Analysis of Fuzzy T-S Observers and Controllers from the Interpolation Perspective

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Abstract

In this paper T-S fuzzy observers and controllers, built for the plants represented by T-S models, are analyzed from the interpolation viewpoint. The notion of fuzzy stability covering condition is defined extending the definition given in [7, 10, 11, 12]. Assuming slow variation of a so-called interpolating variable, sufficient conditions are given in order to generate stable parameter-varying family of observers and controllers that estimate the states and, stabilize nonlinear plants.

Keywords: Fuzzy Control, Fuzzy Observer, Stability, Scheduling, Interpolation.

1 Introduction

In this paper T-S fuzzy observers and controllers, built for the plants represented by T-S models, are analyzed from the interpolation viewpoint. Using T-S models and controllers along with the T-S inference, one obtains a nonlinear model, where nonlinear vector fields and functions result from the interpolation of local A_i, B_i and C_i. The global controllers and observers result from the interpolation, through the membership functions of fuzzy sets, of local linear controllers and observers. The latter are designed to stabilize the process or estimate its states around operating points. It is however well known that interpolating locally stabilizing controllers or observers, does not always guarantee the global stability of the system. Most of the results on the analysis of robustness and stability of T-S controllers, appeared in the literature [1, 3, 4, 13] do not explicitly address the problem. In the literature investigating the stability of T-S-type controllers or observers, very little attention has been directed toward interpolation an important step in the synthesis of gain-scheduled controllers. This is also the problem of "classical" gain scheduling approaches. Detailed elucidation of this problem can be for example found in [5, 6, 9, 10, 11]. The desire to use linear controllers in the synthesis T-S-type gain scheduled controllers or observers demands a careful inspection of the interpolation problem, and this is the main motivation for this work. In this work we extend some "classical" results of scheduling-interpolation methods and particularly those presented in [7, 10, 11, 12].

2 Problem Statement

Consider a nonlinear dynamic process represented by the following equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u - \exp(-\tfrac{x_1^2}{4}) \\ y &= x_1 \end{aligned} \tag{1}$$

The problem now is, to know if this process could be

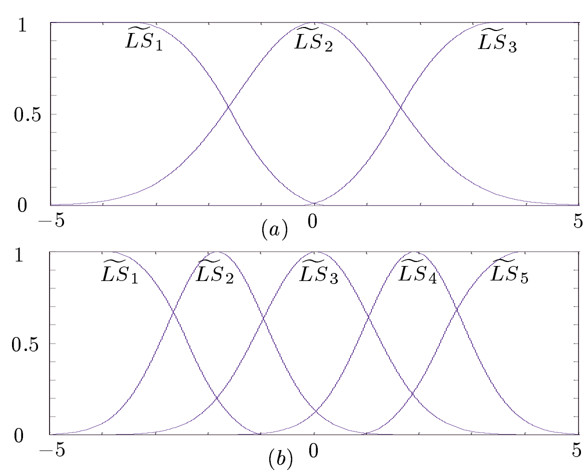

Figure 1:

represented by a TSK fuzzy model and then be controlled the same way. The scheduling variable given by the equilibrium manifold $\left(x_2 = 0, u = \exp(-\frac{x_1^2}{4})\right)$. Let us approximate $\exp(-\frac{x_1^2}{4})$, using a TSK model and so obtain a fuzzy partition on it. If we use only three fuzzy set to perform this approximation, one obtains the partition represented by the figure $1(a)$. Local linear models are obtained linearizing (1) around operating points representing the core[1] of fuzzy sets $\widetilde{LS}_1$, $\widetilde{LS}_2$ and $\widetilde{LS}_3$. The Jacobian linearization corresponding to (1) is given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2}x_1 e^{-\frac{1}{4}x_1^2} & 0 \end{pmatrix}_{\text{op. pt.}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \tag{2}$$

Evaluating (2) for the values of x, corresponding to the cores of $\widetilde{LS}_1$, $\widetilde{LS}_2$ and $\widetilde{LS}_3$ (i.e. $-3.5, 0, 3.5$), we obtain three linear models $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$. We can describe the process by

$$\text{If } x \in \widetilde{LS}^i \text{ then } \mathcal{M} = \mathcal{M}_i, i = 1,2,3 \tag{3}$$

