Generalized spectral factorization problem for discrete time polynomial matrices via quadratic difference forms

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Abstract

In this paper, we address spectral factorization problems for discrete time polynomial matrices. Main concept used in this paper is based on quadratic differential/difference forms and dissipativeness, similarly to [6], [10] and [8] which treat the polynomial matrices with no zeros on the $j\omega$ axis or the unit circle. Here, by using some inherent techniques in discrete time, we expand the spectral factorization algorithms for polynomial matrices with zeros on the unit circle via quadratic difference forms. Moreover, we show that this algorithm is also available to the singular polynomial matrices in discrete time.

Keywords: spectral factorization, discrete time, behavioral approach, quadratic difference forms, dissipativeness.

1 Introduction

1.1 Problem formulation

The spectral factorization problem we consider here is described as follows. Let

$$\Pi(\lambda^{-1},\lambda) = \Pi_n^T\lambda^{-n} + \cdots + \Pi_1^T\lambda^{-1} + \Pi_0 + \Pi_1\lambda + \cdots + \Pi_n\lambda^n \quad (1)$$

denote a given polynomial matrix. Assume that

(A). $\Pi(\lambda,\lambda^{-1})^T = \Pi(\lambda^{-1},\lambda)$,
(B). $\Pi(e^{-j\omega}, e^{j\omega}) \geq 0$ for all $\omega \in [0, 2\pi)$,
(C). $\Pi(\lambda^{-1},\lambda)$ is not a trivial zero matrix.

Then, we consider the following two problems.

Problem 1.1 *Assume (A), (B), (C), and*
(D). $\det(\Pi(\lambda^{-1},\lambda)) \neq 0$ *for almost every* $\lambda \in \mathbf{C}$.

Then, obtain a polynomial matrix $A(\lambda)$ *such that* $\Pi(\lambda^{-1},\lambda) = A(\lambda^{-1})^T A(\lambda)$ *and* $\det(A(\lambda)) \neq 0$ *for all* $\lambda \in \mathbf{C}$ *such that* $|\lambda| < 1$. □

Problem 1.2 *Assume (A), (B), (C), and*
(E). $\det(\Pi(\lambda^{-1},\lambda)) \equiv 0$.

Then, obtain a polynomial matrix $A(\lambda)$ *such that* $\Pi(\lambda^{-1},\lambda) = A(\lambda^{-1})^T A(\lambda)$ *and the rowrank of* $A(\lambda) = r$ *for all* $\lambda \in \mathbf{C}$ *such that* $|\lambda| < 1$*, where* r *is the normal rank of* $\det(\Pi(\lambda^{-1},\lambda))$. □

In this paper, we call these $A(\lambda)$ an *almost anti-Hurwitz* spectral factor. Of course, we can also formalize the almost Hurwitz spectral factorization as in the same way to the above problems. Note that the existence of the above polynomial matrices is guaranteed in e.g., [9].

1.2 Backgrounds and motivations

As is well known, the spectral factorization is one of the central issues in the systems and circuits theory (cf.[3],[9] and so on). Various computational algorithms of the spectral factors were studied and developed. Except [2],[3], [4],[13] and so on, most of those studies assumed that the polynomial matrices (spectral density matrices) are positive definite on the unit circle or $j\omega$ axis. From the practical points of view, however, it is natural to consider that spectral density polynomial matrices often have zeros on these boundaries. Thus, it is important to consider such a spectral factorization problem. As for this problem, literature [2], [3], [4],[13] proposed some algorithms. Here, we will take the different approach to the spectral factorization problem from the standpoint of *application of quadratic difference forms and discrete time dissipativeness* (cf.[8]).

Main concept used in this paper is deeply related to quadratic difference forms and dissipativeness, similarly to [6] and [8]. They are regarded as the application of Theorem 5.7 in [12] and Proposition 4.1 in [8]. These algorithms are equivalent to solving simple LMIs formed directly by coefficient matrices of given polynomial matrices, which implies that they are very useful from the computational points of view. However, these spectral factorization algortithms does not guarantee that they are applicable to polynomial matrices with zeros on the $j\omega$ axis or the unit circle . These restrictions concerning zeros result from the assumptions of Theorem 5.7 in [12] or Proposition 4.1 in [8]; " if a given polynomial matrix is positive definite on the $j\omega$ axis or the unit circle, then the maximum and the minimum storage functions for a supply rate induced by this polynomial matrices can be described by anti-Hurwitz and Hurwitz spectral factors, respectively". From the viewpoint of theoretical importance, it is meaningful to eliminate such an assumption.

From these reasons, by using some inherent techniques in discrete time , we show that the assumption of Proposition 4.1 in [8] can be eliminated, i.e., " the maximum and the minimum storage functions for a supply rate induced by a given polynomial matrices can be described by almost anti-Hurwitz and almost Hurwitz spectral factors as long as this polynomial matrix satisfies (A), (B), (C) and (E) (or (D)) ". Then, as an application of this result, we provide spectral factorization algorithms for discrete time polynomial matrices with zeros on the unit circle.

