Estimation of Perturbation Bounds for Finite Trajectories

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Abstract

The problem of estimating perturbation bounds for finite trajectories of non-autonomous systems is considered. A worst case sensitivity derivative of the trajectory with respect to the uncertainty is used to verify that the perturbed trajectory is within a given neighborhood of the nominal. This gives rise to a robust control problem for linear time-varying systems. It is shown that relaxation using integral quadratic constraints and the solution to a linear quadratic optimal control problem can be used to find bounds on the robust control problem.

1 Introduction

The following robustness problem for finite time trajectories of non-autonomous systems is considered: Find conditions which ensure the existence of a unique solution within a given neighborhood of the nominal solution when the system is perturbed with a class of structured uncertainties. We consider time-varying parametric uncertainty and dynamic uncertainty.

There are a large number of important applications of such a result. For example, it can be used to verify robustness of hybrid automatons where the continuous state is moved from one equilibrium set to another, and for robustness of switched dynamical systems [1, 7, 3, 8]. Another application, which is considered in this paper, is robustness of the step response of a nonlinear control system.

The problem will be addressed by computing the worst case sensitivity derivative of the trajectory with respect to the uncertainty. It is possible to guarantee that the robustness condition is satisfied if a certain robust control problem is feasible. This can be verified by using relaxation in terms of Integral Quadratic Constraints (IQC) and then solve a linear quadratic optimal control problem.

The approach has much in common with the ideas for robustness analysis of periodic trajectories of non-autonomous systems in [5]. However, there are some distinct differences. On one hand, the finite trajectory case is simpler than the periodic trajectory case since stability is not an issue. On the other hand, the treatment of the transient response becomes both hard and critical, in particular for the case of dynamic uncertainty. We will discuss several alternative ways of treating the transient response. It should finally be noted that our approach is different from the classic perturbation method since we obtain an explicit uniform bound on the maximal perturbation of the trajectory instead of an order of magnitude statement, see, for example [6]. Proofs and additional comments for the results of this paper are given in [4].

Notation and Preliminaries

We let $C[0,T]$ be the continuous functions on $[0,T]$ with the usual supremum norm. The operator algebra $\mathcal{A}$ is defined as follows, see [2]. Each $H \in \mathcal{A}$ defines a causal convolution operator on $C[0,T]$[1]

$$(He)(t) = \int_0^t h_c(t-\tau)e(\tau)d\tau + \sum_{t_k \le T} h_k e(t-t_k),$$

where $h_c \in \mathbf{L}_1[0,\infty)$, $\sum_{k=0}^{\infty} |h_k| < \infty$, and $t_k \ge 0$.

Let $\mathcal{S} \subset \mathbf{C}^{n\times n}$ be a fixed uncertainty structure, see, for example [10, 12]. We consider uncertainties of the following types

Time-varying parameters: $\mathcal{B}_{TV}(\mu) = \{\Delta : [0,T] \to \mathbf{R}^{n\times n} : \Delta(t) \in \mathcal{S};\ |\Delta(t)| \le \mu\}$

Dynamic: $\mathcal{B}(\mu) = \{\Delta \in \mathcal{A} : \Delta(s) \in \mathcal{S};\ \|\Delta\| \le \mu\}$

Parametric: $\mathcal{B}_0(\mu) = \{\Delta \in \mathbf{R}^{n\times n} : \Delta \in \mathcal{S};\ |\Delta| \le \mu\}$

where $|\cdot|$ is the largest singular value of a matrix and $\|\Delta\|$ denotes the induced $\mathbf{L}_2$-norm. The functions in the first set are assumed to be continuous. It is

[1] We will in this paper extend $e \in C[0,T]$ to negative times by defining $e(t) = e(0)$, for $t \le 0$.

assumed that the uncertainty structure is such that $\mathcal{B}(\infty)$, $\mathcal{B}_{TV}(\infty)$, and $\mathcal{B}_0(\infty)$ are vector spaces.