Let us compute now for $\mathcal{M}_2$ and $\mathcal{M}_3$ a state feedback placing the eigenvalues of the closed loop system at -0.5. For $\mathcal{M}_2$ and $\mathcal{M}_3$ one obtains $K_2 = (0.25, 1)^T$ and $K_3 = (0.\,33, 1)^T$ respectively. Let us evaluate the linearized model at the intermediate point $\frac{3.5}{2} = 1.75$, denoted by $\mathcal{M}_{2-3}$. Applying TSK method one obtains

$$K_{2-3} = \frac{\mu_{\widetilde{LS}_2}(1.75)\,K_2 + \mu_{\widetilde{LS}_3}(1.75)\,K_3}{\mu_{\widetilde{LS}_2}(1.75) + \mu_{\widetilde{LS}_3}(1.75)} \approx (0.29, 1)^T. \tag{4}$$

A state feedback with this gain places the eigenvalues of $\mathcal{M}_{2-3}$ at 0.11 and -1.11. It shows that interpolating two linear controllers stabilizing the process around operating points does not guaranty local stability of any intermediate operating point. Let us compute now the gains K_2 and K_3 placing the eigenvalues of $\mathcal{M}_2$ and $\mathcal{M}_3$ at -1.5. one obtains $K_2 = (2.25, 3)^T$ and $K_3 = (2.33, 3)^T$ respectively. This time $K_{2-3} \approx (2.29, 3)^T$ and it places the eigenvalues of $\mathcal{M}_{2-3}$ at -0.89 and -2.11. *In the second case the stability is preserved and, in the first case it is not.*

Let us consider the same process with another partition of the steady state characteristics given in the Figure 1(b), and five linear models defined at the core of $\widetilde{LS}_i$, $i = 1, \cdots, 5$. The TSK model is now

$$\text{If } x \in \widetilde{LS}^i \text{ then } \mathcal{M} = \mathcal{M}_i, i = 1, \cdots, 5 \tag{5}$$

Let us compute the state feedback gains for $\mathcal{M}_3$(defined at $x_1 = 0$) and $\mathcal{M}_4$(defined at $x_1 = 1.8$) such that the eigenvalues are placed at -0.5. One obtains $K_3 = (0.25, 1)^T$ and $K_4 = (0.65, 1)^T$. Let us consider the intermediate point $x_1 = 0.9$ and compute the linearized model at this point, denoted by $\mathcal{M}_{3-4}$. Analogous computation to (4) gives $K_{3-4} \approx (0.\,45, 1)^T$. We can verify that this gain places the poles of $\mathcal{M}_{3-4}$ at -0.1 and -0.9.

Is is worth noticing that the in both cases the equilibrium manifold is well approximated by the fuzzy relations defined by (3) and (5). However in the first case, the interpolation does not satisfies the satisfy the *fuzzy stability covering condition* and, in the two last cases it does.

[1] The core of a fuzzy set $\tilde{A} = \left\{x \in \tilde{A}, \mu_{\tilde{A}}(x) = 1\right\}$

Definition 1 *Suppose that $\widetilde{LS}_i$, defines a fuzzy partition of the equilibrium manifold. Suppose that linear controllers $\Lambda_i, \cdots, \Lambda_M$ have been designed at points $p_1, \cdots, p_M$, with $\mu_{\widetilde{LX}^i}(p_i) = 1, 1 \leq i \leq M$. If Λ_i stabilizes $\mathcal{M}_i = \mathcal{M}(p_i)$, it is known that there exists an open neighborhood of p_i, U_{p_i} containing p_i, such that Λ_i stabilizes $\mathcal{M}(p)$, for each $p \in U_{p_i}$. We say that the* fuzzy stability covering condition *is satisfied, if $\bigcup_{i=1}^{M} supp^2 \widetilde{LX}^i \subseteq \bigcup_{i=1}^{M} U_{p_i}$.*

Suppose that the nonlinear plant to be controlled can be represented by

$$\begin{array}{rcl} \dot{X}(t) & = & f(X(t), U(t)) \\ Y(t) & = & h(X(t)) \end{array} \tag{6}$$

with $X \in \mathcal{X} \subset \mathbb{R}^n$, $U \in \mathcal{U} \in \mathbb{R}^m$. Suppose there exists an equilibrium manifold of (6) that can be parameterized by a scheduling variable, $\sigma \in \Sigma \subset \mathbb{R}^l$, Σ a connected compact set. That is, there exist continuous functions, $X^0 : \mathbb{R}^l \to \mathbb{R}^n$ and $U^0 : \mathbb{R}^l \to \mathbb{R}^m$ such that

