Iterative Feedback Tuning of Controllers for a Two-Mass Spring System with Friction

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Abstract

In this paper, we present a two-degree-of-freedom controller tuning for two-mass spring systems with friction based on the IFT (Iterative Feedback Tuning) approach. While two-mass spring systems are widely used, they sometimes have severe friction. In such a case, the existing IFT methods may not work very well because they heavily rely on the linearity of the plants. In order to cope with such cases, we adopt two strategies. One is the separate tuning of the feedback and feedforward controller. The other is to introduce a quasi-Newton method into a parameter renewal law. The effectiveness of the proposed method is demonstrated through numerical simulations and an experiment.

1 Introduction

Most of the controller design methods are based on the plant models. Therefore, because of the model uncertainties, the resultant control systems sometimes cannot achieve the specified control performance. On the other hand, input-output data-based design methods have been proposed by many researchers. These methods do not depend on the plant model, and utilize I/O data only. Therefore, they are inherently robust against the plant model uncertainty, and sometimes give us systematic approach to tune the controller parameters through experiments.

Among various such I/O data-based design methods, one of the most interesting approaches is the Iterative Feedback Tuning(IFT) scheme[2, 4]. This scheme is based on iterative tuning of the controller parameters along the gradient direction of a given cost function, and is applicable when one stabilizing controller parameter is given in advance. Many interesting results on IFT have been reported; the extension to nonlinear systems [1, 8, 9] and to MIMO systems[3, 7, 10], tuning methods for the various specifications[14, 15] and the evaluation of IFT with the numerical simulations and experiments[5, 12, 13]. Moreover, an algorithm of IFT which assure the stability of the closed loop systems is proposed[11].

However, there are some points which are not fully addressed in the former studies. First one is related to the plant nonlinearity. Since IFT basically depend on the linearity of the plants (or the closed loop systems), they may not work very well when the plants have strongly nonlinearity. One of such nonlinearity comes from the existence of friction. There are few reports which fully treat this problem in the frame work of IFT, though it is very important to cope with friction in order to avoid the performance degradation. Second one is concerning to the choice of the cost function for parameter tuning. Some papers adopt two-degree-of-freedom control structure, which is known to be useful because we can design the feedback and the feedforward controller independently[6]. Nevertheless, most of the existing works do not care about the role of each controllers and try to tune the controller parameters simultaneously by using the single cost function. Third one is related to the choice of optimization methods in the design parameters renewal law. The former studies usually adopt the Gauss-Newton method. However in nonlinear programming, there are many other useful methods such as the quasi-Newton method. The authors have witnessed that Broyden-Fletcher-Goldfarb-Shanno (BFGS) method is one of the most powerful alternatives.

In this paper, we consider an IFT design method of two-degree-of-freedom controllers for two-mass spring systems whose friction is not to be neglected, which may overcome the above drawbacks of the existing methods. In order to enjoy the two d.o.f control structure and to cope with the friction, we adopt two strategies. One is the separate tuning of feedback and feedforward controllers. In particular, as for the feedback controller, we concentrate on tuning to achieve low sensitivity instead of tracking property, while the feedforward controller is tuned from the viewpoint of command tracking property. The other is the design parameter renewal law based on the BFGS method. Then we demonstrate the effectiveness through numerical simulations. Finally and most importantly, we evaluate its effectiveness by experiment.

In this paper, the inner product and norm of time domain signals are denoted by

$$\begin{aligned}\langle x_1(t), x_2(t)\rangle &= \int_0^{t_f} x_1^T(t)x_2(t)dt \\ \|x(t)\|^2 &= \langle x(t), x(t)\rangle.\end{aligned}$$

And $\mathcal{L}[x]$ denotes the Laplace transform of $x(t)$.

