Infinite Dimensional Models : Approximation and Realization.

Nicolas Guijarro*, Laurent Lefèvre**,
Geneviève Dauphin-Tanguy*.
* Ecole Centrale de Lille, Cité Scientifique, BP48
59651 Villeneuve d'Ascq Cedex France
e-mail : {guijarro,gdt}@ec-lille.fr
** Laboratoire d'Automatique de Grenoble (I.N.P.G.)
ESISAR, 50 rue Barthélémy de Laffemas, BP 54
26902 Valence Cedex 9 France
e-mail : laurent.lefevre@esisar.inpg.fr

Abstract

In this paper, we introduce a procedure to treat passive functional nodes in bond graphs. This procedure is carried out in three steps. First we approximate the initial infinite dimensional model by a finite one with huge dimension. Then we reduce it to a lower dimension model. Finally we realize this latter model by a lumped parameter electrical network.

1 Problem Statement

In system theory and particularly when using the bond graph approach [5], there exists a lot of tools to analyze, synthesize and control (to mention the principal areas) finite dimensional models. Frequently we have to deal with infinite dimensional models built when studying distributed parameter systems or delayed systems for example. When this kind of system is introduced in a bond graph it can be defined as a functional node [12]. This node prevents us from the use of classical tools. To overcome this difficulty we could develop specific tools for bond graphs with functional nodes or, as it will be set out here, approximate the functional nodes by "classical" nodes (a combination of simple passive elements such as R,C and L), which amounts to build an equivalent electrical circuit composed of R, L and C cells. The models, considered in this paper, are linear (L), time invariant (T.I.) and passive. The passivity property, in the case of an L.T.I. finite dimensional model, can be characterized by the following theorem [4].

Theorem 1 *An L.T.I. finite dimensional model with transfer function $H(s)$ is passive ($H(s)$ is a positive real function) iff*

$$H(s) \in \mathbb{R} \textit{ if } s \in \mathbb{R} \quad \textit{and} \quad \mathrm{Re}(H(s)) \geqq 0 \textit{ if } \mathrm{Re}(s) \geqq 0 \tag{1}$$

The input u and output y signals are chosen here conjugate power variables (voltage and current or force and velocity) so that $u.y$ denotes the power supplied to the system[1] [15].

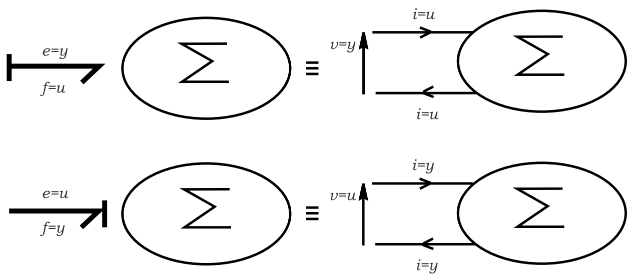

Figure 1 : Bond graph and electrical network representation of a 1-port.

The general procedure we want to apply to the model studied here can be summarized as follows :

1. approximation of the infinite dimensional model by a large scale finite dimensional one (say of dimension N),
2. reduction of the model's dimension to obtain a model of dimension $n << N$,
3. realization of the latter model by a R-L-C network or a bond graph.

Steps *1)* and *2)* of the procedure must conserve the key property of the initial model, i.e. its passivity.

[1] We suppose in the following that the system is a 1-port, that means that the power is supplied to the system by one single access, as shown figure 1.

In the second section we review some techniques to approximate infinite dimensional models by a finite dimensional one. Next, the third section recalls briefly a technique used in reduced-order modeling. Then the fourth section gives a way to realize the low-dimension model by using lumped passive linear elements (*R-L-C* type). Finally in the sequel some numerical experiments illustrate the whole procedure and concluding remarks are stated.

2 Infinite Dimensional Model Approximation

In a few words we can summarize the reduction or approximation problem as follows :

- Given a model, find another one which is fast to compute, reduced-order and accurate.