$$f(X^0(\sigma), U^0(\sigma)) = 0 \tag{7}$$

for all $\sigma \in \Sigma$. *The scheduling variable, σ, can be a function of the state, input, and an exogenous signal.* For each σ, the linearization of the nonlinear plant is written as,

$$\begin{array}{rcl} \dot{x}(t) & = & A(\sigma)x(t) + B(\sigma)u(t), \\ y & = & C(\sigma)x(t) \end{array} \tag{8}$$

with

$$\begin{array}{ll} A(\sigma) = \left.\dfrac{\partial f(X)}{\partial X}\right|_{X=X^0(\sigma)} & B(\sigma) = \left.\dfrac{\partial f(X)}{\partial U}\right|_{X=X^0(\sigma)} \\ C(\sigma) = \left.\dfrac{\partial h(X)}{\partial X}\right|_{X=X^0(\sigma)} & u = U - U^0(\sigma) \end{array} \tag{9}$$

Considering a set of values of $\sigma(.) : \sigma_1(.), \cdots, \sigma_M(.)$ one obtains a set of $(A_i, B_i, C_i)\,, i = 1, \cdots, M$.

Let us assume that (6) can be approximated by the following T-S-type model

$$\begin{array}{ll} R_i^{\mathcal{P}} : & \text{If } \sigma_i(.) \in \widetilde{LS}_i \\ & \text{Then } \mathcal{P}_i : \left\{ \begin{array}{rcl} \dot{x}(t) & = & A_i x(t) + B_i u(t), \\ y & = & C_i x(t) \end{array} \right. \\ & i = 1, \cdots, M \end{array} \tag{10}$$

with $\sigma_i(.) = (\sigma_{i1}(.), \cdots, \sigma_{il}(.))^T$ and $\widetilde{LS}_i = \left(\widetilde{LS}_{i1}, \cdots, \widetilde{LS}_{il}\right)^T$, $A_i = A(\sigma_i)$, $B_i = B(\sigma_i)$, $C_i = C(\sigma_i)$. The common approach to control the nonlinear plant, is to design a series of linear controller designed for linear plants.

$$R_C^i : \text{If } \sigma_i(.) \in \widetilde{LS}_i \text{ Then } u = K_i x \tag{11}$$

[2] Support of a fuzzy set $\tilde{A}$: $\{x, \mu_{\tilde{A}}(x) \neq 0\}$

and the global controller is then built as

$$u(\sigma(.)) = \left[\sum_{i=1}^{M} \mu_{\widetilde{LS}_i}(\sigma(.))u_i\right] \Big/ \left[\sum_{i=1}^{M} \mu_{\widetilde{LS}_i}(\sigma(.))\right]. \tag{12}$$

The theoretical works on the analysis of the global stabilizing property of $u(\sigma(.))$ hide two important issues. The first is how local controller should be designed to guarantee a locally robust stabilization. The second is how the scheduling and interpolation should be performed to guarantee the global stability.

3 Stability preserving fuzzy Scheduling-Interpolation

Suppose that several points have been chosen on the equilibrium manifold parameterized by scheduling variable $\sigma : \sigma_1, \sigma_2, \cdots, \sigma_n$, as shown in the Figure 2. Suppose that each of those points are the core of a fuzzy set, $\widetilde{LS}_i$, in (10). Each fuzzy set $\widetilde{LS}_i$, fuzzily delimitate a re-

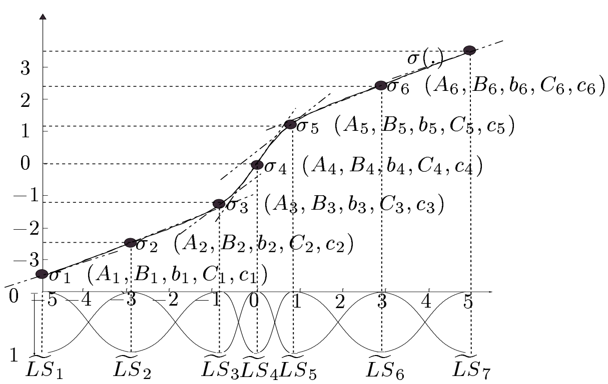

Figure 2:

gion where a local model $\mathcal{M}(\sigma_i) : (A(\sigma_i), B(\sigma_i), C(\sigma_i))$. Suppose that a local controller Λ_i is designed for each $\mathcal{M}_i = \mathcal{M}(\sigma_i)$, such that $(\mathcal{M}_i, \Lambda_i)$ is stable. The crucial question is how a global stabilizing controller Λ, could be obtained interpolating the Λ_i.