2 Preliminaries

2.1 Notations

Let $\mathbf{Z}$, $\mathbf{R}$, and $\mathbf{C}$ denote the set of integers, real numbers and complex numbers, respectively. The notation $\mathbf{R}^q$ ($\mathbf{C}^q$) denotes the set of real (complex, respectively) vectors of size q. For $\lambda \in \mathbf{C}$, $\bar{\lambda}$ denotes the conjugate of λ. For $a \in \mathbf{R}^q$, $\|a\|^2{:=}a^T a$. For $a \in \mathbf{C}^q$, a^* denotes the conjugated transpose of a and $\|a\|^2{:=}a^* a$. Let $\mathbf{R}^{p\times q}$ denote the set of real matrices of size $p \times q$. Let $(\mathbf{R}^q)^Z$ denote the set of real time series vectors of size q. For $w \in (\mathbf{R}^q)^Z$, the shift operator σ is defined by $(\sigma w)(t) := w(t{+}1)$. For $w \in (\mathbf{R}^q)^Z$ and $a \le b \in \mathbf{Z}$, $w|_{[a,b]}$ denote the behavior of w in the time interval $[a, b]$. The notation l_2^q is the set of square summable time series vectors of size q, i.e., $w \in l_2^q$ means that $\sum_{t=-\infty}^{t=\infty} \|w(t)\|^2 < \infty$. Let $l_2^q|_T$ denote the set of square summable time series vector of size q from $-\infty$ to T, i.e., $w|_{(-\infty,T]} \in l_2^q|_T$ means that $\sum_{t=-\infty}^{t=T} \|w(t)\|^2 < \infty$. Similarly, we use the notation $l_2^q|^T$ to denote the set of square summable time series vector of size q from T to ∞. Let $\mathbf{R}[\lambda]$ denote the set of polynomials in the indeterminate λ with coefficients in $\mathbf{R}$. Similarly, $\mathbf{R}[\zeta, \eta]$ denote the set of two-variable polynomials in the indeterminates ζ and η with coefficients in $\mathbf{R}$. The powers of these indeterminates may be not only nonnegative but also negative in discrete time. Particularly, $\mathbf{R}[\lambda^{-1}, \lambda]$ denote the set of two-sided polynomials which are obtained by regarding the indeterminates ζ and η in $\mathbf{R}[\zeta, \eta]$ as λ^{-1} and λ, respectively. Similarly, the set of matrix version of them are written respectively by $\mathbf{R}^{q\times q}[\lambda]$, $\mathbf{R}^{q\times q}[\zeta, \eta]$, and $\mathbf{R}^{q\times q}[\lambda^{-1}, \lambda]$ for real coefficient matrices of size $p \times q$. For a nonsingular polynomial matrix $D(\lambda)$, we call it Hurwitz (anti-Hurwitz) if $\det(D(\lambda)) \ne 0$ for all $\lambda \in \mathbf{C}$ such that $|\lambda| \ge 1$ ($|\lambda| \le 1$, respectively). For a constant matrix A, let $rank(A)$ denote the rank of A. Finally, let I_q and $0_{p\times q}$ denote the unit matrix of size $q \times q$ and the zero matrix of size $p \times q$, respectively.

2.2 Quadratic difference forms

Quadratic difference or differential forms are appropriate mathematical tools related to dissipativeness. See [12] and [8] for more details of the continuous time case and the discrete time case, respectively.

An element of $\mathbf{R}^{q\times q}[\zeta, \eta]$ is described by

$$\Phi(\zeta, \eta) = \sum_{k,l} \Phi_{kl} \zeta^k \eta^l. \qquad (2)$$

The sum in Eq.(2) ranges over the integers and is assumed to be finite, and $\Phi_{kl} \in \mathbf{R}^{q\times q}$. For $\Phi(\zeta, \eta) \in \mathbf{R}^{q\times q}[\zeta, \eta]$, let $\mathbf{R}_s^{q\times q}[\zeta, \eta]$ denote the set of two-variable polynomial matrices satisfying $\Phi(\zeta, \eta) = \Phi(\eta, \zeta)^T$. For all $w \in (\mathbf{R}^q)^Z$, $\Phi(\zeta, \eta) \in \mathbf{R}_s^{q\times q}[\zeta, \eta]$ induces a quadratic difference form $Q_\Phi : (\mathbf{R}^q)^Z \to \mathbf{R}^Z$ as defined by

$$Q_\Phi(w)(t) := \sum_{k,l} w(t+k)^T \Phi_{kl} w(t+l). \qquad (3)$$

For a given arbitrary $\Phi(\zeta, \eta) = \sum_{k,l=0}^{n} \Phi_{kl} \zeta^k \eta^l \in \mathbf{R}_s^{q\times q}[\zeta, \eta]$, we define a constant matrix $\tilde{\Phi} \in \mathbf{R}^{q(n+1)\times q(n+1)}$ as follows

$$\tilde{\Phi} := \begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \cdots & \Phi_{0,n} \\ \Phi_{1,0} & \Phi_{1,1} & \cdots & \Phi_{1,n} \\ \vdots & & \ddots & \\ \Phi_{n,0} & \Phi_{n,1} & \cdots & \Phi_{n,n} \end{bmatrix}. \qquad (4)$$

