighted by a new vector $\tilde{\mu}$. Taking into account that $x_{n+1} = 1$, it is straightforward to verify that $\mu_i = x_j^{i^*-i}$, $1 \leq i \leq i^*$, and hence that (14) can be rewritten as

$$v = \sum_{i=1}^{i^*} x_j^{i^*-i} \tilde{v}_i. \tag{16}$$

This allows us to write any vector belonging to $\mathcal{K}^*$ as a polynomial function of the variable x_j only. In this way, using one relation on the monomials of $y^{\{m\}}$ given by the rows of $\tilde{V}$ not used yet, we can write a polynomial equation in the variable x_j that all solutions of system S_x must satisfy. In order to obtain an equation with the lowest degree, we have just to consider the rows r_a and r_b corresponding, respectively, to monomials x_k and $x_j x_k$, for any $k \neq j$. Then, we find for x_j the following equation of degree not greater than i^*

$$x_j \left(\sum_{i=1}^{i^*} x_j^{i^*-i} \tilde{V}_{a,i} \right) = \sum_{i=1}^{i^*} x_j^{i^*-i} \tilde{V}_{b,i}.$$

Once x_j has been found, the other variables can be read on the vector v in (16), in the elements corresponding to monomials x_h, $h \neq j$, in $y^{\{m\}}$. This procedure provides all the solutions of system S_x but, in principle, it may also introduce spurious solutions. The feasibility of the solution candidates can be checked by building the vector $\hat{y}^{\{m\}}$ for every candidate $\hat{y}$ and testing whether $\hat{y}^{\{m\}}$ belongs to $\mathcal{K}^*$ or not, in accordance with Lemma 3.

If it is not possible to reach the form (15), this means that the solutions of system S_x can be found just involving a smaller number of vectors $\tilde{v}_i$ and, consequently, by solving an equation of lower degree. In fact, let us start the pivot procedure from the row r_1 (corresponding to $x_{n+1}^m = 1$). If we cannot operate the pivot for the monomial x_j^k after the pivot for $1, \ldots, x_j^{k-1}$, this means that the actual row $\tilde{r}_{k+1}$ has zeros on the columns $1, \ldots, i^* - k$, that is on the columns not considered yet. Thus, we can write the following equation in the variable x_j

$$x_j^k = \sum_{i=0}^{k-1} \tilde{V}_{i+1, i^*-i} x_j^i \tag{17}$$

where $\tilde{V}$ is the matrix obtained after the last pivot step. Once x_j has been found, we can reduce $\tilde{V}$ to a smaller matrix $\tilde{V}_r$, because the rows corresponding to the monomials x_j^i, $i = k, \ldots, m$ and the columns of index $i^* - k + 1, \ldots, i^*$ are known. Note that such reduced matrix corresponds to a homogeneous form of degree m in $n-1$ variables. Hence, we are still in case I (see Example 2 in Section 5).

CASE **II:** $i^* > m + 1$

In this case, the kernel dimension is greater than the maximum number of monomials containing one only variable. This means that we are able to write an equation in one variable only if the pivot procedure described for the previous case stops at monomial x_j^k with $k \leq m$, that is if we cannot operate the pivot for a such monomial. Hence, there are cases for which the procedure described for case I does not lead to a straightforward characterization of the solution set $\mathcal{Y}$. However, we formulate the following conjecture.

Conjecture 1 *If $i^* > m+1$, it is always possible to increase the degree of the system to $\tilde{m} = m + \delta m$, without adding new solutions to the solution set of the original system, so that for the new system the dimension $\tilde{i}^*$ of the kernel $\mathcal{K}^*$ satisfies $\tilde{i}^* \leq \tilde{m}$.* ■

Hence, it is always possible to increase the degree of the system, and then apply the procedure described for case I to the new system. Since no new solution has been introduced, one can characterize the solution set $\mathcal{Y}$ by solving an equation in one variable, of degree not greater than the degree of the system (see Example 3 in Section 5).

5 Examples

In this section we present some examples in which our approach is applied to the solution of polynomial systems. Results are compared to those given by the polynomial resultants method, which is the most popular approach among algebraic geometry methods (see e.g. [1, 9, 8]).