2 Two-mass spring system and problem Setting

2.1 Two-mass spring system

Two-mass spring system is illustrated in Fig.1. The purpose of this system is to control the load angle θ by

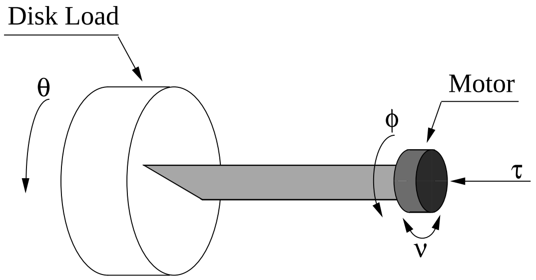

Fig.1 Two-mass spring system

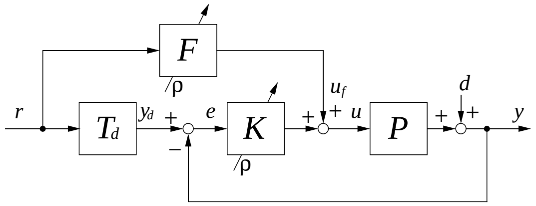

Fig.2 Two-degree-of-freedom system

adjusting the input torque τ based on the measurement of a load angle.

The kinetic equations of this system are given by

$$J_d\ddot{\theta} + D_d\dot{\theta} + k(\theta - \phi) = 0 \tag{1}$$

$$J_m\ddot{\phi} + D_m\dot{\phi} + k(\phi - \theta) = \tau - \nu \tag{2}$$

where θ and ϕ denote the rotate angle of load and motor respectively, and $J_d,\ J_m, D_d,\ D_m,\ k$ denote the moment of inertia of load and motor [kgm^2], the viscosity in load and motor [Nms/rad] and the spring constant [Nm/rad] respectively. And τ denotes the input torque [Nm] and ν denotes the friction of motor [Nm].

From the kinetic equations eq.(1) and (2), the linearized system is given by neglecting the friction ν. When the friction is not neglected, we may not obtain the desired controller from the linearized system. Hence we proposed the IFT method which overcomes such a friction. The model of the friction is introduced in section 4.

2.2 Problem Setting

In this paper, we consider the controlled system shown in Fig.2, which has a two-degree-of-freedom(Two-DOF) controller configuration. It is known that any Two-DOF control system can be described in this way[6].

In this figure, P denotes the continuous-time linear time invariant SISO system[1]. Time interval is finite. We assume that P is the unknown plant. The scalars $u(t)$, $y(t)$ and $u_f(t)$ denote the control input, the observed output and the feedforward input, respectively. K and F denote a feedback and feedforward controller, respectively. These controllers are represented by the transfer functions. T_d denotes the transfer function which satisfies the equation $\mathcal{L}[y_d] = T_d\mathcal{L}[r]$ where $y_d(t)$ and $r(t)$ denote the target trajectory signal and the reference signal which is the step signal in this paper. The error signal $y_d(t) - y(t)$ is denoted by $e(t)$.

We suppose that K and F are uniquely determined by the design parameter $\rho \in \mathbf{R}^p$ which is a column vector. We use the symbol $K(\rho)$ and $F(\rho)$ in order to indicate its dependence on ρ explicitly. For I/O signals, we also use $u(\rho), y(\rho), u_f(\rho)$ and $e(\rho)$.

The signal $d(t)$ denotes the test signal which is used for the estimation of the performance of the closed loop system. We assume that we can choose d arbitrarily. This is the key of our proposed method. Note that all subsystems are known except the plant P, and all the signals are observable.

For this controlled system (Fig.2), we consider the following specifications.

1. Tracking control for the target trajectory y_d.
2. The feedback controller has the low sensitivity performance.

From the second aim, we can expect the controller still has better performance though the friction exists.

From these specifications, we state the following controller design problem.

[**Controller design problem**] Assume the initial feedback and feedfoward controllers which stabilize the system as Fig.2 are given. Find the optimal parameter ρ^* which minimizes the cost function.

$$J(\rho) = \|y(\rho) - y_d\|^2 + \lambda\|u(\rho)\|^2 \tag{3}$$

by the iteration of the experiments. In eq.(3), λ denote a weighting positive constant scalar. □

3 Tuning the Two-DOF controller via IFT

3.1 Subproblems for the feedback and feedforward controllers

In this section, from the controller design problem in the previous section, we state the subproblems for the feedback and feedforward controllers which are suitable for the role of each controller.

Feedback performance: Let r be 0 in Fig.2, then the controlled system is shown as Fig.3. The system as depicted in Fig.3 is the part of the original system (Fig.2) with respect to feedback performance. For this feedback system, we state the feedback controller design subproblem as follows.