In a similar way to the constant symmetric matrix case, the nonnegativity of quadratic difference form induced by $\Phi(\zeta, \eta) \in \mathbf{R}_s^{q\times q}[\zeta, \eta]$ is defined as $Q_\Phi(w)(t) \ge 0$ for all $w \in (\mathbf{R}^q)^Z$ and for all $t \in Z$, and denoted by $\Phi(\zeta, \eta) \ge 0$. As shown in Proposition 2.1 in [8], $\Phi(\zeta, \eta) \ge 0$ is equivalent to $\tilde{\Phi} \ge 0$.

2.3 Discrete time dissipativeness

Next, we formalize the notion of discrete time dissipativeness by using quadratic difference forms and two-variable polynomial matrices. Of course, these notions are parallel with the continious time case (cf. [12]). At first, we can regard $Q_\Phi(w)$ induced by $\Phi(\zeta,\eta) \in \mathbf{R}_s^{q\times q}[\zeta,\eta]$ as the power entering into the dynamical system $\Sigma = (Z, \mathbf{R}^q, (\mathbf{R}^q)^Z)$ (the trivial dynamical system), i.e., a suppy rate. Then, we can formalize dissipativeness of discrete time dynamical system as follows (cf. Definition 3.1 in [8] and Definition 5.1 in [12]).

Definition 2.1 *Let $\Phi(\zeta,\eta) \in \mathbf{R}_s^{q\times q}[\zeta,\eta]$ induce a quadratic supply rate $Q_\Phi(w)$.*

1. *$Q_\Psi(w)$ induced by $\Psi(\zeta,\eta) \in \mathbf{R}_s^{q\times q}[\zeta,\eta]$ is said to be a storage function for the supply rate $Q_\Phi(w)$, if $Q_\Psi(w)(t+1) - Q_\Psi(w)(t) \leq Q_\Phi(w)(t)$ for all t and for all $w \in l_2^q$.*
2. *$Q_\Delta(w)$ induced by $\Delta(\zeta,\eta) \in \mathbf{R}_s^{q\times q}[\zeta,\eta]$ is said to be a dissipation rate for the supply rate $Q_\Phi(w)$, if $\sum_{t=-\infty}^{t=\infty} Q_\Phi(w)(t) = \sum_{t=-\infty}^{t=\infty} Q_\Delta(w)(t)$, and $Q_\Delta(w)(t) \geq 0$ for all t and for all $w \in l_2^q$.* □

As for dissipation rates, it follows from Lemma 3.1 in [8] that $\sum_{t=-\infty}^{t=\infty} Q_\Phi(w)(t) = \sum_{t=-\infty}^{t=\infty} Q_\Delta(w)(t)$ for all $w \in l_2^q$ is equivalent to saying

$$\Phi(\lambda^{-1},\lambda) = \Delta(\lambda^{-1},\lambda) \tag{5}$$

for all nonzero $\lambda \in \mathbf{C}$.

The relation between a supply rate, a storage function, and a dissipation rate can be formalized as follows (cf. Proposition 3.3 in [8], Proposition 5.2 in [12]). Similarly to Theorem 5.3 in [6] in continuous time , the above theorem plays one of the important roles in order to derive the spectral factorization.

Theorem 2.1 *Let $\Phi(\zeta,\eta) \in \mathbf{R}_s^{q\times q}[\zeta,\eta]$ induce a quadratic supply rate $Q_\Phi(w)$. Then, the following four conditions are equivalent.*

1). For all $w \in l_2^q$, $\sum_{t=-\infty}^{\infty} Q_\Phi(w)(t) \geq 0$.

2). $\Phi(e^{-j\omega}, e^{j\omega}) \geq 0$ for all $\omega \in [0, 2\pi)$.

3). $\Phi(\zeta,\eta)$ admits a storage function.

4). $\Phi(\zeta,\eta)$ admits a dissipation rate.