The (lower) linear fractional transformation between a block structured operator and an uncertainty is defined as, [10, 12]

$$\mathcal{F}_l(H, \Delta) = H_{11} + H_{12}\Delta(I - H_{22}\Delta)^{-1}H_{21}.$$

Let V_1, V_2 be normed vector spaces and let $U \in V_1$ be open. A nonlinear operator $F : U \to V_2$ is said to be *continuously differentiable* (C^1) if there exists a bounded linear operator $DF : U \to \mathcal{L}(V_1, V_2)$ such that

$$\lim_{u \to u_0} \frac{\|F(u) - F(u_0) - DF(u_0)(u - u_0)\|_{V_2}}{\|u - u_0\|_{V_1}} = 0$$

for each $u_0 \in U$. We let $DF(u_0) \cdot v$ denote $DF(u_0)$ applied to $v \in V_1$ and if F depends on several arguments then we denote the partial derivative with respect to u as $D_u F$. We sometimes use the alternative notation $\frac{dF}{du}$ for the derivative $D_u F$.

2 Parametric Uncertainties

We will first consider systems with parametric uncertainties. Let the system be [2]

$$\dot{x} = Ax + (B_1 + B_2\Delta(t))\varphi(Cx, t), \quad x(0) \in X, \qquad (1)$$

where A is Hurwitz, X denotes the set of initial conditions, $\Delta \in \mathcal{B}_{TV}(\mu)$, and where $\varphi : \mathbf{R}^m \times [0, T] \to \mathbf{R}^p$ is continuously differentiable in z and continuous in t. We will use the notation $\varphi_z(z, \cdot) = \frac{\partial \varphi(z, \cdot)}{\partial z}$. The system is called *nominal* when $\Delta = 0$ and we assume there exists a unique solution $x_0 \in C[0, T]$ to the nominal system. We are interested in verifying that the perturbed system, i.e., when $\Delta \neq 0$, has a unique solution within some given tolerance of the nominal solution. Our development is based on using the mean value theorem on the worst case sensitivity derivative $\frac{dx}{d\Delta}$ over the uncertainty set, which includes the structured uncertainty and uncertainty in the trajectory. It turns out that $z = C_1 x$ is the most crucial signal because it appears in the derivative of φ. This leads us to formulate the following robustness problem

Tube Condition: Given the nominal trajectory $z_0 = Cx_0$ and the tube

$$B_{z_0}(r) = \{z \in C[0, T] : |z(t) - z_0(t))| \leq r(t),\ t \in [0, T]\},$$

check if for each $\Delta \in \mathcal{B}_{TV}(\mu)$, system (1) has a unique solution $z \in B_{z_0}(r)$. See Figure 1 for an illustration.

[2] This is a large class of systems. It includes, for example, $\dot{x} = f_0(x, t) + \sum_{i=1}^{N} \delta_i(t) f_i(x, t)$, where δ_i are time-varying uncertainties.

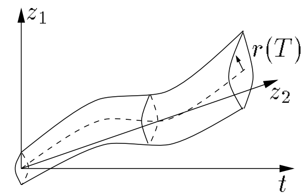

Figure 1: The nominal trajectory is given in dashed line. We will give conditions for existence of a unique solution to the perturbed system within the indicated tube.

Our method of solution in Proposition 1 can easily be adapted to obtain bounds for other linear combinations of the state vector.

Next we define the initial conditions set. For each $\Delta \in \mathcal{B}(\mu)$, the set X needs to specify a unique $x(0)$, which satisfies the tube constraint and a bound on its sensitivity derivative. We propose the following assumption.

Assumption on X: For each $\Delta \in \mathcal{B}_{TV}(\mu)$ there exists a unique $x(0) \in X$, with $|C(x(0) - x_0(0))| \leq r(0)$ and $\frac{dx(0)}{d\Delta}(z, \Delta) \cdot \widehat{\Delta} \in \chi_0(B_{z_0}(r), \Delta, \widehat{\Delta})$, where the set $\chi_0(B_{z_0}(r), \Delta, \widehat{\Delta})$ can be obtained in several ways. For example:

1. If $x(0) = x_0(0)$ for all $\Delta \in \mathcal{B}_{TV}(\mu)$ then $\chi_0(B_{z_0}(r), \Delta, \widehat{\Delta}) = \{0\}$