For the sake of clearness, we first consider the interpolation of two state feedback gains with a scalar scheduling variable. We will associate to each fuzzy set $\widetilde{LS}_i$ a membership function $\mu_{\widetilde{LS}^i}$ and suppose that:

Hypothesis 1 $\mu^r_{\widetilde{LS}_i}(\sigma) = 1 - \mu^l_{\widetilde{LS}_{i+1}}(\sigma), \forall \sigma \in \Sigma$ *and* $i = 1, \cdots, M-1$,

- $0 \leq \mu_{\widetilde{LS}_i}(\sigma) \leq 1, i = 1, \cdots, M$ *and* $\sum_i \mu_{\widetilde{LS}_i}(\sigma) = 1, \forall \sigma \in \Sigma$.
- $\mu\mu_{\widetilde{LS}_i}(\sigma_j) = \delta_{ij}$.

The above hypothesis suggest the following remarks:

Remark 1 *Under the above hypothesis, we can associate to the* scheduling variable σ, *an* interpolating variable μ^f. *Two particular examples of* μ^f, *are shown in the Figure 3:* μ^{f1} *and* μ^{f2} *with* $\mu^{f1} = 1 - \mu^{f2}$. *They are obtained by connecting* $\widetilde{LS}_i$, $\widetilde{LS}_{i+2}$, $\widetilde{LS}_{i+4}$, $\cdots$ *This connection along with the above hypothesis guaranty the continuity of* μ^{f1} *and* μ^{f2}. *We also assume their smoothness. For the sake of clearness we drop the superscript and, the scheduling variable is just denoted* μ. Also, in the following μ_i as on the figure 4, will denote a particular extreme value (0 or 1*)* of the interpolating variable μ, and is different of the function $\mu_{\widetilde{LS}^i}$. We will also distinguish μ_i and μ_{l_i} (l=$a, b, c, d, \cdots$). while μ_i only represents an extreme point (0 or 1), μ_{l_i} takes its values between 0 and 1.

- *Under the above hypothesis and remark, one can show that a bijective relation exists between scheduling and interpolating variables. This allows us to address the problem of stability preserving interpolation directly in terms of interpolating variable. This variable has also the advantage of being bounded by* 0 *and* 1.

Let us consider now a particular portion of μ in the Figure 2. Suppose K_2 and K_3 both asymptotically stabilize $\mathcal{M}_2 = \mathcal{M}(\mu_2)$ and $\mathcal{M}_3 = \mathcal{M}(\mu_3)$: Re[eig $(A(\mu_i) + B(\mu_i)K_i)] < 0, i = 2, 3$. It is well known that there exists a neighborhood of $\mu_i, \mathbb{M}_i$ containing μ_i such that Re[eig $(A(\mu) + B(\mu)K_i)] < 0, i = 2, 3$, for each $\mu \in \mathbb{M}_i$, $i = 2, 3$. Our goal is to find a parameterized state feedback gain, $K(\mu)$, such that Re[eig $(A(\mu) + B(\mu)K(\mu))] < 0$, for each $\mu \in [\mu_2, \mu_3] = [0, 1]$. Suppose that Re[eig $(A(\mu) + A(\mu)K_2)] < 0$, for $\mu \in [0, \mu(a)) = [0, \mu_a)$ and, Re[eig $(A(\mu) + B(\mu)K_3)] < 0$, for $\mu \in (\mu(b), 1] = (\mu_b, 1]$ with $\mu_2 \leq \mu_b < \mu_a \leq \mu_3$. Obviously Re[eig $(A(\mu) + B(\mu)K_i)] < 0, i = 2, 3$ for $\mu \in (\mu_b, \mu_a)$. *In this situation, one says that* K_2 *and* K_3 *cover* $[\mu_2, \mu_3]$.

Since K_2 and K_3 stabilize the system, for $\mu \in (\mu_b, \mu_a)$, there exist matrices $P_i = P_i^T > 0, i = 2, 3$, and an interval $[\mu(d), \mu(c)] = [\mu_d, \mu_c] \subset (\mu_b, \mu_a)$ such that

$$P_i \left[A(\mu) + B(\mu)K_i\right]^T + \left[A(\mu) + B(\mu)K_i\right] P_i < 0, i = 2, 3 \tag{13}$$

for all $\mu \in [\mu_d, \mu_c]$. We can show that the interpolated gain,

$$K(\mu) = \begin{cases} K_2, & \mu \in [0, \mu_d) \\ \left(\dfrac{\mu_c - \mu}{\mu_c - \mu_d} K_2 P_2 + \dfrac{\mu - \mu_d}{\mu_c - \mu_d} K_3 W_3 \right. \\ \left. \dfrac{\mu_c - \mu}{\mu_c - \mu_d} P_2 + \dfrac{\mu - \mu_d}{\mu_c - \mu_d} P_3 \right)^{-1} & , \mu \in [\mu_d, \mu_c] \\ K_{32}, & \mu \in (\mu_c, 1] \end{cases}$$

that is continuous and, it stabilizes $(A(\mu), B(\mu))$, for each fixed $\mu \in [\mu_2, \mu_3] = [0, 1]$. So far the stability pre-