Moreover, for the supply rate induced by $\Phi(\zeta,\eta)$ there is a one-one relation between storage functions $Q_\Psi(w)$ induced by $\Psi(\zeta,\eta)$ and dissipation rates $Q_\Delta(w)$ induced by $\Delta(\zeta,\eta)$, which is described by

$$\begin{aligned} &Q_\Psi(w)(t+1) - Q_\Psi(w)(t) \\ &\qquad = Q_\Phi(w)(t) - Q_\Delta(w)(t) \end{aligned} \tag{6}$$

for all time $t \in \mathbf{Z}$, or equivalently,

$$(\zeta\eta - 1)\Psi(\zeta,\eta) = (\Phi(\zeta,\eta) - \Delta(\zeta,\eta)). \tag{7}$$

3 Main result

3.1 Storage functions and spectral factors

There exist maximum and minimum storage fucntions for a given supply rate. The following theorem is used to guarantee that the desired spectral factors for Problem 1.1 and 1.2 are obtained from the maximum and minimum storage functions for the supply rate induced by a given polynomial matrix.

Theorem 3.1 *Let $\Phi(\zeta,\eta) = \sum_{k,l=0}^{n} \Phi_{kl}\zeta^k\eta^l \in \mathbf{R}_s^{q\times q}[\zeta,\eta]$ induce a supply rate. Assume that $\Phi(\lambda^{-1},\lambda) \neq 0$ and $\Phi(e^{-j\omega}, e^{j\omega}) \geq 0$ for all $\omega \in [0, 2\pi)$. Then, for this supply rate and $\Sigma = (\mathbf{Z}, \mathbf{R}^q, (\mathbf{R}^q)^Z)$, there exist storage functions induced by $\Psi_+(\zeta,\eta)$ and $\Psi_-(\zeta,\eta) \in \mathbf{R}_s^{q\times q}[\zeta,\eta]$ that satisfy*

$$\Psi_-(\zeta,\eta) \leq \Psi(\zeta,\eta) \leq \Psi_+(\zeta,\eta) \tag{8}$$

for any other storage function induced by $\Psi(\zeta,\eta) \in \mathbf{R}_s^{q\times q}[\zeta,\eta]$. In addition, $\Psi_+(\zeta,\eta)$ and $\Psi_-(\zeta,\eta)$ are written by

$$\Psi_+(\zeta,\eta) = (\Phi(\zeta,\eta) - A^T(\zeta)A(\eta))/(\zeta\eta - 1) \tag{9}$$

and

$$\Psi_-(\zeta,\eta) = (\Phi(\zeta,\eta) - H^T(\zeta)H(\eta))/(\zeta\eta - 1) \tag{10}$$

where $A(\lambda)$, $H(\lambda)$ satisfies that

$$\Phi(\lambda^{-1},\lambda) = A^T(\lambda^{-1})A(\lambda) = H^T(\lambda^{-1})H(\lambda) \tag{11}$$

and the rank of $A(\lambda)$ ($H(\lambda)$) is invariant for all $\lambda \in \mathbf{C}$ s.t. $|\lambda| < 1$ ($|\lambda| > 1$, respectively). □

Proof: The proof of the theorem under (E) allows us to claim that the statement holds under (D), so we focus on the former case. By using the well known result stated in e.g., [9], it follows from $\Phi(\lambda^{-1},\lambda) \neq 0$, $\det(\Phi(\lambda^{-1},\lambda)) \equiv 0$ and $\Phi(e^{-j\omega}, e^{j\omega}) \geq 0$ for all $\omega \in [0, 2\pi)$ that there exist a matrix $V_a(\lambda) \in \mathbf{R}^{g\times q}[\lambda]$ and a nonsingular diagonal matrix $\Gamma(\lambda^{-1},\lambda) \in \mathbf{R}^{g\times g}[\lambda^{-1},\lambda]$ such that

$$\Phi(\lambda^{-1},\lambda) = V_a(\lambda^{-1})^T\Gamma(\lambda^{-1},\lambda)V_a(\lambda) \tag{12}$$

where $\Gamma(e^{-j\omega}, e^{j\omega}) \geq 0, ^\forall \omega \in [0, 2\pi)$, $V_a(\lambda)$ is row full rank for all $\lambda \in \mathbf{C}$, and g is the normal rank of $\Phi(\lambda^{-1},\lambda)$. Moreover, $\Gamma(\lambda^{-1},\lambda)$ can be described by

$$\Gamma(\lambda^{-1},\lambda) = \Gamma_a^T(\lambda^{-1})\Gamma_c^T(\lambda^{-1})\Gamma_c(\lambda)\Gamma_a(\lambda). \tag{13}$$