[**FB Controller Design Problem**] Assume the initial controller $K(\rho^{(0)})$ which stabilizes the system as Fig.3 is given. The signal d is given by $\mathcal{L}[d] = W\mathcal{L}[n]$ where n is noise signal which is Gaussian, white, of zero mean and whose covariance is determined in advance. Find the optimal parameter ρ^* which minimizes the cost function

$$J_{fb}(\rho) = \|y(\rho)\|^2 + \lambda_b\|u(\rho)\|^2 \tag{4}$$

by the iteration of the experiments, where $y(\rho)$ and $u(\rho)$ are the responses corresponding to input d. In eq.(4), λ_b denotes the weighting positive constant scalar. □

The transfer function from d to y is equal to sensitivity function S. Therefore if experiment time interval

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[1] In this section and next, we regard P as linear system for simplicity

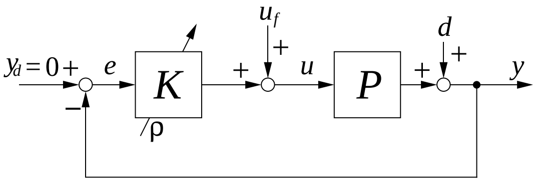

Fig.3 Feedback loop system

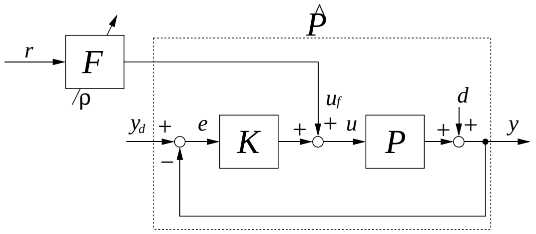

Fig.4 Feedfoward system

is infinite, then $||y(\rho)||^2$ is equal to square of H_2 norm of WS because of whiteness of the signal n. Hence we expect the low sensitive performance to make $J_{fb}(\rho)$ small. The second term of right hand of eq.(4) restricts the optimal input signal not to be large. By achieving the low sensitivity, we can expect to cope with friction.

Feedforward performance: Next, we state the subproblem for feedfoward performance. For this aim, we fix the feedback controller K in Fig.2, then the system is shown in Fig.4. For this system, we state the following controller design subproblem.

[**FF Controller Design Problem**] Assume the initial controllers K and $F(\rho^{(0)})$ which stabilize the system from r to y in Fig.4 is given. Find the optimal parameter ρ^* which minimizes the cost function

$$J_{ff}(\rho) = ||y(\rho) - y_d||^2 + \lambda_f ||u(\rho)||^2 \tag{5}$$

by the iteration of the experiments, where $y(\rho)$ and $u(\rho)$ are the responses corresponding to step reference input r. In eq.(5), λ_f denote the weighting positive constant scalar. □

We regard the solutions of the previous two subproblems as the one of the original controller design problem in 2.1. The procedure of the proposed method is as follows. At first, the feedback performance is tuned. Then next, with the obtained feedback controller K, feedforward performance is tuned. We regard the obtained controller K and F from this procedure as the solution of the original problem.

3.2 Minimization via IFT

For the above controller design subproblems, we explain how to solve those subproblems via IFT in this subsection [2].

Feedback performance: The renewal law of the design parameters is given as follows.

$$\rho^{(i+1)} = \rho^{(i)} + \gamma^{(i)} B(\rho^{(i)})^{-1} J'_{fb}(\rho^{(i)}) \tag{6}$$

The derivation of the cost function J_{fb} with respect to $\rho_j (j = 1, \cdots, p)$ is as follows.

$$J'_{fb,j}(\rho) = 2\langle y'_j(\rho), y(\rho)\rangle + 2\lambda_b \langle u'_j(\rho), u(\rho)\rangle \tag{7}$$

gradient J'_{fb} is obtained from these equations. The symmetric and positive definite matrix B is obtained by using the BFGS method which is described in Appendix A (see eq.(13)). and the initial matrix $B^{(0)}$ is given by $B^{(0)}_{ij} := 2\langle y'_i \cdot y'_j\rangle + 2\lambda_b \langle u'_i \cdot u'_j\rangle$

From eq.(7) and (13), we can see that $J'_{fb}(\rho)$ and $B(\rho)$ are constructed by $y(\rho)$, $u(\rho)$, $y'_j(\rho)$ and $u'_j(\rho)$ $(j = 1, \cdots, p)$. From the assumptions, it is clear that $y(\rho)$ and $u(\rho)$ is obtained by experiment.