Example 1. Let us consider the polynomial system

$$\begin{cases} (x_1 - 1)(x_1^2 + x_2^2 + 1) = 0 \\ (x_2 - 1)(x_2^2 + 2x_2 x_3 + x_3^2 + 2) = 0 \\ (x_3 - 1)(x_1^2 + 2x_3^2 + 1) = 0 \end{cases}$$

This system has been constructed so that the unique solution corresponds to $(x_1, x_2, x_3) = (1, 1, 1)$. In this case, $n = 3$, $m = 3$. Hence, $y \in \mathbb{R}^4$ (with $x_4 = 1$) and the dimension of $y^{\{m\}}$ in (2) is $d = 20$. We use the MATLAB LMI Control Toolbox [15] to solve the optimization problem (12) and we obtain a kernel $\mathcal{K}^* =$

Ker$(M(\alpha^*))$ of dimension $i^* = 1$. The base matrix V in (13) is given by

$$V = [v_1] = -0.2236 \cdot \underline{1}_{20}$$

where $\underline{1}_h \in \mathbb{R}^h$ is a vector whose entries are all equal to 1. Since $i^* \leq 4$ we are in case I. Now, recalling that the vector $y^{\{m\}}$, with $x_4 = 1$, takes on the form

$$y^{\{m\}} = \left[x_1^3, x_1^2x_2, x_1^2x_3, x_1^2, x_1x_2^2, x_1x_2x_3, x_1x_2, \ldots, 1\right]'$$

the procedure described in Section 4 just requires to scale vector v_1 in order to obtain 1 on the last entry. Then, one can read directly the solution $(x_1, x_2, x_3) = (1, 1, 1)$ in the entries of v_1 corresponding to x_1, x_2, x_3 respectively.

Let us consider now the polynomial resultants method applied to this case. We obtain a polynomial equation in the variable x_3 of degree 27. This gives 22 complex solutions and 5 real solutions, the closest to 1 being $x_3 = 1.032$. Similar results are obtained by solving with respect to x_1 or x_2.

Example 2. Let us consider the polynomial system

$$\begin{cases} (x_1 - 1)^3(x_2 - 1) = 0, \\ (x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2)(x_2 - \pi)^2 = 0. \end{cases}$$

It is easy to see that the solutions of this system are $(1, 1)$ and $(1, \pi)$. Here, $n = 2$, $m = 4$ and $d = 15$. The obtained kernel $\mathcal{K}^*$ has dimension $i^* = 3$. Recall that if (x_1, x_2) is a solution of the system S_x, then $y^{\{4\}} = (x_1, x_2, 1)'^{\{4\}}$ must belong to $\mathcal{K}^*$ and, hence, there exists $\mu \in \mathbb{R}^3$ such that $V\mu = y^{\{4\}}$. Since $i^* \leq 5$, we are in case I. Now, let us start the pivot procedure and perform just the first step. Matrix V is transformed into the following matrix

$$\tilde{V} = \left(\begin{array}{ccc}
0.3434 & 1.0846 & 1.7799 \\
-0.0209 & 0.0141 & 1.0131 \\
-0.0001 & 0.0000 & 1.0002 \\
-0.0860 & 0.0580 & 1.0528 \\
-0.0208 & 0.0140 & 1.0128 \\
-0.0000 & 0.0000 & 1.0000 \\
-0.2902 & 0.1960 & 1.1778 \\
-0.0858 & 0.0579 & 1.0526 \\
-0.0207 & 0.0140 & 1.0127 \\
0.0000 & -0.0000 & 1.0000 \\
-0.9330 & 0.6304 & 1.5706 \\
-0.2904 & 0.1962 & 1.1777 \\
-0.0858 & 0.0580 & 1.0526 \\
-0.0207 & 0.0140 & 1.0127 \\
0 & 0 & 1.0000
\end{array}\right)
\begin{array}{cc}
\rightarrow & \text{row corresp. to } x_1^4 \\
\rightarrow & x_1^3x_2 \\
\rightarrow & x_1^3 \\
\rightarrow & x_1^2x_2^2 \\
\rightarrow & x_1^2x_2 \\
\rightarrow & x_1^2 \\
\rightarrow & x_1x_2^3 \\
\rightarrow & x_1x_2^2 \\
\rightarrow & x_1x_2 \\
\rightarrow & x_1 \\
\rightarrow & x_2^4 \\
\rightarrow & x_2^3 \\
\rightarrow & x_2^2 \\
\rightarrow & x_2 \\
\rightarrow & 1
\end{array}$$