Among the three properties that define the model built through an approximation or reduction process, the second (reduced-order) is the only one that does not need to be developed. In fact it is easy to quantify the order decrease. On the other hand, the rapidity of the computation depends on the numerical methods involved, typically it can be measured by the complexity. Last but not least, the accuracy is the most subjective feature to define. Depending on the way we want to quantify the error between the original model and the reduced-order one, we have got to choose a different norm (e.g. $\mathcal{H}_\infty$ for the supremum). Furthermore, the model will be called accurate if it retains some properties like stability or passivity. For example, balanced truncation retains the largest singular values and gives an evaluation of the $\mathcal{H}_\infty$ norm of the error through the sum of the remaining ones. In the reduction process used in this procedure, moment matching is the key feature.

The in-between model is the result of a finite dimension approximation of the initial model. Furthermore some properties, like stability and passivity, must remain. The most important one is passivity. As a matter of fact, passivity has a closure characteristic. Unlike stability, in the interconnection of passive networks, the passivity property hold. According to the sort of infinite dimensional model the plan to follow will differ. Three principal classes of models can be shown up :

- *Partial Differential Equations (P.D.E.) models* : for this kind of models we can for example use a modified finite differences spatial discretization to obtain the finite dimensional model. This scheme gives a large sparse structured matrix (tridiagonal for a second order centered differences schema).

- *Delay models* : a reduction process like the one presented in [14] could be used but in general passivity is not preserved. Certain classes of delay models can be equivalently described by P.D.E. models (like wave equations). We can also expand in several points the irrational transfer function to obtain a multipoint *Padé* approximant [10].

- *Fractal models* : the *Numerical Experiments* section presents two ways to produce a finite dimensional reduced-order model. The first method gives in one stage at a low computational cost the reduced-order model. The second one gives in two stages a reduced-order model at a more expensive cost than the first but with lower order for comparable accuracy.

3 Large Scale Model Reduction

In this section we will recall the *Krylov* subspaces techniques used in reduced-order modeling.

The previous section methods provide us with an approximate but large scale finite dimensional model. In state-space formalism we can describe it as follows [7] :

$$\left\{ \begin{array}{l} \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{c}^T\mathbf{x}(t) \end{array} \right. \tag{2}$$

In (2) $\mathbf{E},\ \mathbf{A} \in \mathbb{R}^{N\times N},\ \mathbf{b} \in \mathbb{R}^N$, and $\mathbf{c} \in \mathbb{R}^N$ are given matrices. The matrices $\mathbf{E}$ and $\mathbf{A}$ are allowed to be singular but we assume that the pencil $(s\mathbf{E} - \mathbf{A})$ is regular (there exists points, $s_0 \in \mathbb{C}$, such that the matrix $(s_0\mathbf{E} - \mathbf{A})$ is regular). In the case of a singular matrix $\mathbf{E}$ then the first equation in (2) represents a set of linear differential-algebraic equations (D.A.E.).

Applying the *Laplace* transform on (2), one obtains

$$H(s) = \mathbf{c}^T(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{b} \tag{3}$$

Similarly for the reduced-order model (n) :

$$\left\{ \begin{array}{l} \mathbf{E}_n\dot{\mathbf{z}}(t) = \mathbf{A}_n\mathbf{z}(t) + \mathbf{b}_n\mathbf{u}(t) \\ \hat{\mathbf{y}}(t) = \mathbf{c}_n^T\mathbf{z}(t) \end{array} \right. \tag{4}$$

and

$$H_n(s) = \mathbf{c}_n^T(s\mathbf{E}_n - \mathbf{A}_n)^{-1}\mathbf{b}_n \tag{5}$$

The model (4) must, in a certain sense, approximate the model (2). The classical reduction methods, like balanced truncation or *Hankel*-norm optimal approximations, do not take advantage of the structure of the matrices. In our case, the matrices are large scale, sparse and/or structured. The reduction method presented makes use of *Krylov* subspaces techniques which exploit the characteristics of the matrices. The reduced-order transfer function must match $2n$ moments of certain series expansions of the initial model transfer function. A table which summarizes the different matchings can be found in [8].