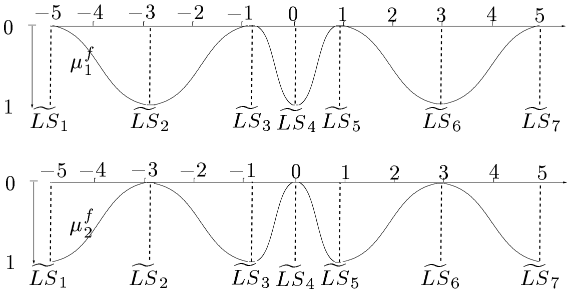

Figure 3:

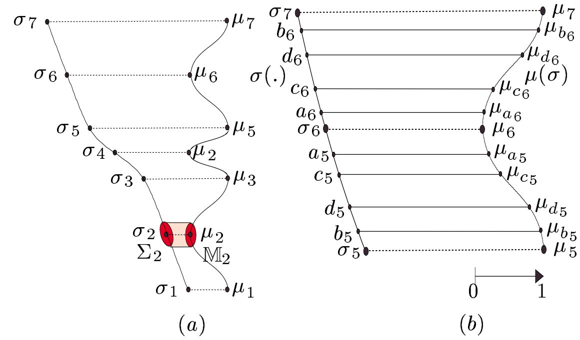

Figure 4:

serving interpolation has been derived in terms of *frozen values* of the interpolating variable μ. *When the interpolating variable is time–varying, stability is established by imposing a bound on the rate of variation of* μ.

Theorem 1 *Suppose the state feedback gains K_i have been designed for $\mathcal{M}_i : (A_i, B_i, C_i), i = 1, \cdots, M$. If for each i, there exists a matrix $P_i = P_i^T > 0, \gamma > 0$ and an open set $\mathbb{M}_i$ containing μ_i, such that*

$$P_i \left[A(\mu) + B(\mu)K_i\right]^T + \left[A(\mu) + B(\mu)K_i\right] P_i \leq -\gamma I, \quad \forall \mu \in \mathbb{M}_i, i = 1, \cdots, M \tag{14}$$

if $\mathbb{M}_i \cap \mathbb{M}_{i+1} \neq \emptyset$, then there exist pairs of pints μ_{d_i}, μ_{c_i} (see the Figure 4 (b)),

$$[m, M] \subset \mathbb{M}_i \cap \mathbb{M}_{i+1} \cap \left[\min(\mu_j, \mu_{j+1}), \max(\mu_j, \mu_{j+1})\right], \quad i = 1, \cdots, M \tag{15}$$

with $m = \min\left(\mu_{d_i}, \mu_{c_i}\right)$ and $M = \max\left(\mu_{d_i}, \mu_{c_i}\right)$, such that if $\mu(t)$ satisfies

$$\dot{\mu}(t) < \min_{i=1,\cdots,M-1} \frac{\left|\mu_{d_i} - \mu_{c_i}\right|}{\|P_{i+1} - P_i\|}, t \geq 0 \tag{16}$$

there exist a continuous state feedback gain,

$$K(\mu) = \begin{cases} K_j, & \mu \in [0, m) \\ \left(\dfrac{M - \mu}{M - m} K_j P_j + \dfrac{\mu - m}{M - m} K_l P_l\right) P(\mu)^{-1}, & \mu \in [m, M] \\ K_l, & \mu \in (M, 1] \end{cases}$$

$$P(\mu) = \begin{cases} P_j, & \mu \in [0, m) \\ \left(\dfrac{M - \mu}{M - m} P_j + \dfrac{\mu - m}{M - m} P_l\right)^{-1} & \mu \in [m, M] \\ P_l, & \mu \in (M, 1] \end{cases} \tag{17}$$

with

$$j = i, l = i + 1 \text{ if } \max(\mu_i, \mu_{i+1}) = \mu_{i+1}$$
$$j = i + 1, l = i \text{ if } \max(\mu_i, \mu_{i+1}) = \mu_i$$

such that for all $\mu(t) \in \Gamma = \bigcup_{i=1}^{M} \mathbb{M}_i$, the closed–loop time–varying system,

$$\dot{X}(t) = \left(A(\mu(t)) + B(\mu(t)) K(\mu(t))\right) X(t), t \geq 0, \tag{18}$$

is exponentially stable.