2. If the system is at stationarity for $t \leq 0$ ($\Delta(t) = \Delta(0)$ and $\varphi(z(t), t) = \varphi(z(0), 0)$ for $t \leq 0$) then use[3]

$$\chi_0 = \{(I - \widetilde{H}_\Delta(0)\varphi_z(z, 0)C)^{-1}A^{-1}B_2\widehat{\Delta}\varphi(z, 0) : \Delta, \widehat{\Delta} \in \mathcal{B}_0(\mu),\ |z| \leq r(0)\}. \qquad (2)$$

where $\widetilde{H}_\Delta(s) = (sI - A)^{-1}\begin{bmatrix} B_1 & B_2\Delta \end{bmatrix}$

3. If we are considering switched systems then we can obtain χ_0 as in Remark 1 below.

Proposition 1. *Let the set X satisfy the above assumption and let ($\mathcal{E} \subset C[0, T]$)*

$$\mathcal{E} = \{e(t) = \widehat{\Delta}(t)\varphi(z(t), t) : \widehat{\Delta} \in \mathcal{B}_{TV}(\mu),\ z \in B_{z_0}(r)\}.$$

[3] The inverse must generally be interpreted as the inverse image. However, if the matrix is singular then the well-posedness condition in Proposition 2 is violated.

If the linear system

$$\begin{aligned}
&\dot{\chi} = A\chi + B_1\omega_1 + B_2\omega_2 + B_2\varepsilon, \quad \chi(0) \in \mathcal{X}_0, \ \varepsilon \in \mathcal{E}, \\
&\zeta = C\chi, \\
&\omega_1 = \varphi_z(z(t), t)C\chi, \qquad (3)\\
&\omega_2 = \Delta(t)\omega_1,
\end{aligned}$$

is well-posed[4] *and if* $|\zeta(t)| \leq r(t)$ *for all* $\Delta \in \mathcal{B}_{TV}$, $z \in B_{z_0}(r)$, *and* $\varepsilon \in \mathcal{E}$, *then for each* $\Delta \in \mathcal{B}_{TV}(\mu)$, *system (1) has a unique solution* $z \in B_{z_0}(r)$.

Remark 1. The result can be used for analysis of switched and hybrid systems. If the system change vector field at time T then $\mathcal{X}_0 = \{\chi(T) : \chi$ is a solution of (3)$\}$ can be used in the initial condition specification for analysis over the next time interval.

3 IQC Relaxation of the uncertain system

We will next discuss how the condition in Proposition 1 can be verified. The uncertainty in the linear system (3) is due to

- (i) the structured uncertainty $\Delta \in \mathcal{B}_{TV}(\mu)$,
- (ii) uncertain "linearization point" $z \in B_{z_0}(r)$,
- (iii) the noise signal $\varepsilon \in \mathcal{E}$,

As a first step we let $\varphi(z,t) = \varphi(z_0,t) + \phi(t)$, where

$$\phi(t) \in \Theta := \{\phi(t) = \varphi(z,t) - \varphi(z_0,t) : z \in B_{z_0}(r)\}.$$

The next step is to do IQC relaxation of the system (3). Let $\omega = \begin{bmatrix}\omega_1^T & \omega_2^T\end{bmatrix}$, where $\omega_1 = \phi(t)C\chi$, $\phi \in \Theta$, and $\omega_2 = \Delta\omega_1$. It is possible to find quadratic forms σ_k such that the constraints

$$\int_0^t \sigma_k(\chi,\omega)d\tau \geq 0, \quad t \in [0,T], \ k = 1,\dots,N \qquad (4)$$

hold for all pairs $(\chi,\omega) \in \mathbf{L}_2[0,T]$, where ω is defined as above. Such IQC relations can be obtained by exploiting properties (such as norm bounds) of Θ and $\mathcal{B}_{TV}(\mu)$. We also assume that we have an energy bound $\sup_{\varepsilon\in\mathcal{E}} \int_0^t |\varepsilon(\tau)|^2 d\tau \leq \eta(t), \ t \in [0,T]$, where η is a suitable continuous function on $[0,T]$.