We can obtain $y'_j(\rho)$ $(j = 1, \cdots, p)$ with the following procedure.

Step 1 Obtain $w(\rho)$ as the output response $y(\rho)$ corresponding to input d.

Step 2 Reform the experiment under the condition of $u_f = w(\rho)$, then we obtain $f(\rho)$ as the output y.

Step 3 Calculate $y'(\rho)$ with $f(\rho)$ and $K'_j(\rho)$.

Hence with two experiments and one off-line calculation, we obtain $y'(\rho)$. As for $u'(\rho)$, it is obtained by the similar procedure.

Substituting y, u, y' and u' for eq.(7), J'_{fb} is obtained. And so is B, because eq.(13) can be calculated from ρ and the gradient J'_{fb}.

Feedforward performance: The fundamental calculation is the same as the one for the feedback controller design, but the cost function eq.(7) is different. In this case, the gradient of the cost function is given by

$$J'_{ff,j}(\rho) = 2\langle y'_j(\rho), \tilde{y}(\rho)\rangle + 2\lambda_f \langle u'_j(\rho), u(\rho)\rangle \tag{8}$$

where $\tilde{y}(\rho) = y(\rho) - y_d$, hence $\tilde{y}'(\rho) = y'(\rho)$. We need $y(\rho)$, $y'(\rho)$, $u(\rho)$ and $u'(\rho)$ for the gradient. $y'_j(\rho)$ is given from $F'(\rho)$ and the step response of the system $P/(1 + PK)$. Because K is fixed, this step response can be obtained as the signal y of the one experiment when $u_f = r$ and $y_d = 0$. The other calculation are done by off-line. In the case of $u'(\rho)$, we use the signal u_f of the same experiment.

4 Evaluation of the proposed method

In this section, we evaluate the effectiveness of the proposed method by numerical simulations. The system to be considered here is a two-mass spring system which has the friction nonlinearity and our aim is to consider the robust tuning method against this friction. We compare the performance of the proposed method with one of the conventional method.

4.1 Control specifications

In the Two-DOF system depicted in Fig.2, P is the model of the two-mass spring system with friction. We assume the model of friction ν is as follows.

$$\nu = \begin{cases} \nu_0(\dot{\phi})\mathrm{sgn}(\dot{\phi}) & \text{, if } |\dot{\phi}| > 0 \\ \tau & \text{, if } \dot{\phi} = 0 \text{ and } |\tau| \le f_s \\ f_s \cdot \mathrm{sgn}(\tau) & \text{, otherwise} \end{cases}$$

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[2] Note that we apply the algorithm which is proposed by Hjalmarsson et. al. [10] and Hamamoto et. al. [7].

Table 1 The initial parameter $\rho^{(0)}$

Feedback controller:

ρ_1	2.0×10^3	ρ_2	7.5×10^4	ρ_3	1.6×10^7
ρ_4	4.7×10^8	ρ_5	9.5×10^6	ρ_6	4.7×10^4
ρ_7	3.6×10^2	ρ_8	7.2×10^4	ρ_9	8.5×10^6
ρ_{10}	5.2×10^7	ρ_{11}	5.2×10^6		

Feedforward controller:

ρ_{12}	4.1×10^{-6}	ρ_{13}	6.7×10^{-4}	ρ_{14}	3.4×10^{-2}
ρ_{15}	9.6×10^{-1}	ρ_{16}	0		

where $\nu_0(\phi) = f_d(1 + \alpha \sin(\omega\phi + \psi))$, f_d and f_s are the value of the dynamical friction and statical friction, α and ω are the parameters which represents the dependence of the friction on the angle of motor. And ψ is the Gaussian random number of zero mean and whose covariance is $(0.1)^2$, which represents the gap of the initial angle of motor.

We use the following values of the physical parameters; J_d : 9.24×10^{-4} [kgm^2], J_m : 4.45×10^{-3} [kgm^2], D_d : 5.24×10^{-3} [Nms/rad], D_m : 0.0 [Nms/rad], k : 6.16 [Nm/rad], and f_d : 0.42[Nm], f_s : 0.49[Nm], α : 0.03[Nm] and $\omega = 3.00$.