For an expansion point $s_0 = \sigma$, it follows that

$$\begin{aligned} H(s) &= H_n(s) + \mathcal{O}((s-\sigma)^{2n}) && (6)\\ &= \sum_{i=0}^{2n-1} m_i (s-\sigma)^i + \mathcal{O}((s-\sigma)^{2n}) \end{aligned}$$

According to [8]

$$m_i = \mathbf{c}^T . \underbrace{(-(\boldsymbol{\sigma}\mathbf{E}-\mathbf{A})^{-1}\mathbf{E})}_{\mathbf{F}}{}^i \underbrace{(\boldsymbol{\sigma}\mathbf{E}-\mathbf{A})^{-1}\mathbf{b}}_{\mathbf{r}} \quad (7)$$

$$m_i = \mathbf{c}^T.(\mathbf{F}^i\mathbf{r}) \quad (8)$$

Rewriting (7) for odd and even indexes, it yields

$$m_{2i} = ((\mathbf{F}^T)^i\mathbf{c})^T.(\mathbf{F}^i\mathbf{r}) \quad (9)$$

and

$$m_{2i+1} = ((\mathbf{F}^T)^i\mathbf{c})^T.\mathbf{F}(\mathbf{F}^i\mathbf{r}) \quad (10)$$

for $i = 0, ..., n$.

A *Krylov* space is defined by :

$$\mathcal{K}_j(\mathbf{G},\mathbf{g}) := span\{\mathbf{g}, \mathbf{G}\mathbf{g}, \mathbf{G}^2\mathbf{g}, \ldots, \mathbf{G}^{j-1}\mathbf{g}\} \quad (11)$$

where G is a square matrix and g a vector.

The two *Krylov* spaces

$$\begin{aligned} \mathcal{K}_n(\mathbf{F},\mathbf{r}) &= span\{\mathbf{r}, \mathbf{F}\mathbf{r}, \mathbf{F}^2\mathbf{r}, \ldots, \mathbf{F}^{n-1}\mathbf{r}\} && (12)\\ &= span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_n\}\\ \mathcal{K}_n(\mathbf{F}^T,\mathbf{c}) &= span\{\mathbf{c}, \mathbf{F}^T\mathbf{c}, (\mathbf{F}^T)^2\mathbf{c}, \ldots, (\mathbf{F}^T)^{n-1}\mathbf{c}\}\\ &= span\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, ..., \mathbf{w}_n\} \end{aligned}$$

allow the retranscription of (9) and (10) as follows

$$\begin{aligned} m_{2i} &= \mathbf{w}_{i+1}^T.\mathbf{v}_{i+1} && (13)\\ m_{2i+1} &= \mathbf{w}_{i+1}^T.\mathbf{F}\mathbf{v}_{i+1} \end{aligned}$$

for $i = 0, ..., n$.

Classical *Padé* approximant calculation uses explicit moment matching. But explicit calculation of the m_i's generates the vector sequence $\mathbf{r}, \mathbf{F}\mathbf{r}, \mathbf{F}^2\mathbf{r}, \ldots, \mathbf{F}^n\mathbf{r}$ which converges, sometimes rapidly, to the eigenvector associated with the predominant eigenvalue of $\mathbf{F}$. So the resulting *Padé* approximant will firstly exhibit the model behavior corresponding to the largest eigenvalue of $\mathbf{F}$. Consequently we need a method to implicitly calculate the moments such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_n\}$ form a basis of $\mathcal{K}_n(\mathbf{F},\mathbf{r})$ and $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, ..., \mathbf{w}_n\}$ a basis of $\mathcal{K}_n(\mathbf{F}^T,\mathbf{c})$. The Lanczos algorithm is an efficient method to solve this problem.

The following gives a sketch of the algorithm, it yields the iterative construction of the basis $\{\mathbf{v}_i\}_{i=1}^n$ and $\{\mathbf{w}_i\}_{i=1}^n$.

The vectors $\{\mathbf{v}_i\}_{i=1}^{q+1}$ and $\{\mathbf{w}_i\}_{i=1}^{q+1}$ are biorthogonal $(0 \leq q \leq n-1)$:

$$\mathbf{w}_j^T\mathbf{v}_k = \begin{cases} \delta_j, & \text{if } j = k\\ 0, & \text{if } j \neq k \end{cases} \qquad j,k = 1,2,\ldots,q+1. \quad (14)$$

Setting $\mathbf{W}_q = [\mathbf{w}_1 ... \mathbf{w}_q]$ and $\mathbf{V}_q = [\mathbf{v}_1 ... \mathbf{v}_q]$ we have

$$\mathbf{W}_q^T\mathbf{V}_q = \mathbf{D}_q = diag(\delta_1, \delta_2, \ldots, \delta_q) \quad (15)$$

0. Set $\rho_1 = ||\mathbf{r}||_2$, $\eta_1 = ||\mathbf{c}||_2$, $\mathbf{v}_1 = \mathbf{r}/\rho_1$ and $\mathbf{w}_1 = \mathbf{c}/\eta_1$.