An IQC relaxation of the system (3) is now obtained if we replace the uncertain operators ϕ and Δ by any signal pair $(\chi,\omega) \in \mathbf{L}_2[0,T]$ that satisfies (4). Then an upper bound on ζ in (3) at time t can be obtained by solving the optimization problem

$$\sup |C\chi(t)|^2 \quad \text{subject to}$$
$$\begin{cases} \dot{\chi} = A(t)\chi + B_1\omega + B_2\varepsilon, \quad \chi(0) \in \mathcal{X}_0, \\ \int_0^t \sigma_k(\chi,\omega)d\tau \geq 0, \ k = 1,\dots,N, \\ \int_0^t |\varepsilon(\tau)|^2 d\tau \leq \eta(t), \ (\chi,\omega) \in \mathbf{L}_2[0,T] \end{cases} \qquad (5)$$

where $A(t) = A + B_1\varphi_z(z_0(t),t)C$. If we can show that the optimal objective of (5) is less than $r(t)^2$ for all $t \in [0,T]$ then the condition in Proposition 1 is satisfied. However, (5) is generally a nonconvex optimization problem that we need to solve for all $t \in [0,T]$. We will make the problem tractable by relaxing the problem further using the S-procedure [11, 9] and then sample the time-interval $[0,T]$ on a suitably chosen grid.

S-Procedure: Let $t \in [0,T]$, $v = \begin{bmatrix}\varepsilon^T & \omega^T\end{bmatrix}^T$, $B = \begin{bmatrix}B_1 & B_2\end{bmatrix}$, and define

$$\mathcal{Q}_1 = \{(\chi,v) \in \mathbf{L}_2[0,t] : \dot{\chi} = A\chi + Bv; \ \chi(0) \in \mathcal{X}_0\},$$
$$\mathcal{Q}_2 = \{(\chi,v) \in \mathcal{Q}_1 : \int_0^t \sigma_k(\chi,\omega)d\tau \geq 0, \ k = 1,\dots,N, \ \int_0^t |\varepsilon|^2 d\tau \leq \eta(t)\}.$$

Then for any $\lambda \geq 0$ and $\gamma \geq 0$ we have the relation

$$\begin{aligned}(5) &= \sup_{\mathcal{Q}_2} |C\chi(t)|^2 \\ &\leq \gamma\eta(t) + \sup_{\mathcal{Q}_1}\left\{|C\chi(t)|^2 + \int_0^t \sigma(\chi,v,\lambda,\gamma)d\tau\right\} \qquad (6)\end{aligned}$$

where

$$\sigma(\chi,v,\lambda,\gamma) = \sum_{k=1}^N \lambda_k\sigma_k(\chi,\omega) - \gamma|\varepsilon|^2.$$

Sparse sampling: If the quadratic optimization problem (6) has a finite solution then we can use the solution to the corresponding Riccati equation to generate a bound for ζ in (3) over an interval. This gives a rigorous way of obtaining a "sparse" sampling of the time interval $[0,T]$.
Based on these two ideas we can prove

Proposition 2. *Let the sampling points* $0 < T_1 < T_2 < \ldots < T_M = T$ *be given. Assume that* $\lambda \geq 0$ *and* $\gamma \geq 0$ *are chosen such that*

$$\sigma(\chi,v,\lambda,\gamma) =: \begin{bmatrix}\chi \\ v\end{bmatrix}^T \begin{bmatrix}Q & F \\ F^T & R\end{bmatrix}\begin{bmatrix}\chi \\ v\end{bmatrix} \qquad (7)$$

have $Q \geq 0$ *and* $R < 0$. *If there exist solutions* $P_m(t) = P_m(t)^T$ *(defined over* $[0,T_m]$*) to the Riccati equations*

$$\dot{P}_m + A^TP_m + P_mA + Q = (P_mB + F)^TR^{-1}(P_mB + F),$$

[4] Well-posed will here mean that there exists a unique solution for each $\Delta \in \mathcal{B}_{TV}(\mu)$, $z \in B_{z_0}(r)$ and $\varepsilon \in C[0,T]$.