For this system, we consider the following control specifications. Let the cost function be

$$J(\rho) = \|y(\rho) - y_d\|^2, \tag{9}$$

where y_d is the step response of the following desired model.

$$T_d(s) = \frac{50^4}{(s+50)^4}. \tag{10}$$

Each feedback and feedforward controller has the structure given by the following equations with respect to the design parameter ρ. The transfer function of the feedback controller is denoted by

$$K(\rho) = \frac{\rho_1 s^5 + \rho_2 s^4 + \rho_3 s^3 + \rho_4 s^2 + \rho_5 s + \rho_6}{s^6 + \rho_7 s^5 + \rho_8 s^4 + \rho_9 s^3 + \rho_{10} s^2 + \rho_{11} s} \tag{11}$$

which has an integrator, because we consider the servo problem. And the transfer function of the feedforward controller is given by

$$F(\rho) = T_d \cdot \frac{\rho_{12} s^4 + \rho_{13} s^3 + \rho_{14} s^2 + \rho_{15} s + \rho_{16}}{k}. \tag{12}$$

The initial design parameter $\rho^{(0)}$ are given as Table 1.

4.2 Simulation results

We consider the following specifications for each subproblem which is described in section 3.

FB Controller Design Problem

The cost function J_{fb} is given by eq.(4) with $\lambda_b = 0.001$. And the frequency weighting function W is defined by $W = 10^4/(s^2 + 10s)$. White noise n is Gaussian of zero mean and has covariance $(0.3)^2$. The simulation time interval is 10[s], and sampling period for the simulation is 1[ms].

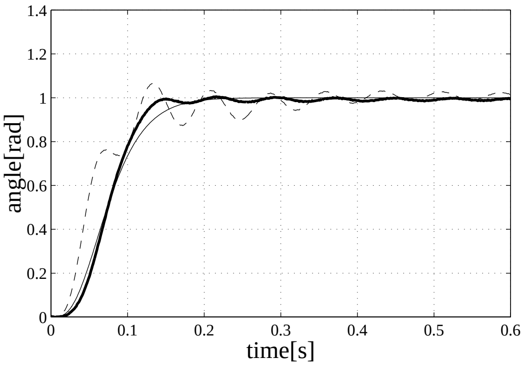

Fig.5 The output responses when $r = 1$

FF Controller Design Problem

The cost function J_{ff} is given by eq.(9). The simulation time interval is 10[s], and sampling period for the simulation is 1[ms]. Note that in this problem, the feedback controller is given already by FB controller design problem.

And then in the case of the conventional method, the simulation time interval is 20[s], and the sampling period for simulation is 1[ms].

We use the renewal law based on BFGS method for both the proposed and the conventional method.

The cost by the initial controllers is $J(\rho^{(0)}) = 1.7 \times 10^{-2}$. With the conventional method, the cost is $J = 2.8 \times 10^{-4}$ by 13 iterations. And we cannot optimize the cost with more iterations for tuning. With the proposed method, the cost is $J = 4.1 \times 10^{-5}$ by 22 iterations for feedback performance and 6 iterations for feedforward performance. Fig.5 shows the output responses for the step reference $r(t) = 1$. The target trajectory, the response by the conventional method and the response by the proposed method are depicted by thin solid line, dashed line and thick solid line, respectively. From Fig.5, the tracking performance of the proposed method is better and achieves smaller vibration. We can find that the controller obtained by the proposed method has the good tracking performance against the friction. This is achieved by the separate tuning.

5 Experiment

In this section, the effectiveness of the proposed method is demonstrated by the experiment with the two-mass spring system. At first, we describe the experimental system. And then we show the tuning of the controllers by proposed method.

5.1 Experimental system

Fig.6 shows the appearance of the two-mass spring system. As for the experimental system, we used a DC servo motor with reduction planetary gear of the reduction ratio 1/11, as an actuator, and a spring board as a flexible joint between the actuator and the load.