$\Gamma_a(\lambda) \in \mathbf{R}^{g\times g}[\lambda]$ is an anti-Hurwitz diagonal matrix, and $\Gamma_c(\lambda)$ is a diagonal matrix whose determinat has zeros on the unit circle. In addition to Eq.(12), it is clear that

$$V_a(\zeta)\Gamma_a^T(\zeta)\Gamma_c^T(\zeta)\Gamma_c(\eta)\Gamma_a(\eta)V(\eta) \geq 0,$$

so this is one of the dissipation rates for $Q_\Phi(w)$. Let $\Psi_+(\zeta,\eta)=\sum_{k,l=0}^{h}\Psi_{(+)kl}\,\zeta^k\eta^l\in\mathbf{R}_s^{q\times q}[\zeta,\eta]$ induce the corresponding storage function. Note that we do not know whether $Q_{\Psi+}(w)$ induced by $\Psi_+(\zeta,\eta)$ is the maximum storage function or not at this point. From Eq.(7), we can write the relation of them as

$$Q_{\Psi_+}(w)(t+1)-Q_{\Psi_+}(w)(t)= Q_\Phi(w)(t)-\|(\Gamma_c(\sigma)\Gamma_a(\sigma)V_a(\sigma)w)(t)\|^2 \quad (14)$$

for all time t. Consider another storage function induced by $\Psi(\zeta,\eta)=\sum_{k,l=0}^{h'}\Psi_{kl}\zeta^k\eta^l\in\mathbf{R}_s^{q\times q}[\zeta,\eta]$ and its corresponding dissipation rate induced by $\Delta(\zeta,\eta)=\sum_{kl=0}^{m'}\Delta_{kl}\zeta^k\eta^l\in\mathbf{R}_s^{q\times q}[\zeta,\eta]$. Subtracting the dissipation relation for these $\Psi(\zeta,\eta)$ and $\Delta(\zeta,\eta)$ from Eq.(14) yields

$$\begin{aligned}Q_{\Psi_+}(w)(t+1)-Q_{\Psi_+}(w)(t)&\\ -Q_\Psi(w)(t+1)+Q_\Psi(w)(t)&\\ =Q_\Delta(w)(t)-\|(\Gamma_c(\sigma)\Gamma_a(\sigma)V_a(\sigma)w)(t)\|^2.&\quad (15)\end{aligned}$$

We consider the following mathematical model

$$d=\Gamma_a(\sigma)v \text{ and } v=V_a(\sigma)w. \quad (16)$$

In the following discussion, we will show that there exists a $w\in l_2^q|_0$ satisfying $d=0$ in Eq.(16). Since $V_a(\lambda)\in\mathbf{R}^{g\times q}[\lambda]$ is row full rank for all $\lambda\in\mathbf{C}$, there exists a $V_s(\lambda)\in\mathbf{R}^{(q-g)\times q}[\lambda]$ such that $\begin{bmatrix}V_a(\lambda)^T & V_s(\lambda)^T\end{bmatrix}^T$ is a unimodular matrix in $\mathbf{R}^{q\times q}[\lambda]$. Using this unimodular matrix, we can rewrite $v=V_a(\sigma)w$ in Eq.(16) as

$$\begin{aligned}v=V_a(\sigma)w&=\begin{bmatrix}I & 0\end{bmatrix}\begin{bmatrix}V_a(\sigma)\\ V_s(\sigma)\end{bmatrix}w\\ &=\begin{bmatrix}I & 0\end{bmatrix}\begin{bmatrix}w_a\\ w_s\end{bmatrix}=w_a \quad (17)\end{aligned}$$

where

$$\begin{bmatrix}w_a\\ w_s\end{bmatrix}:=\begin{bmatrix}V_a(\sigma)\\ V_s(\sigma)\end{bmatrix}w. \quad (18)$$

In order to clarifying the following proof, suppose that $\Gamma_a(\lambda)$ and $\begin{bmatrix}V_a(\lambda)^T & V_s(\lambda)^T\end{bmatrix}^T$ are described by $\Gamma_a(\lambda)=:\Gamma_{a0}+\Gamma_{a1}\lambda+\cdots+\Gamma_{ap}\lambda^p$ and

$$\begin{bmatrix}V_a(\lambda)\\ V_s(\lambda)\end{bmatrix}:=\begin{bmatrix}V_{a0}\\ V_{s0}\end{bmatrix}+\begin{bmatrix}V_{a1}\\ V_{s1}\end{bmatrix}\lambda+\cdots\begin{bmatrix}V_{ar}\\ V_{sr}\end{bmatrix}\lambda^r$$

respectively. Moreover, notice that $\det|\Gamma_{a0}|\neq 0$ because $\Gamma_a(\lambda)$ is anti-Hurwitz.

Next, consider Eq.(17) and Eq.(18) at time $t=1$. Then it is clear that $w_a(1)$ and $w_s(1)$ are uniquely determined by $w(1)$, $w(2)$, $\cdots$, $w(r+1)$. Due to $v(1)=w_a(1)$, $v(1)$ is also determined by $w(1)$, $w(2)$, $\cdots$, $w(r+1)$. Since $v(t)$ is also determined by $w(t)$, $w(t+1)$, $\cdots$, $w(t+r)$ for all $t\geq 2$ similarly, $v(1),\cdots,v(p)$ are determined by $w(1),w(2),\cdots,w(r+p)$. By using Eq.(16), Eq.(17), and nonsingularity of Γ_{a0}, we can observe that there exists $v(0)(=w_a(0))$ for arbitrary $v(1),v(2),\cdots,v(p)$ such that