where $P_m(T_m) = C^TC$, *then we have (for* $t \in [T_{m-1}, T_m]$*)*

$$|C\chi(t)|^2 \leq \rho_m(t)\left(\gamma\eta(t) + \chi(0)^T P_m(0)\chi(0)\right),$$

where

$$\rho_m(t) = \begin{cases} \dfrac{|C|^4}{CP_m(t)C^T}, & C \neq 0, \\ 0, & C = 0. \end{cases}, \quad t \in [T_{m-1}, T_m]$$

Hence, the criterion in Proposition 3 is satisfied if

$$\rho_m(t)\left(\gamma \sup_{\varepsilon\in\mathcal{E}} \int_0^t |\varepsilon(\tau)|^2 d\tau + \sup_{\chi_0\in\mathcal{X}_0} \chi_0^T P_m(0)\chi_0\right) \leq r(t)^2, \tag{8}$$

for $t \in [T_{m-1}, T_m]$ *and* $m = 1, \ldots, M$.

Remark 2. If we minimize the right-hand side of (6) over $\gamma \geq 0$ and $\lambda \geq 0$ then under special conditions it is sometimes possible to get equality in (6), see [9].

Remark 3. The sampling points will be determined in reverse order according to the following algorithm. First let $T_1' = T$. We let T_2' be the minimal $t \in [0, T)$ such that (8) holds. We continue to define $T_3', T_4', \ldots$ in this manner until (8) holds over $[0, T_M']$. Use $T_m = T'_{M+1-m}$ $(m = 1, \ldots, M)$ as sampling points.

4 Dynamic Uncertainties

We now consider systems on the form

$$\begin{aligned} \dot{x} &= Ax + B_1\varphi(z,t) + B_2 w, \\ z &= C_1 x, \quad w = \Delta v, \\ v &= C_2 x + D_{21}\varphi(z,t) + D_{22} w, \end{aligned} \tag{9}$$

where A is Hurwitz, $\Delta \in \mathcal{B}(\mu)$, and $\varphi : \mathbf{R}^m \times [0, T] \to \mathbf{R}^p$ is defined as before. The system is called *nominal* when $\Delta = 0$ and we assume there exists a unique solution $x_0 \in C[0, T]$ to the nominal system. We are again interested in verifying the existence of a unique solution to (9) in a neighborhood of the nominal solution. The output z will again be the most important signal.
Tube Condition: Given the nominal trajectory $z_0 = C_1 x_0$ and the tube

$$B_{z_0}(r) = \{z \in C[0, T] : |z(t) - z_0(t))| \leq r(t)\},$$

check if for each $\Delta \in \mathcal{B}(\mu)$, system (9) has a unique solution $z \in B_{z_0}(r)$.

It is no restriction to assume that the nominal system satisfies[5] $x_0(0) = 0$, and thus $z_0(0) = 0$. We will consider an equivalent integral equation for the output z in our further development. Let $H_\Delta(s) = \mathcal{F}_l(H, \Delta)$, where

$$H(s) = \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix}$$

where $H_{ij} = C_i(sI - A)^{-1}B_j + D_{ij}$, $i, j = 1, 2$ (we use $D_{11} = 0$ and $D_{12} = 0$). We assume that H_Δ is stable, i.e., the corresponding weighting function $h_\Delta \in \mathbf{L}_1$, for all $\Delta \in \mathcal{B}(\mu)$. The output z in (9) can then equivalently be defined by the integral equation

$$\begin{aligned} z(t) &= \int_{-\infty}^t h_\Delta(t-\tau)\varphi(z(\tau),\tau)d\tau \\ &= \int_0^t h_\Delta(t-\tau)\varphi(z(\tau),\tau)d\tau + z^0(t), \end{aligned} \tag{10}$$

where the initial conditions response is

$$z^0(t) = \int_{-\infty}^0 h_\Delta(t-\tau)\varphi(z(\tau),\tau)d\tau. \tag{11}$$

The initial conditions response in (11) is rather complicated in general. We will first show that it can be split up into two terms, one that corresponds to an uncertain initial condition and another that is resulting from the energy stored in the uncertainty