It obvious that μ satisfies $\tilde{V}\mu = y^{\{4\}}$ if and only if $\mu_3 = 1$. Now, if we look at row 10 (corresponding to the monomial x_1), we see that the first two entries of this row are zero. Hence, from $[0, 0, 1]\mu = x_1$ we can conclude that $x_1 = 1$ is the only solution in the variable x_1. Since we already know x_1, we can reduce matrix $\tilde{V}$ by eliminating the rows containing x_1 and keeping just the last five rows. Completing the pivot with respect to x_2, we obtain the final matrix

$$\tilde{V}_r = \left(\begin{array}{ccc}
0.0000 & 45.023 & -44.024 \\
0.0000 & 14.012 & -13.012 \\
0.0000 & 4.1417 & -3.1417 \\
0 & 1.0000 & 0 \\
0 & 0 & 1.0000
\end{array}\right)
\begin{array}{cc}
\rightarrow & x_2^4 \\
\rightarrow & x_2^3 \\
\rightarrow & x_2^2 \\
\rightarrow & x_2 \\
\rightarrow & 1
\end{array}$$

As we can see from the third row (corresponding to x_2^2), we cannot terminate the procedure being zero the first entry of this row. Hence, according to (17), we write the following equation for x_2 (notice that $\mu_2 = x_2$)

$$x_2^2 = 4.1417x_2 - 3.1417$$

which gives $x_2 = 1$ and $x_2 = 3.1417$.

Use of the polynomial resultants method for this case, provides polynomial equations of degree 14 in x_1 or x_2, whose roots are all complex and quite far from the true solutions. This is not surprising, because it is well known that algebraic geometry methods introduce many spurious solutions, and suffer from the presence of multiple solutions.

Example 3. Let us consider the system

$$\begin{cases} (x_1 - 1)(x_2 + 1) = 0, \\ (x_1 + x_2 - 3)(x_1 - 2x_2 - 7) = 0. \end{cases} \tag{18}$$

Here: $n = 2$, $m = 2$, $d = 6$, but from the optimization (12) we get $i^* = 4$. Therefore, we are in case II. Moreover, the pivoting procedure described in section 4 does not stop at any monomial of type x_j^k with $k \leq m$, and hence we do not obtain any equation in one only variable.

Now, according to Conjecture 1, let us consider the following system

$$\begin{cases} (x_1 - 1)(x_2 + 1)(x_1^2 + x_2^2 + 1) = 0, \\ (x_1 + x_2 - 3)(x_1 - 2x_2 - 7)(x_1^2 + x_2^2 + 1) = 0. \end{cases}$$

Notice that no new solution has been introduced with respect to (18), but the degree has been increased to $\tilde{m} = 4$. From LMI optimization we obtain $\tilde{i}^* = 4$ and hence the new system satisfies the condition of case I. By applying the pivoting procedure with respect to the variable x_1, one obtains the basis $\hat{V} = [\hat{v}_1, \hat{v}_2, \hat{v}_3, \hat{v}_4]$ and the equation

$$x_1^3 = 10x_1^2 - 29x_1 + 20.$$

Three solutions are found, namely $x_1 = 1$, $x_1 = 4$ and $x_1 = 5$. Then, we can consider the reduced matrix $V_r(x_1) = [\hat{v}_1, x_1^2\hat{v}_2 + x_1\hat{v}_3 + \hat{v}_4]$ for each value obtained for x_1. When $x_1 = 1$, further pivoting provides the equation $x_2^2 = -x_2 + 6$, leading to the solutions $x_2 = 2$ and $x_2 = -3$. The same can be done for $x_1 = 4$ (obtaining $x_2 = -1$) and for $x_1 = 5$ ($x_2 = -1$). Hence, the solutions of the initial system are $(x_1, x_2) = (1, 2)$, $(x_1, x_2) = (1, -3)$, $(x_1, x_2) = (4, -1)$ and $(x_1, x_2) = (5, -1)$. Notice that the four solutions of the system have been found by solving an equation of degree 3. This example clearly shows that the presence of coincident solutions can be easily handled by the proposed technique.