Proof: see the Appendix A.

Theorem 1 shows that it is desirable to find solutions of (14) that are "close together" in norm. If the solutions of (14) are all the same, then there is an infinite stability margin on the rate of variation of the scheduling parameter [11]. In this case, the stability preserving interpolation reduces to linear interpolation. A continuous, interpolated state observer gain, $L(\mu)$, can be built the same way.

Theorem 2 *Suppose the state observer gains L_i have been designed for the linear plants, $(A_i, C_i) = (A(\mu_i), C_i(\mu_i))$, where, $\mu_i \in \Gamma, i = 1, \cdots, M$. Suppose also that for each i, there exists a matrix $Q_i = Q_i > 0$ and an open interval $\mathbb{M}_i$, such that*

$$\left(A(\mu) + L_i C(\mu)\right)^T Q_i + Q_i \left(A(\mu) + L_i C(\mu)\right) \leq -\gamma I, \quad \mu \in \mathbb{M}_i, \gamma > 0, i = 1, \cdots, M \tag{19}$$

with $\Gamma \subseteq_i \mathbb{M}_i$ and $\mathbb{M}_i \cap \mathbb{M}_{i+1} \neq \emptyset$, then there exists pairs of points μ_{d_i}, μ_{c_i},

$$[m, M] \subset \mathbb{M}_i \cap \mathbb{M}_{i+1} \cap \left[\min(\mu_i, \mu_{i+1}), \max(\mu_i, \mu_{i+1})\right] \quad j = 1, \cdots, M - 1$$

with $m = \min\left(\mu_{d_i}, \mu_{c_i}\right)$ and $M = \max\left(\mu_{d_i}, \mu_{c_i}\right)$, such that if $\mu = \mu(t)$ satisfies

$$\dot{\mu}(t) < \min_{i=1,\cdots,M-1} \frac{\left|\mu_{c_i} - \mu_{d_i}\right|}{\|Q_{i+1} - Q_i\|}, t \geq 0 \tag{20}$$

there exists a continuous state observer gain,

$$L(\mu) = \begin{cases} L_j, & \mu \in [0, m) \\ \left(\dfrac{M - \mu}{M - m} L_j Q_j + \dfrac{\mu - m}{M - m} L_l Q_l\right) Q(\mu)^{-1}, & \mu \in [m, M] \\ L_l, & \mu \in (M, 1] \end{cases} \tag{21}$$

$$Q(\mu) = \begin{cases} Q_j, & \mu \in [0, m) \\ \left(\dfrac{M-\mu}{M-m} Q_j + \dfrac{\mu - m}{M-m} Q_l \right)^{-1} & \mu \in [m, M] \\ Q_l, & \mu \in (M, 1] \end{cases}$$

with

$$j = i, l = i+1 \text{ if } \max(\mu_i, \mu_{i+1}) = \mu_{i+1}$$
$$j = i+1, l = i \text{ if } \max(\mu_i, \mu_{i+1}) = \mu_i$$

such that the closed–loop time–varying system,

$$\dot{\hat{X}}(t) = [A(\mu(t)) + L(\mu(t)) C(\mu(t))] \hat{X}(t) \tag{22}$$

is exponentially stable.

Proof: similar to the proof of 1, via minor modifications.

The above theorems show that stable T-S state feedback controllers and observers can be built separately by interpolating local stables state feedback controllers and observers, in such a way that the fuzzy stability covering conditions is satisfied and $\mu(t)$ is slowly varying with time. The very important problem is now to know, how would behave the parameter-varying closed-loop system, when the parameter-varying state feedback controller uses the *estimated states* from the parameter-varying observer, both assumed to be exponentially stable.

Theorem 3 *If the hypotheses of theorems 1 and 2 are satisfied. If the gains $K(\mu)$ and $L(\mu)$ are given by (17) and (21), respectively. If $\mu = \mu(t)$ satisfies the bounds, (16) and (20), then*

$$\dot{z}(t) = \begin{pmatrix} A(\mu) + B(\mu) K(\mu) & -B(\mu) K(\mu) \\ 0 & A(\mu) + L(\mu) C(\mu) \end{pmatrix} z(t) \tag{23}$$

is exponentially stable, with $z(t) = \left[X^T(t), \hat{X}^T(t) \right]^T$.