$$0=\Gamma_{a0}v(0)+\Gamma_{a1}v(1)+\cdots+\Gamma_{ap}v(p). \quad (19)$$

In addition, we can set $w_s(0)=0$. Considering Eq.(18) at time $t=0$ and substituting $w_a(0)$, $w_s(0)$ yields

$$\begin{aligned}\begin{bmatrix}w_a(0)\\ w_s(0)\end{bmatrix}&=\begin{bmatrix}V_{a0}\\ V_{s0}\end{bmatrix}w(0)+\begin{bmatrix}V_{a1}\\ V_{s1}\end{bmatrix}w(1)+\\ &\cdots+\begin{bmatrix}V_{ar}\\ V_{sr}\end{bmatrix}w(r). \quad (20)\end{aligned}$$

Since $\begin{bmatrix}V_{a0}^T & V_{s0}^T\end{bmatrix}^T$ is nonsingular due to the unimodularity of $\begin{bmatrix}V_a(\lambda)^T & V_s(\lambda)^T\end{bmatrix}^T$ in $\mathbf{R}^{q\times q}[\lambda]$, there exists a unique $w(0)$ for the left hand side and the second term of the right hand side of Eq.(20). Similarly, repeat the above process from -1 to $-\infty$. As a result, we can claim that there exists w such that

$$0=\Gamma_a v(t)=\Gamma_a(\sigma)V_a(\sigma)w(t),t=0,\cdots,-\infty \quad (21)$$

for arbitrary $w(1),w(2),\cdots,w(r+p)$. At the same time, $w_a=v$ is an anti-Hurwitz trajectory and w_s is 0 trajectory. Thus

$$w=\begin{bmatrix}V_a(\sigma)\\ V_s(\sigma)\end{bmatrix}^{-1}\begin{bmatrix}w_a\\ w_s\end{bmatrix} \quad (22)$$

is also anti-Hurwitz trajectory.

Define $m:=r+p$. Moreover, define $\Psi_d(\zeta,\eta)=\Psi^+(\zeta,\eta)-\Psi(\zeta,\eta)$ and h^* is the degree of $\Psi_d(\zeta,\eta)$. In the case of $h^*+1\geq m$, we can take $w\in l_2^q|_{h^*+1}$ such that $w(h^*+1),\cdots,w(1)\in\mathbf{R}^q$ are arbitrary and $w(t)$ in $t\leq 0$ satisfies the difference equation $\Gamma_a(\sigma)V_a(\sigma)w(t)=0$. Substituting this w into Eq.(15) and summing it from $t=-\infty$ to 0 yield

$$\begin{aligned}\begin{bmatrix}w(1)^T & \cdots & w(h^*+1)^T\end{bmatrix}\left\{\tilde{\Psi}_d\right\}\begin{bmatrix}w(1)\\ \vdots\\ w(h^*+1)\end{bmatrix}&\\ =\sum_{t=-\infty}^{0}Q_\Delta(w)(t)\geq 0.&\quad (23)\end{aligned}$$

where $\tilde{\Psi}_d$ is defined for $\Psi_d(\zeta,\eta)$ as in the same way to Eq.(4). (Here, in the case of $m'>h^*+1$, it suffices to consider $w\in l_2^q|_{m'}$ by connecting arbitrary $w(h^*+2),\cdots,w(m')\in\mathbf{R}^q$ to this $w\in l_2^q|_{h^*+1}$). In the case of $h^*+1\leq m$, we can take $w\in l_2^q|_m$ such that $w(m),\cdots,w(1)\in\mathbf{R}^q$ are arbitrary and $w(t)$ in $t\leq 0$ satisfies the difference equation $\Gamma_a(\sigma)V_a(\sigma)w(t)=0$. Substituting this w into Eq.(15) and summing it from $t=-\infty$ to 0 also yield Eq.(23). (In the case of $m'>m$, it also suffices to consider $w\in l_2^q|_{m'}$ by connecting arbitrary $w(m+1),\cdots,w(m')\in\mathbf{R}^q$ to this $w\in l_2^q|_m$) Thus, whether $h^*+1\le m$ or $h^*+1\ge m$, Eq.(23) holds for arbitrary $w(h^*+1),\cdots,w(1)\in\mathbf{R}^q$. Since Eq.(23) means that $a^T\tilde{\Psi}_d a\ge 0$ for all $a\in\mathbf{R}^{q(h^*+1)}$which implies $\Psi_d\ge 0$, we can conclude that $\Psi^+(\zeta,\eta)\ge\Psi(\zeta,\eta)$.