Example 4. This example is taken from [2]. Let us consider the bilinear system of differential equations

$$\begin{cases} \dot{x}_1 &= a_1x_1 + a_3x_2 + b_6x_1^2 + b_7x_1x_2 \\ \dot{x}_2 &= a_2x_1 + a_4x_2 + a_6x_3 + b_8x_1x_2 \\ &\quad +b_9x_1x_3 + b_{12}x_1 \\ \dot{x}_3 &= a_5x_2 + a_7x_3 + b_{10}x_1x_3 + b_{15}x_1 \end{cases} \tag{19}$$

where $a \in \mathbb{R}^7$ and $b \in \mathbb{R}^{15}$ are given by

$$\begin{array}{rcl} a & = & (-1.02e-2, 2.98e-3, 10.66e-2, -3.29e-2, \\ & & 3.12e-2, 26.7, -27.49)' \\ b & = & (-0.768, 8.8e-3, -9.2e-2, 9.2e-2, -0.768, \\ & & -8.8e-3, 1.47e-3, -1.47e-3, 23.04, -22.27, \\ & & 5.2e-2, -1.52e-2, 0.38, 2.5e-2, -0.61)' \end{array}$$

In order to compute the equilibrium points of (19), we must solve the quadratic system

$$\left\{ \begin{array}{l} a_1x_1 + a_3x_2 + b_6x_1^2 + b_7x_1x_2 = 0 \\ a_2x_1 + a_4x_2 + a_6x_3 + b_8x_1x_2 + b_9x_1x_3 + b_{12}x_1 = 0 \\ a_5x_2 + a_7x_3 + b_{10}x_1x_3 + b_{15}x_1 = 0 \end{array} \right. \tag{20}$$

By applying the technique introduced, one finds that $\mathcal{K}^*$ has dimension $i^* = 5$ and therefore we are in case II. Nevertheless, after two steps of the pivoting procedure with respect to x_3, the matrix $\tilde{V}$ turns out to be

$$\tilde{V} = \left(\begin{array}{ccccc} -0.000 & -0.000 & 0.000 & 3.134 & -0.002 \\ -0.000 & -0.001 & 0.001 & -0.062 & -0.018 \\ -0.000 & -0.000 & 0.000 & -1.160 & -0.000 \\ -0.000 & -0.000 & -0.000 & -2.706 & 0.000 \\ 32.70 & 664.9 & 746.4 & -18.68 & -155.3 \\ 9.981 & -0.749 & 0.220 & 0.796 & 4.802 \\ 0.000 & 0.000 & -0.000 & 0.000 & 0.000 \\ -0.008 & -0.075 & 0.068 & -0.047 & -0.147 \\ 0 & 0 & 0 & 1.000 & 0 \\ 0 & 0 & 0 & 0 & 1.000 \end{array} \right) \begin{array}{cc} \rightarrow & x_1^2 \\ \rightarrow & x_1x_2 \\ \rightarrow & x_1x_3 \\ \rightarrow & x_1 \\ \rightarrow & x_2^2 \\ \rightarrow & x_2x_3 \\ \rightarrow & x_2 \\ \rightarrow & x_3^2 \\ \rightarrow & x_3 \\ \rightarrow & 1 \end{array}$$

From rows 3, 4 and 7, we get the equations

$$\left\{ \begin{array}{rcl} x_1x_3 & = & -1.1603x_3 \\ x_1 & = & -2.7055x_3 \\ x_2 & = & 0.0001x_3 \end{array} \right.$$

that immediately provide the solutions $(x_1, x_2, x_3) = (0, 0, 0)$ and $(x_1, x_2, x_3) = (-1.1603, 1.2e-4, 0.4289)$. Hence, in this case it is possible to find the solutions directly, after suitable pivoting, without increasing the degree of the system, even if we are in case II.

6 Conclusions

In this paper a novel approach for determining the solution set of polynomial systems is developed. First, an equivalent representation of the solution set is introduced by means of a quadratic homogeneous form, defined by a symmetric matrix with entries depending affinely on a parameter vector. It is shown that all the solutions of the polynomial system belong to the kernel of such a symmetric matrix. Moreover, it is pointed out that the matrix with the smallest kernel can be computed via the solution of a suitable LMI. Successively, the problem of computing the solutions from the obtained kernel is addressed. Here, the main result is that the computation is quite straightforward if the dimension of the smallest kernel is less than the degree of the polynomial system.

Based on these results and the analysis of several application examples, it appears that the proposed approach can be considered an appealing alternative to algebraic geometry and homotopy methods. In particular, it is expected that the method developed here performs much better, especially when the number of the solutions is small with respect to the degree of the polynomial system.

Another interesting conjecture concerning the proposed method is that it always provides a polynomial equation in one variable, whose degree is equal to the number of distinct solutions for that variable (i.e., spurious solutions are never introduced). This has never been contradicted by all the examples worked out. Further investigations in this respect will be the subject of future research.

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