A Proofs

Lemma 4 *([12])*

Suppose , $W(\rho) : \Gamma \subset \mathbb{R} \to \mathbb{R}^{n \times n}$ is a continuous, piecewise affine matrix-valued function with corner points $\{c_1, \cdots, c_r\}$. For any $\varepsilon > 0$, there exists $\delta > 0$ and a continuouslydifferentiable function, $\hat{W}(\rho) : \Gamma \subset \mathbb{R} \to \mathbb{R}^{n \times n}$, and $\delta > 0$, such that

$$\begin{array}{ll} \left\| W(\rho) - \hat{W}(\rho) \right\| < \varepsilon, & \rho \in (c_i - \delta/2, c_i + \delta/2) \\ W(\rho) = \hat{W}(\rho), & \rho \notin (c_i - \delta/2, c_i + \delta/2) \\ \left\| \dfrac{d}{d\rho} \hat{W}(\rho) \right\| \le \displaystyle\max_{\rho \in \Gamma, \rho \neq c_i} \left\| \dfrac{d}{d\rho} W(\rho) \right\| & \end{array} \tag{24}$$

A.1 Proof of Theorem 1

Define $A_{cl}(\mu) = A(\mu) + B(\mu) K(\mu)$ then from (14) and (17),

$$P(\mu) A_{cl}^T(\mu) + A_{cl}(\mu) P(\mu) \le -\gamma I, \gamma > 1, \mu \in \Gamma \tag{25}$$

Since $P(\mu) = P(\mu)^T > 0$ for each $\mu \in \Gamma$ and Γ is compact, there exists $\delta_P > 0$ such that

$$P \ge \delta_P I \tag{26}$$

Let us choose

$$\varepsilon < \min \left\{ (\gamma - 1) / \left(2 \max_{\mu \in \Gamma} \| A_{cl}(\mu) \| \right), \delta_P \right\} \tag{27}$$

and a continuously differentiable approximation, $\hat{P}(\mu)$, as in Lemma 4. The approximation error, $\tilde{P}(\mu) = P(\mu) - \hat{P}(\mu)$ satisfies

$$\max_{\mu \in \Gamma} \tilde{P}(\mu) < \varepsilon < (\gamma - 1) / \left(2 \max_{\mu \in \Gamma} \| A_{cl}(\mu) \| \right) \tag{28}$$

This expression along with the Shwartz inequality give

$$\begin{array}{l} \left\| \tilde{P}(\mu) A_{cl}^T(\mu) + A_{cl}(\mu) \tilde{P}(\mu) \right\| \le \\ 2 \left(\displaystyle\max_{\mu \in \Gamma} \left\| \tilde{P}(\mu) \right\| \right) \left(\displaystyle\max_{\mu \in \Gamma} \| A_{cl}(\mu) \| \right) \le \gamma - 1 \end{array} \tag{29}$$

From (29) and (14), we obtain

$$\hat{P}(\mu) A_{cl}^T(\mu) + A_{cl}(\mu) \hat{P}(\mu) \le -I \tag{30}$$

From (24), (26) and (27) we deduce $P(\mu) - \hat{P}(\mu) < \varepsilon I < \delta_P I$, and then $\hat{P}(\mu) > 0$. For any given $\mu(t)$, let choose the Lyapunov function, $V(X, t) = X^T \hat{P}(\mu)^{-1} X$. Since Γ is compact, there exist positive constants, δ_1 and δ_2, such that

$$\delta_1 \|X\|^2 \le V(X, t) \le \delta_2 \|X\|^2 \tag{31}$$

Using again, the compactness of Γ and computing the derivative of $V(X, t)$along the trajectories of (18)

$$\begin{array}{rcl} \frac{d}{dt} V(X, t) & = & X^T \left[A_{cl}^T(\mu) \hat{P}(\mu)^{-1} + \hat{P}(\mu)^{-1} A_{cl}(\mu) \right. \\ & & \left. + \frac{d}{dt} \hat{P}(\mu)^{-1} \right] X \\ & \le & X^T \hat{P}(\mu)^{-1} \left[-I - \frac{d}{dt} \hat{P}(\mu) \right] \hat{P}(\mu)^{-1} X \end{array}$$

if $\max_{\mu \in \Gamma} \left\| \frac{d}{dt} \hat{P}(\mu) \right\| < 1$, there exists $\delta_3 > 0$ such that $\frac{d}{dt} V(X, t) \le -\delta_3 \|X\|^2 (\Gamma \text{ compact})$. From Lemma 4,

$$\max_{\mu \in \Gamma} \left\| \tfrac{d}{dt} \hat{P}(\mu) \right\| = \max_{\mu \in \Gamma} \left\{ \left\| \tfrac{d}{dt} P(\mu) \right\|, \mu(t) \neq \mu_{c_i}, \mu_{d_i}, \; i = 1, \cdots, M-1 \right\}$$