As for the minimum storage function, the proof can be shown by applying the similar discussion as in the case of the maximum storage function, so we omit it here. □

Remark 3.1 *The basic technique of the proof of Theorem 3.1 is the solvability of difference equations and the backward (forward) shifting , which are inherent techniques in discrete time. That is, e.g., if $w(1),\cdots,w(n),d(0)$ are given and D_0 is row full rank in a difference equation $D_0w(0)+D_1w(1)+\cdots+D_nw(n)=d(0)$, we can claim that there exists $w(0)$ satisfying this difference equation. Once the existence of such a $w(0)$ can be claimed, the backward shifting yields the trajectory satisfying this difference equation from 0 to $-\infty$. This is one of main different points from the continuous time case obtained in Theorem 5.7 in [12].* □

Remark 3.2 *The above proof of Theorem 3.1 is related to Problem 1.2. By specializing thsi proof, we can also solve Problem 1.1.*

3.2 The spectral factorization

By using Theorem 3.1 we obtain the spectral factorization algorithm as answer for Problem 1.1 and Problem 1.2 as follows. The proof are similar to that of Theorem 4.1 in [8], so we omit it here.

Theorem 3.2 *Let $\Pi(\lambda^{-1},\lambda)\in\mathbf{R}^{q\times q}[\lambda^{-1},\lambda]$ described by Eq.(1). Assume that (A),(B) and (C). Moreover, we assume (D) or (E).*

First, define the constant matrix $\tilde{\Pi}=\tilde{\Pi}^T\in\mathbf{R}^{(n+1)q\times(n+1)q}$ by

$$\tilde{\Pi}:=\begin{bmatrix}\frac{\Pi_0}{n+1} & \frac{\Pi_1}{n} & \cdots & \frac{\Pi_{n-1}}{2} & \Pi_n\\ \frac{\Pi_1^T}{n} & \frac{\Pi_0}{n+1} & \frac{\Pi_1}{n} & \ddots & \frac{\Pi_{n-1}}{2}\\ & & \ddots & \ddots & \\ \frac{\Pi_{n-1}^T}{2} & & \ddots & \ddots & \frac{\Pi_1}{n}\\ \Pi_n^T & \frac{\Pi_{n-1}^T}{2} & \cdots & \frac{\Pi_1^T}{n} & \frac{\Pi_0}{n+1}\end{bmatrix} \quad (24)$$

Next, consider the following inequality $\Pi(P)\ge 0$, where $\Pi(P)$ is defined by

$$\tilde{\Pi}+\begin{bmatrix}P & 0_{nq\times q}\\ 0_{q\times nq} & 0_{q\times q}\end{bmatrix}-\begin{bmatrix}0_{q\times q} & 0_{q\times nq}\\ 0_{nq\times q} & P\end{bmatrix} \quad (25)$$

with unknown matrix $P=P^T\in\mathbf{R}^{nq\times nq}$. Then the following properties hold.

1). *There exist a maximum solution P_+ and a minimum solution P_- of Eq.(25).*
2). *In the case of (D), there exist row full rank constant matrices $\tilde{A}$ and $\tilde{H}\in\mathbf{R}^{q\times(n+1)q}$ (in the case of (E), $\mathbf{R}^{g\times(n+1)q}$ where g is the normal rank of $\Pi(\lambda^{-1},\lambda)$) such that $\Gamma(P_+)=\tilde{A}^T\tilde{A}$ and $\Gamma(P_-)=\tilde{H}^T\tilde{H}$.*
3). *In the case of (D), the matrices $H(\lambda)$ and $A(\lambda)\in\mathbf{R}^{q\times q}[\lambda]$ (in the case of (E),$\mathbf{R}^{g\times q}[\lambda]$) defined by $A(\lambda):=\tilde{A}\begin{bmatrix}I & \lambda I & \cdots & \lambda^n I\end{bmatrix}^T$ and $H(\lambda):=\tilde{H}\begin{bmatrix}I & \lambda I & \cdots & \lambda^n I\end{bmatrix}^T$ form an almost Hurwitz spectral factor and an almost anti-Hurwitz spectral factor of $\Pi(\lambda^{-1},\lambda)$, respectively.*□

Remark 3.3 *In continuous time case, Theorem 5.7 in [12] and Theorem 5.3 in [6] give the spectral factorization algorithm for polynomial with no zeros on $j\omega$ axis. We expect that this assumption about zeros on the $j\omega$ axis may be eliminated by using the continuous time version of the proof of Theorem 3.1. This point is one of the further studies.* □

4 Numerical example

4.1 Example: Problem 1.1

As one example, we consider the following polynomial matrix $\Pi(\lambda^{-1},\lambda)=$

$$\begin{bmatrix}\lambda^{-1}+2+\lambda & \lambda^{-1}-\lambda & 0\\ -\lambda^{-1}+\lambda & -\lambda^{-1}+6-\lambda & 0.5\lambda^{-1}-0.5\lambda\\ 0 & -0.5\lambda^{-1}+0.5\lambda & \lambda^{-1}+2+\lambda\end{bmatrix}$$

as treated in [13]. Clearly, it is easy to verify that $\Phi(e^{-j\omega},e^{j\omega})\ge 0$ for all $\omega\in[0,2\pi)$ and the roots of $\det(\lambda\Pi(\lambda^{-1},\lambda))$ are equal to $(-1,-1,-1,-1,\ -13.922820325,\ -0.071824528)$. Notice that the multiple zero of -1 yields the singularity of $\Pi(e^{-j\omega},e^{j\omega})$ at $\omega=\pi$.