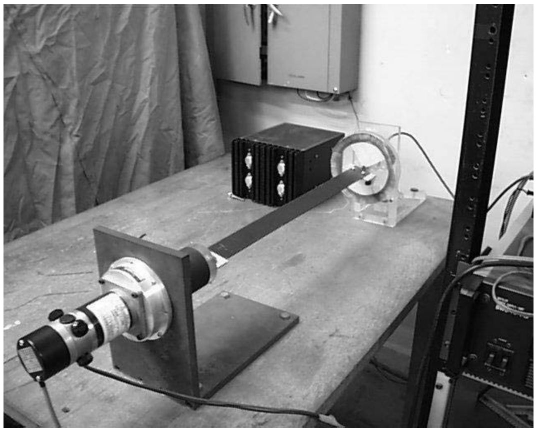

Fig.6 The experimental setup

Table 2 The optimal parameter for the feedback controller and the feedforward controller

Feedback controller:

ρ_1	3.5×10^4	ρ_2	2.4×10^5	ρ_3	3.8×10^8
ρ_4	2.5×10^9	ρ_5	1.0×10^{10}	ρ_6	-2.3×10^9
ρ_7	1.2×10^2	ρ_8	5.6×10^4	ρ_9	4.1×10^7
ρ_{10}	2.7×10^8	ρ_{11}	7.4×10^8		

Feedforward controller:

ρ_{12}	2.3×10^{-6}	ρ_{13}	1.7×10^{-4}	ρ_{14}	2.4×10^{-2}
ρ_{15}	5.4×10^{-1}	ρ_{16}	1.9×10^{-1}		

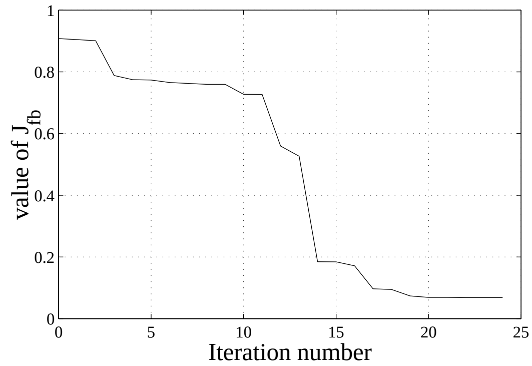

Fig.7 The cost J_{fb} v.s. iteration number

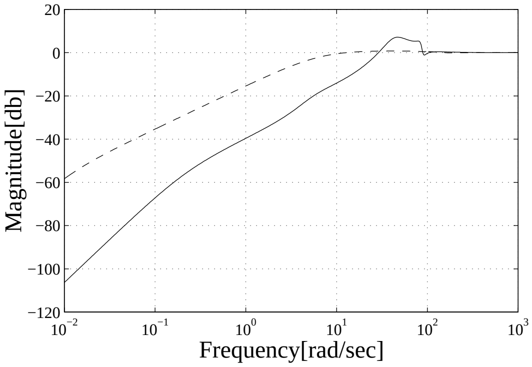

Fig.8 The gain plots of sensitivity function

5.2 Design specifications for the controllers

The cost functions, The structure of feedback and feedforward controller, and the initial controllers ($K^{(0)}$ and $F^{(0)}$) are the same as the given in simulation (subsection 4.1).

For the tuning of the feedback controller, the data sampling period is 4[ms]. From the two experiment whose interval is 12[s] (6000 data), gradient and the estimation of the Hessian are calculated. The controller is implemented by the personal computer with the control sampling period 2[ms]. For the tuning of the feedforward controller, the control and data sampling period are 2[ms], and experiment time interval is 4[s].

5.3 Results

[**Feedback controller**] By 24 iterations, the optimal cost is $J_{fb}(\rho^{(24)}) = 6.8 \times 10^{-2}$(the initial cost $J_{fb}(\rho^{(0)}) = 9.1 \times 10^{-1}$). Table 2 shows the optimal parameter ρ. Fig.7 shows the transition of the cost J_{fb} by iterations. From this figure, we can find the cost J_{fb} converge to the optimal value well. The gain plots of sensitivity functions are shown by Fig.8. These gain plots are calculated with the linearized plant model. The gain plot of the initial controller and the optimal controller are depicted by dashed line and solid line, respectively. The optimal controller achieves satisfactory properties at the low frequency domain. All the figures show that the obtained controller achieves desired performance.