Thus,

$$\left\| \frac{d}{dt} \hat{P}(\mu) \right\| \le \begin{cases} 0, & \mu \in \left[0, \mu_{d_i}\right) \cup \left(\mu_{c_i}, 1\right] \\ \dfrac{\dot{\mu}(t)}{\left| \mu_{c_i} - \mu_{d_i} \right|} \| P_{i+1} - P_i \|, & \mu \in \left[\mu_{d_i}, \mu_{c_i} \right] \\ & i = 1, \cdots, M-1 \end{cases}$$

and (18) is exponentially stable if (16) is satisfied (Lemma 4).

This is similar to those presented for the "classical" interpolation in the [10, 11, 12]

A.2 Proof of the Theorem 3

Let us consider the Lyapunov function

$$V(z,t) = z^T(t) \underbrace{\begin{pmatrix} \alpha \hat{P}^{-1}(\mu(t)) & 0 \\ 0 & \hat{Q}^{-1}(\mu(t)) \end{pmatrix}}_{M} z^T(t) \tag{32}$$

with $\hat{P}$ and $\hat{Q}$ defined in theorems 1 and 2 and, $\alpha > 0$. It is obvious that $M = M^T > 0$. In the following, for the sake of notational clearness, the dependencies with respect to $\mu(t)$ will be omitted.

$$\frac{d}{dt}V(z,t) = z^T \begin{bmatrix} \alpha\left(A_{cl}^T\hat{P}^{-1} + \hat{P}^{-1}A_{cl} + \frac{d}{dt}\hat{P}^{-1}\right) \\ -\alpha(BK)^T\hat{P}^{-1} \\ \quad -\alpha\hat{P}^{-1}(BK) \\ \quad \left(A_{clo}\hat{Q}^{-1} + \hat{Q}^{-1}A_{clo}^T + \frac{d}{dt}\hat{Q}^{-1}\right) \end{bmatrix} z \triangleq z^T N z.$$

$N < 0$ if and only if (LMI Lemma [2])

$$\begin{aligned} \alpha\left(A_{cl}\hat{P}^{-1} + \hat{P}^{-1}A_{cl}^T + \tfrac{d}{dt}\hat{P}^{-1}\right) &< 0 \\ \alpha\hat{P}^{-1}(BK)\left(A_{clo}\hat{Q}^{-1} + \hat{Q}^{-1}A_{clo}^T + \tfrac{d}{dt}\hat{Q}^{-1}\right)^{-1} &. \\ (BK)^T\hat{P}^{-1} - \left(A_{cl}^T\hat{P}^{-1} + \hat{P}^{-1}A_{cl} + \tfrac{d}{dt}\hat{P}^{-1}\right) &< 0 \end{aligned}$$

The first inequality is satisfied by hypothesis, and the second is satisfied if

$$\lambda_M < \lambda_m \tag{33}$$

with

$$\begin{aligned} \lambda_m =& \inf_{\mu\in\Gamma} \operatorname{eig}\Big(A_{cl}^T(\mu)\hat{P}^{-1}(\mu) + \\ & \qquad \hat{P}^{-1}(\mu)A_{cl}(\mu) + \tfrac{d}{dt}\hat{P}^{-1}(\mu)\Big) \\ \lambda_M =& \sup_{\mu\in\Gamma} \operatorname{eig}\Big[\alpha\hat{P}^{-1}(B(\mu)K(\mu))\Big(A_{clo}(\mu)\hat{Q}^{-1}(\mu) + \\ & \hat{Q}^{-1}(\mu)A_{clo}^T(\mu) + \tfrac{d}{dt}\hat{Q}^{-1}(\mu)\Big)^{-1}(BK)^T\hat{P}^{-1}\Big] \end{aligned}$$

where eig represents the eigenvalue. From the LMI Lemma and the compactness of Γ we know that there always exists an α such that (33) is satisfied.

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