Set $\mathbf{v}_0 = \mathbf{w}_0 = 0$ and $\delta_0 = 1$.

For $n = 1, ..., q$ do :

1. Compute $\delta_n = \mathbf{w}_n^T\mathbf{v}_n$.
2. Set
$$\alpha_n = \frac{\mathbf{w}_n^T\mathbf{F}\mathbf{v}_n}{\delta_n}, \quad \beta_n = \frac{\delta_n}{\delta_{n-1}}\eta_n, \quad \gamma_n = \frac{\delta_n}{\delta_{n-1}}\rho_n. \quad (16)$$
3. Set
$$\mathbf{v} = \mathbf{F}\mathbf{v}_n - \mathbf{v}_n\alpha_n - \mathbf{v}_{n-1}\beta_n, \quad (17)$$
$$\mathbf{w} = \mathbf{F}^T\mathbf{w}_n - \mathbf{w}_n\alpha_n - \mathbf{w}_{n-1}\gamma_n. \quad (18)$$
4. Set $\rho_{n+1} = ||\mathbf{v}||_2$, $\eta_{n+1} = ||\mathbf{w}||_2$ and
$$\mathbf{v}_{n+1} = \frac{\mathbf{v}}{\rho_{n+1}}, \; \mathbf{w}_{n+1} = \frac{\mathbf{w}}{\eta_{n+1}} \quad (19)$$

This algorithm yields to the construction of two matrices

$$\mathbf{T}_q = \begin{bmatrix} \alpha_1 & \beta_2 & 0 & \cdots & 0 \\ \rho_2 & \alpha_2 & \beta_3 & \ddots & \vdots \\ 0 & \rho_3 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \beta_q \\ 0 & \cdots & 0 & \rho_q & \alpha_q \end{bmatrix} \tag{20}$$

and

$$\tilde{\mathbf{T}}_q = \begin{bmatrix} \alpha_1 & \gamma_2 & 0 & \cdots & 0 \\ \eta_2 & \alpha_2 & \gamma_3 & \ddots & \vdots \\ 0 & \eta_3 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \gamma_q \\ 0 & \cdots & 0 & \eta_q & \alpha_q \end{bmatrix} \tag{21}$$

that approximate, in a certain sense, respectively $\mathbf{F}$ and $\mathbf{F}^T$. The following equations exhibit how they are related.

$$\mathbf{F}\mathbf{V}_q = \mathbf{V}_q\mathbf{T}_q + [0 \cdots 0\, \mathbf{v}_{q+1}]\rho_{q+1} \tag{22}$$

$$\mathbf{F}^T\mathbf{W}_q = \mathbf{W}_q\tilde{\mathbf{T}}_q + [0 \cdots 0\, \mathbf{w}_{q+1}]\eta_{q+1} \tag{23}$$

Using the biorthogonality property (14) with (22) and (23), it yields

$$\mathbf{W}_q^T\mathbf{F}\mathbf{V}_q = \underbrace{\mathbf{W}_q^T\mathbf{V}_q}_{\mathbf{D}_q}\,\mathbf{T}_q + \underbrace{\mathbf{W}_q^T[0 \cdots 0\, \mathbf{v}_{q+1}]}_{\mathbf{O}}\,\rho_{q+1} \tag{24}$$

$$\begin{aligned} \mathbf{V}_q^T\mathbf{F}^T\mathbf{W}_q &= \underbrace{\mathbf{V}_q^T\mathbf{W}_q}_{\mathbf{D}_q}\,\tilde{\mathbf{T}}_q + \underbrace{\mathbf{V}_q^T[0 \cdots 0\, \mathbf{w}_{q+1}]}_{\mathbf{O}}\,\eta_{q+1} && (25) \\ &= \left(\mathbf{W}_q^T\mathbf{F}\mathbf{V}_q\right)^T && (26) \end{aligned}$$

From (24) and (25) we deduce (27)

$$\tilde{\mathbf{T}}_q^T\mathbf{D}_q = \mathbf{D}_q\mathbf{T}_q, \quad \tilde{\mathbf{T}}_q^T = \mathbf{D}_q\mathbf{T}_q\mathbf{D}_q^{-1} \tag{27}$$