[**Feedforward Controller**] By 8 iterations, the optimal cost is $J_{ff}(\rho^{(8)}) = 1.3 \times 10^{-4}$ (the initial cost $J_{ff}(\rho^{(0)}) = 7.3 \times 10^{-4}$). Table 2 shows the optimal parameter ρ. Fig.9 shows the transition of the cost J_{ff} by iterations. From this figure, we can find the cost J_{ff} converges to the optimal value smoothly by the proposed method. Fig.10 shows the output response for the unit step reference. The target trajectory, the output of $K^{(24)}$ and $F^{(8)}$, the output of $K^{(24)}$ and $F^{(0)}$, and the output of initial controllers ($K^{(0)}$ and $F^{(0)}$) are depicted by thin solid line, thick solid line, dashed line and dash-dot line. The doted line depicts the response in the case where the controller is tuned with respect to feedforward performance only and whose feedback controller is fixed at $K^{(0)}$.

From these results, the performance of the controller tuned by the proposed method is better than the other controller.

6 Conclusion

In this paper, we have considered an IFT design method of two-degree-of-freedom controllers for two-mass spring systems whose friction is not to be neglected. In order to enjoy the two d.o.f control structure and to cope with the friction, we have adopted two strategies. One is the two stage tuning of the feedback and feedforward controller. The other is the design pa-

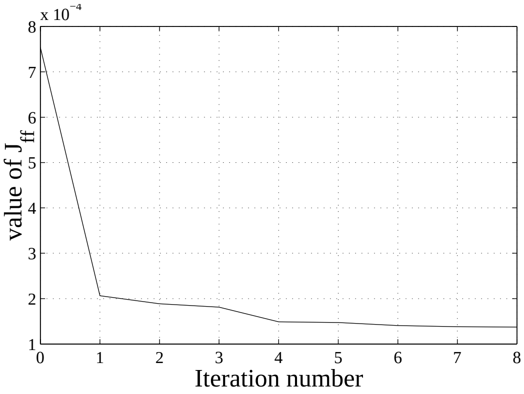

Fig.9 The cost J_{ff} v.s. iteration number

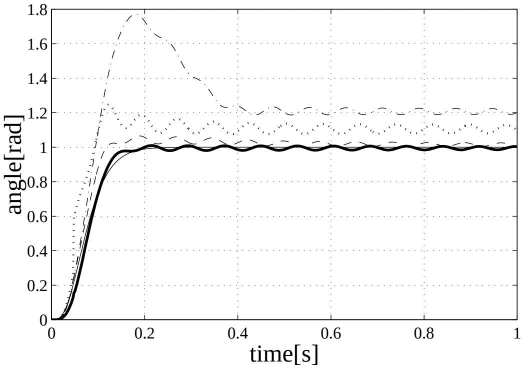

Fig.10 The output responses of Two-DOF system

rameter renewal law based on the BFGS method. From the numerical simulations and the experiment, the effectiveness of the proposed method have been shown.

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A BFGS method

Broyden-Fletcher-Goldfarb-Shanno(BFGS) method is the one of the quasi-Newton methods. One of the merit of the quasi-Newton methods is that the good estimation of Hessian matrix is given from the gradient of the cost function J and the design parameter ρ. BFGS method is well known as a good optimization method. The renewal law to estimate the Hessian based on BFGS method is given as follows.

$$B^{(k+1)} := B^{(k)} + \frac{z^{(k)}(z^{(k)})^T}{(z^{(k)})^T s^{(k)}} - \frac{B^{(k)} s^{(k)} (s^{(k)})^T B^{(k)}}{(s^{(k)})^T B^{(k)} s^{(k)}} \tag{13}$$

where $B^{(k)} := B(\rho^{(k)})$, $s^{(k)} := \rho^{(k+1)} - \rho^{(k)}$ and $z^{(k)} := J'(\rho^{(k+1)}) - J'(\rho^{(k)})$. The superscript (k) denotes the k-th iteration. The initial matrix value of $B^{(0)}$ is an arbitrary positive definite matrix. Usually, $B^{(0)}$ is chosen to be an identity matrix. The following facts are well known about BFGS method.

a) If $B^{(k)}$ is symmetric then $B^{(k+1)}$ is symmetric.

b) If $B^{(k)}$ is positive definite and $(z^{(k)})^T s^{(k)} > 0$, then $B^{(k+1)}$ is positive definite.

When $(z^{(k)})^T s^{(k)} > 0$ is not satisfied, let $B^{(k+1)} := B^{(k)} > 0$. However such a case seldom occurs. BFGS method uses the same data of Gauss-Newton method. And the following properties are expected; global convergence and super-linear convergence. Moreover, we can expect BFGS method is more numerically stable than Gauss-Newton method.