Lemma 1. *Let*

$$\widehat{H}_\Delta(s) = \Delta(I - H_{22}\Delta)^{-1}H_{21}. \tag{12}$$

Then the initial conditions response, (11), can equivalently be written as

$$z^0(t) = C_1 e^{At}x(0) + \int_0^t h_{12}(t-\tau)e(\tau)d\tau$$

where

$$\begin{aligned} e(t) &= \int_{-\infty}^{\min(t,0)} \widehat{h}_\Delta(t-\tau)\varphi(z(\tau),\tau)d\tau \\ x(0) &= \int_{-\infty}^0 e^{-A\tau}(B_1\varphi(z(\tau),\tau) + B_2 e(\tau))d\tau. \end{aligned}$$

We now get the following sensitivity derivative of the initial conditions response

$$\frac{dz^0}{d\Delta}(t) = C_1 e^{At}\frac{dx(0)}{d\Delta} + \int_0^t h_{12}(t-\tau)\frac{de}{d\Delta}(\tau)d\tau. \tag{13}$$

It is generally not possible to get any rigorous bounds on these derivatives unless we have good prior information of $z(t)$ for $t \leq 0$ and the memory properties of the uncertainty. We will assume that we have sets $\mathcal{X} \subset \mathbf{R}^n$ and $\mathcal{E}_2 \subset C[0, T]$ such that

$$\frac{dx(0)}{d\Delta} \cdot \widehat{\Delta} \in \mathcal{X} \tag{14}$$

$$(\frac{de}{d\Delta} \cdot \widehat{\Delta})(t) \in \mathcal{E}_2 \tag{15}$$

[5] Just transform the state of the system according to $x(t) - x_0(0) \to x(t)$, where $x_0(0)$ is a solution of the fixed point equation $x = -A^{-1}B_1\varphi(Cx, 0)$. The new dynamics is obtained by replacing B_1, D_1 and φ in (9) by $\widehat{B}_1 = \begin{bmatrix} B_1 & A \end{bmatrix}$, $\widehat{D}_{21} = \begin{bmatrix} D_{21} & C_2 \end{bmatrix}$, $\widehat{\varphi}(z,t) = \begin{bmatrix} \varphi(z + z_0(0), t)^T & x_0^T \end{bmatrix}^T$.

for all $\Delta, \widehat{\Delta} \in \mathcal{B}(\mu)$, $z \in B_{z_0}(r)$. Typically we let (14) be an ellipsoidal bound and (15) an energy bound. These bounds should reflect our assumptions on the size of the transient response for different uncertainties. One simple case is when the system initially is in stationarity and $z(0) = 0$ uniquely solves $z(0) = H_\Delta(0)\varphi(z(0), 0)$ for all $\Delta \in \mathcal{B}(\mu)$. Then $\mathcal{E}_2 = \{0\}$ and $\mathcal{X} = \{0\}$.

The next result is analogous to Proposition 1.

Proposition 3. *Assume we have sets $\mathcal{X}$ and $\mathcal{E}_2$ such that (14) and (15) hold and that $H_\Delta = \mathcal{F}_l(H, \Delta)$ is stable. Let*

$$\widetilde{A} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad \widetilde{B}_1 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad \widetilde{B}_2 = \begin{bmatrix} B_2 \\ 0 \end{bmatrix}, \quad \widetilde{B}_3 = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$

$$\widetilde{C}_1 = \begin{bmatrix} C_1 & C_1 \end{bmatrix}, \quad \widetilde{C}_2 = \begin{bmatrix} C_2 & 0 \end{bmatrix},$$

$\mathcal{X}_0 = \{(0, \chi) : \chi \in \mathcal{X}\}$*, and finally* $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$ *where*

$$\mathcal{E}_1 = \{\varepsilon(t) = (N_{\Delta,\widehat{\Delta}}\varphi(z, \cdot))(t); \Delta, \widehat{\Delta} \in \mathcal{B}(\mu),\ z \in B_{z_0}(r)\}$$

and where $n_{\Delta,\widehat{\Delta}}$ is the impulse response function corresponding to the transfer function

$$N_{\Delta,\widehat{\Delta}} := \frac{d\widehat{H}_\Delta}{d\Delta} \cdot \widehat{\Delta} = (I - \Delta H_{22})^{-1}\widehat{\Delta}(I - H_{22}\Delta)^{-1}H_{21}.$$