The matrix $\mathbf{T}_q$ is the best approximation of the matrix $\mathbf{F}$ in the sense of an oblique projection onto $\mathcal{K}_q(\mathbf{F}, \mathbf{v}_1)$ orthogonally to $\mathcal{K}_q(\mathbf{F}^T, \mathbf{w}_1)$. Moreover $\mathbf{T}_q$ is the best approximation of the matrix $\mathbf{F}$ in the sense of matching the maximal number of moments [7]. Consequently we obtain

$$\left\{ \begin{array}{c} \mathbf{E}_n = \mathbf{D}_n\mathbf{T}_n,\ \mathbf{A}_n = \mathbf{D}_n(\mathbf{I}_n + \sigma\mathbf{T}_n), \\ b_n = \gamma_1 e_1,\ c_n = \beta_1 e_1 \end{array} \right. \tag{28}$$

and the final model (4) is the result of a *Krylov-Lanczos* reduction [11].

4 Realization

Knowing a state-space representation of the reduced-order model we can realize a network or a bond graph.

In the 1-port case the determination of the electrical network or bond graph component's values can be achieve thanks to the *Brune* process [4]. In the m-port case various processes described in [3] could be used.

Z designates the impedance at a port of the studied system, i.e. in the electrical domain voltage drop v and current i are in relation through $V(s) = Z(s).I(s)$. Similarly Y designates the admittance, i.e. $I(s) = Y(s).V(s)$. The transfer function in the complex frequency domain (*Laplace* transform of the system's impulsive response) $H(s)$ can be an impedance or an admittance so we call it an immittance.

In the general case, realization methods (like *Foster* extraction process [6]) based on the extraction of poles and zeros of the transfer function are not efficient. These methods can only extract poles on the imaginary axis or on the negative real axis. In fact, the extraction process will stop when a minimum function appears. The *Brune* process precisely achieves the realization of this type of function.

5 Numerical Experiments

To illustrate the procedure exposed in the previous sections we make some numerical experiments involving a fractal type model. Such models appear in several applications like relaxation behavior of polarized impedances, dielectrics and interfaces, transmission lines, cardiac rythm [1] and fractal hydraulic impedance [12] to state a few.

Fractional power pole model is expressed as :

$$H(s) = \frac{1}{\left(1+\frac{s}{p_T}\right)^m} \qquad \begin{array}{rl} p_T & : \text{ relaxation time constant} \\ m & : \text{ power factor, } 0 < m < 1 \end{array} \tag{29}$$

For numerical computations we use $p_T = 5$ and $m = 0.5$.

A large scale ladder network (*Cauer* type [9]) proposed in [12], consisting of one hundred and fifty capacitors and as many resistors, yields to an approximant of (29) with an absolute gain error less than 10^{-15}dB through the frequency range of interest $0 - 100Hz$. The parameters values correspond to a *Padé* approximant of order [149/150] at the steady state frequency. This 150^{th} model is our starting point for reduction and realization.

To obtain the reduced order model we apply the *Krylov-Lanczos* algorithm with interpolation points[2] : $\omega_1 = 0$, $\omega_2 = 160$ and $\omega_3 = 375$. We match two moments per point, consequently the approximant order is six.

Remark 1 *We choose* 3 *interpolation points (order* 2 *for each point) because we reach an accuracy better than with* 2 *points (order* 3 *for each point).*

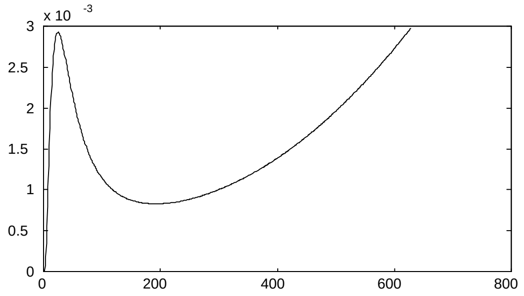

Figure 2 : Relative error on $H_6(i\omega)$ (%) versus ω(rad.s^{-1}).