Firstly, the constant matrix defined by Eq.(24) is obtained as

$$\tilde{\Pi}:=\begin{bmatrix}1 & 0 & 0 & 1 & -1 & 0\\ 0 & 3 & 0 & 1 & -1 & -0.5\\ 0 & 0 & 1 & 0 & 0.5 & 1\\ 1 & 1 & 0 & 1 & 0 & 0\\ -1 & -1 & 0.5 & 0 & 3 & 0\\ 0 & -0.5 & 1 & 0 & 0 & 1\end{bmatrix}.$$

for this example. Secondly, by using MATLAB LMI Control Toolbox ([5]), we solve Eq.(25) and find P_- and P_+. Thirdly, we factorize $\Gamma(P_+)$ and $\Gamma(P_-)$ and construct $A(\lambda)$ and $H(\lambda)$ as stated in Theorem 3.2. For example, we obtain

$$A(\lambda) = \begin{bmatrix} 0.6547 + 0.6547\lambda & -0.7559 - 0.7559\lambda & 2.3441 \times 10^{-6} \\ 0.2673 + 0.2673\lambda & 0.2315 + 0.2315\lambda & 0.9354 + 0.9354\lambda \\ -0.7071 - 0.7071\lambda & -2.0266 + 0.8018\lambda & 0.3536 + 0.3536\lambda \end{bmatrix}$$

as an almost anti-Hurwitz spectral factor of $\Pi(\lambda^{-1}, \lambda)$. Indeed, it is easy to verify that the above $A(\lambda)$ is the desired spectral factor of $\Pi(\lambda^{-1}, \lambda)$ as follows. $(A_0^T + A_1^T \lambda^{-1})(A_0 + A_1\lambda)$ is equal to $\Pi(\lambda^{-1}, \lambda)$ in precision of 8 decimal places. Moreover, the roots of $\det(A_0 + A_1\lambda)$ is -1, -1 and -13.922820325.

4.2 Example: Problem 2

Next, we consider the following polynomial matrix $\Pi(\lambda^{-1}, \lambda) =$

$$\begin{bmatrix} 0.5\lambda^{-1} + 1.25 + 0.5\lambda & 0.5\lambda^{-1} + 1.25 + 0.5\lambda & 0 \\ 0.5\lambda^{-1} + 1.25 + 0.5\lambda & 0.5\lambda^{-1} + 5.25 + 0.5\lambda & 4 \\ 0 & 4 & 4 \end{bmatrix}$$

The normal rank of $\Pi(\lambda^{-1}, \lambda)$ is equal to 2 and the rank of $\Pi(\lambda^{-1}, \lambda)$ is equal to 1 for $\lambda = -2$ and -0.5. Firstly, the constant matrix defined by Eq.(24) is obtained as

$$\tilde{\Pi} := \begin{bmatrix} 0.625 & 0.625 & 0 & 0.5 & 0.5 & 0 \\ 0.625 & 2.625 & 2 & 0.5 & 0.5 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0.625 & 0,625 & 0 \\ 0.5 & 0.5 & 0 & 0.625 & 2.625 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 \end{bmatrix}.$$

for this example. As in the same way to the previous example, we obtain an almost Hurwitz and an almost anti-Hurwitz spectral factors. For example, we obtain $A(\lambda) =$

$$\begin{bmatrix} -0.9510 - 0.4755\lambda & -0.3325 - 0.4755\lambda & 0.6185 \\ 0.3092 + 0.1546\lambda & 0.2211 + 0.1546\lambda & 1.9020 \end{bmatrix}$$

as the almost anti-Hurwitz spectral factor of $\Pi(\lambda^{-1}, \lambda)$. Indeed, $(A_0^T + A_1^T \lambda^{-1})(A_0 + A_1\lambda)$ is equal to $\Pi(\lambda^{-1}, \lambda)$ in precision of 8 decimal places. In addition, the row rank $A_0 + A_1\lambda$ is equal to 1 for $\lambda = -2$.

5 Conclusion

In this paper, we have provided spectral factorization algorithms for discrete time polynomial matrices by using some inherent techniques in discrete time. The key point of this paper is Theorem 3.1. This theorem says; as long as $\Phi(\zeta, \eta)(\neq 0) \in \mathbf{R}_s^{q\times q}[\zeta, \eta]$ that induces a suppy rate satisfies $\Phi(\lambda^{-1}, \lambda) \geq 0$ on the unit circle, its maximum (minimum) storage function corresponds to the dissipation rate formed by almost anti-Hurwitz (Hurwitz, respectively) spectral factor.

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