If the system

$$\begin{aligned} \dot{\chi} &= \widetilde{A}\chi + \widetilde{B}_1\omega_1 + \widetilde{B}_2\omega_2 + \widetilde{B}_3\varepsilon, \quad \chi(0) \in \mathcal{X}_0, \ \varepsilon \in \mathcal{E} \\ \zeta &= \widetilde{C}_1\chi \\ \omega_1 &= \varphi_z(z(t), t)\widetilde{C}_1\chi \\ \omega_2 &= \Delta(\widetilde{C}_2\chi + D_1\omega_1 + D_2\omega_2) \end{aligned} \tag{16}$$

is well-posed[6] *and if $|\zeta(t)| \le r(t)$ for all $\Delta \in \mathcal{B}_{TV}$, $z \in B_{z_0}(r)$, and $\varepsilon \in \mathcal{E}$, then for each $\Delta \in \mathcal{B}_{TV}(\mu)$, system (9) has a unique solution $z \in B_{z_0}(r)$.*

Verification of the condition in the above proposition can be done using the same ideas as in the previous section.

5 Example

We will consider step response analysis of the control system in Figure 2. It is assumed that the saturation nonlinearity is

$$\mathrm{sat}(z) = \frac{z}{\sqrt{1 + z^2}},$$

[6] Well-posed again means that there exists a unique solution for each $\Delta \in \mathcal{B}_{TV}(\mu)$, $z \in B_{z_0}(r)$ and $\varepsilon \in C[0, T]$.

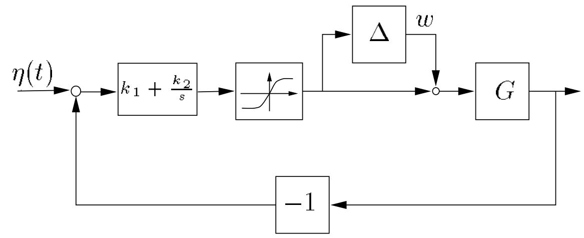

Figure 2: PI-control system.

and $G(s) = C(sI - A)^{-1}B$, where A is Hurwitz. The system has the state space realization

$$\begin{aligned} \dot{x} &= \begin{bmatrix} A & 0 \\ -C & -1 \end{bmatrix} x + \begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix} \varphi(z, t) + \begin{bmatrix} B \\ 0 \end{bmatrix} w, \\ z &= \begin{bmatrix} -k_1 C & k_2 \end{bmatrix} x, \\ w &= \Delta(\begin{bmatrix} 1 & 0 \end{bmatrix} \varphi(z, t)), \end{aligned}$$

where Δ is a scalar uncertainty in any of $\mathcal{B}_0(\mu)$, $\mathcal{B}(\mu)$ and $\mathcal{B}_{TV}(\mu)$, and

$$\varphi(z, t)) = \begin{bmatrix} \mathrm{sat}(z + k_1\eta(t)) \\ \begin{bmatrix} 0_{\mathrm{size(C)}} & 1 \end{bmatrix} x + \eta(t) \end{bmatrix}.$$

The reference input is assumed to be a unit step function, i.e., $\eta(t) = \theta(t)$, which means that we have $x(0) = 0$ ($z(0) = 0$) for all $\|\Delta\| \le \mu$. The initial conditions response is thus zero. Linearization gives

$$\begin{aligned} \dot{\chi} &= \begin{bmatrix} A - k_1 B\psi(t)C & B\psi(t)k_2 \\ -C & 0 \end{bmatrix} \chi + \begin{bmatrix} B & B \\ 0 & 0 \end{bmatrix} \omega + \begin{bmatrix} B \\ 0 \end{bmatrix} \varepsilon, \\ \zeta &= \begin{bmatrix} -k_1 C & k_2 \end{bmatrix} \chi, \end{aligned}$$

where

$$\psi(t) = \mathrm{sat}'(z_0(t) + k_1\eta(t)) = \frac{1}{(1 + (z_0(t) + k_1\eta(t))^2)^{3/2}},$$

and $\omega = \begin{bmatrix} \omega_1 & \omega_2 \end{bmatrix}$. The first component is $\omega_1 = \phi(t)\zeta$, where

$$\phi(t) = \{\mathrm{sat}'(z + k_1\eta)(t) - \mathrm{sat}'(z_0 + k_1\eta)(t) : z \in B_{z_0}(r)\}.$$