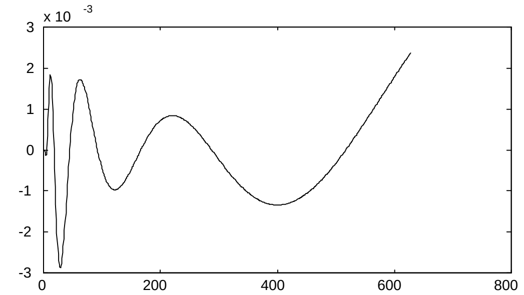

Figure 3 : Absolute phase error (rad) versus ω (rad.s^{-1}).

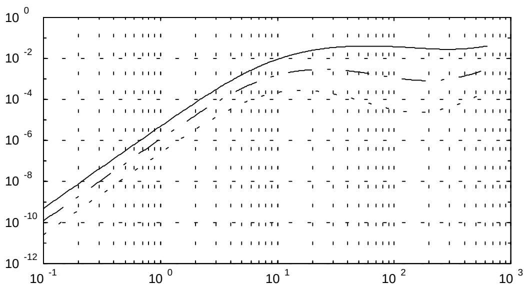

Figure 4 : Relative Gain Error for 4^{th} order (2pts : ! $_1$and ! $_2$)(-), 6^{th}order (3pts : ! $_1$, ! $_2$and ! $_3$)(-.) and 8^{th}order (2pts : ! $_1$and ! $_2$)(..) approximants.

The verification, *a posteriori*, of the passivity of the reduced-order model is achieved thanks to a positive realness test on the transfer function $H_n(s)$ [2].

[2]The points choice is heuristic.

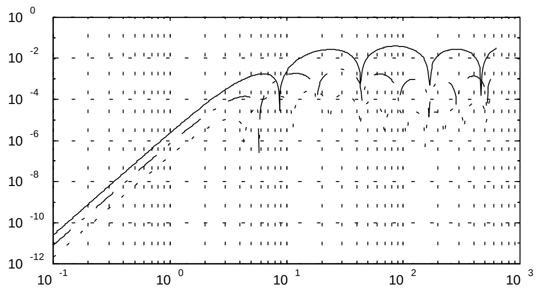

Figure 5 : Absolute Phase Error for a 4^{th} order (-), 6^{th}order (-.) and 8^{th}order (..) approximant.

Poles (p_i)	**Residues** (r_i)
-5922.0150	223.5425
-557.5193	26.8780
-138.3182	11.0672
-37.5070	6.1679
-11.06025	3.2938
-5.46583	2.0321

Table 1: Partial Fraction Expansion

Table 1 gives the result of the partial fraction expansion of $H_6(s)$.

$$H_6(s) = \sum_{i=1}^{6} r_i.\frac{1}{s - p_i} \tag{30}$$

The sum (30) can be realized by a series connection of parallel *R*-*C* cells. The impedance of a parallel *R*-*C* cell is given by :

$$Z_{R_iC_i}(s) = \frac{R_i}{\tau_i}.\frac{1}{s + \dfrac{1}{\tau_i}}, \quad \tau_i = R_iC_i. \tag{31}$$

As a matter of comparison, we used the method developed by *A. Charef, H.H. Sun, Y.Y. Tsao and B. Onaral* [1]. It appears that we need a 12^{th} order approximant to achieve a relative gain error of the same magnitude.

6 Concluding Remarks

In this paper we have exposed an approximation-reduction-realization process. This process applies potentially to a large class of infinite dimensional models through various approximation schemes (first stage of the process). Concerning those schemes, important issues are under investigation. For example, a delay model could be approximated using the method developed by *C. Hwang* and *M.-Y. Chen* [10], but generally it does not preserve passivity. There also exists methods, to approximate irrational functions by rational ones, which could be useful in lumped approximation of the characteristic impedance of *RLGC* transmission lines [13].

At the second stage of the process, which needs to be carried out if the approximant order is important, a multipoint *Padé* approximant (P.V.L. like algorithm [11], [7]) is used to reduce the large scale finite dimensional model stemmed from the approximation of the infinite dimensional model. To ensure the passivity of the reduced-order model we make a positive realness test on its transfer function. It will be interesting to find at least sufficient conditions ensuring passivity preservation.

The last stage gives a realization method of the reduced-order model through the *Brune* process (in the 1-port case).

Finally, we have given a numerical example which illustrates the whole process and provides a bench-mark for comparison with another method.

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