The second component is $\omega_2 = \Delta(\mathrm{sat}'(\zeta + k_1\eta)\zeta)$. Finally, we have

$$\varepsilon \in \mathcal{E} = \{(\widehat{\Delta}(\mathrm{sat}(z + k_1\eta))(t) : z \in B_{z_0}(r); \widehat{\Delta} \in \mathcal{B}(\mu)\}.$$

We need to find a suitable IQC for the uncertain signal ω. We can use

$$\int_0^t \sigma(\chi, \omega)d\tau \ge 0, \quad t \in [0, T],$$

where the quadratic form is

$$\sigma(\chi, \omega) = \lambda_1(\nu_1(t)|\zeta|^2 - |\omega_1|^2) + \lambda_2(\mu^2\nu_2(t)|\zeta|^2 - |\omega_2|^2).$$

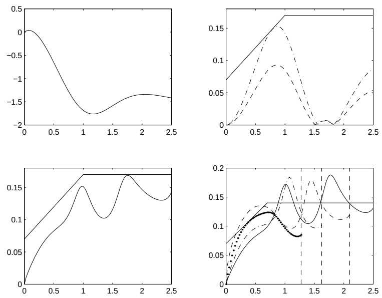

Figure 3: Simulations and numerical bounds.

Here $\lambda_1, \lambda_2 \geq 0$, and

$$\nu_1(t) = \sup_{z \in B_{z_0}(r)} |\text{sat}'(z + k_1\eta)(t) - \text{sat}'(z_0 + k_1\eta)(t)|^2,$$

$$\nu_2(t) = \sup_{z \in B_{z_0}(r)} |\text{sat}'(z(t) + k_1\eta(t))|^2.$$

Finally, we use

$$\int_0^t \varepsilon^2(\tau)d\tau \leq \mu^2 \sup_{z \in B_{z_0}(r)} \int_0^t \text{sat}(z(\tau) + k_1\eta(\tau))^2 d\tau,$$

which can be estimated numerically.

Let us now consider the case when $G(s) = \dfrac{8}{s^2+3s+4}$, $\eta(t) = \theta(t)$ (the unit step function), $k_1 = 2$, and $k_2 = 1$, $T = 2.5$. The nominal response of $z(t)$ is given in Figure 3. We want to show that the perturbed trajectory is within $B_{z_0}(r)$ for all $\|\Delta\| \leq 0.05$ (i.e., $\mu = 0.05$), where $r(t)$ is the solid line in the upper right plot in Figure 3. The plot also shows the deviations $|z - z_0|$ when $\Delta = -0.05$ and $\Delta = -0.081$. We see that the tube condition is violated for a rather small constant parameter, $\Delta = -0.081$, and it is not at all clear that no dynamic perturbation satisfying $\|\Delta\| \leq 0.05$ will move the trajectory out from the tube. However, we will show that the system actually has the desired robustness property.

We choose $\lambda_1 = 70$, $\lambda_2 = 120$ and $\gamma = 7.6$ and one sampling time $T_m = 2.5$. The Riccati equation in Proposition 2 has a solution for this choice of parameters and the bound $\gamma\rho(t)\sup_{\varepsilon\in\mathcal{E}} \int_0^T |\varepsilon|^2 d\tau \leq r(t)^2$, $t \in [0, 2.5]$ is satisfied, as is illustrated in the lower left plot in Figure 3. The gap between our proved robustness bound $\mu = 0.05$ and the upper bound $\mu = 0.081$, obtained by simulation, is only a factor 1.62. This appears to be rather good, in particular if we consider that the upper bound is only based on a non-dynamic perturbation and we have not optimized the IQCs.

In the lower right plot of Figure 3 we consider a tighter $r(t)$. In this case we use the same values of λ_1 and λ_2 but four sampling times. Simulation shows that the system violates the tube constraint when $\Delta = -0.074$. Hence, the conservatism is at most $0.074/0.05 = 1.48$.

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