ON ESTIMATORS FOR NONLINEAR SYSTEMS IN $\mathcal{L}_p$ SPACES

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Abstract

Estimation problems are addressed for continuous-time, nonlinear dynamic systems in a general $\mathcal{L}_p$ framework. In this setting, the connection between the observation and the filtering problems is investigated. Under some regularity assumptions for the nonlinearities and suitable bounds on the $\mathcal{L}_p$ norm of the noises, it is proved that the same hypotheses sufficient to design an exponential observer for a system without noises enable one to design a filter which is $\mathcal{L}$-stable with respect to system and measurement noises. An illustrative example is finally presented.

1 Introduction

In the wide literature on estimation, the problems of designing state observers and filters are usually considered. An observer is a state estimator for a dynamic system without noises, where uncertainties are due to the unknown initial state and to inaccessible components of the state vector. In filtering the problem is made harder for the presence of noises, acting on the system and on the measurement channel.

The design of observers for nonlinear dynamic systems may rely on various approaches, but the fundamental requirement is stability: a Lyapunov function is usually considered, in order to ensure convergence of the estimation error. The simplest choice is a constant-gain observer: suitable conditions on the gain allow to obtain stability of the error dynamics. Constant-gain observers with a moderate computational burden have been introduced to perform state estimation for nonlinear systems, where the nonlinearity is accounted for by considering its Lipschitz constant [1].

Another solution to the observer problem consists in applying a canonical state-space transformation [2, 3]. This representation of the system dynamics enables one to easily find an observer with linear error dynamics in the transformed state-space. The so-called high-gain observers have been considered, for example, in [4], where a fast convergence of the estimation error is obtained for general nonlinear systems by performing a state transformation. Following a different approach, variable-structure observers have been proposed to deal with the problem of uncertainties [5]. The design of sliding-mode observers with nonlinear dynamics has been faced in [6] and [7], where the focus is more on robustness with respect to bounded nonlinearities/uncertainties. An analysis of the noise effects in sliding-mode observers has been made in [8].

Filtering refers to the case in which disturbances affect both the system and the measurement channel. The noises can be regarded either as unknown deterministic inputs or as stochastic random variables within a probabilistic framework. The former approach enables one to design recursive estimators, according to a least-squares estimation criterion. The probabilistic approach has brought to different kinds of solutions; the reader is referred to [9] for an introduction to this subject.

In this paper the estimation problem is addressed for continuous-time nonlinear dynamic systems in a general framework of $\mathcal{L}_p$ signals; in such a context, the connection between the observation and the filtering problems is investigated. Basic notations and definitions are summarized in Section 2. In Section 3 we show that, under some regularity assumptions on the nonlinearities and suitable bounds on the $\mathcal{L}_p$ norm of the noises, the same hypotheses sufficient to design an exponential observer for a system without noises allow to design a filter that is $\mathcal{L}$-stable with respect to system and measurement noises. In other words, we prove the input-to-state $\mathcal{L}$ stability of the error dynamics, where the inputs are the process and channel noises. Section 4 provides an illustrative example. A conclusive discussion and prospects for future work are contained in Section 5.

2 Notations and definitions

The following notations and definitions will be used throughout the paper.

- Given $a \in \mathbb{R}^n$, for any $n \in \mathbb{N}^+$, $\|a\|$ denotes its Euclidean norm; for $A \in \mathbb{R}^{n\times m}$, $\|A\| \triangleq \sup_{a\in\mathbb{R}^n,\, \|a\|=1} \|A\,a\|$ denotes the induced matrix norm.
- $\mathcal{L}_p^m$, for $p \in [1,\infty]$ and $m \in \mathbb{N}^+$, is the set of all piecewise-continuous functions $u : [0,\infty) \to \mathbb{R}^m$ with the $\mathcal{L}_p$ norm, defined as

$$\|u\|_{\mathcal{L}_p} \triangleq \left(\int_0^\infty \|u(t)\|^p\, dt\right)^{1/p},\ p \in [1,\infty),$$

  and
$$\|u\|_{\mathcal{L}_\infty} \triangleq \sup_{t\geq 0} \|u(t)\| .$$

- The extended space $\mathcal{L}_{p\,e}^m$ is defined as $\mathcal{L}_{p\,e}^m \triangleq \{u | u_\tau \in \mathcal{L}_p^m, \forall \tau \geq 0\}$, where

$$u_\tau(t) \triangleq \begin{cases} u(t) & , \quad t \leq \tau \\ 0 & , \quad t > \tau\,. \end{cases}$$

  For every $p \in [1,\infty]$ and $m \in \mathbb{N}^+$, $\mathcal{L}_{p\,e}^m$ is a subset of $\mathcal{L}_p^m$; it allows to consider unbounded signals, and then to deal with unstable dynamic systems. Note that $\mathcal{L}_p^m$ is a Banach (i.e., complete normed linear) space for every $p \in [1,\infty]$, while $\mathcal{L}_{p\,e}^m$ is linear, but not even normed.

- A continuous function $\alpha : [0,\infty) \to [0,\infty)$ belongs to class $\mathcal{K}_\infty$ if it is strictly increasing, $\lim_{r\to\infty} \alpha(r) = \infty$, and $\alpha(0) = 0$.
- A mapping $F : \mathcal{L}_{p\,e}^m \to \mathcal{L}_{q\,e}^n$ is *$\mathcal{L}$-stable* if there exist a function $\alpha : [0,\infty) \to [0,\infty)$ of class $\mathcal{K}_\infty$ and a nonnegative constant β such that

$$\|(Fu)_\tau\|_{\mathcal{L}_q} \leq \alpha\left(\|u_\tau\|_{\mathcal{L}_p}\right) + \beta \tag{1}$$

  for all $u \in \mathcal{L}_{p\,e}^m$ and $\tau \in [0,\infty)$. Moreover, it is *finite-gain $\mathcal{L}$-stable* if there exist nonnegative constants γ and β such that

$$\|(Fu)_\tau\|_{\mathcal{L}_q} \leq \gamma\,\|u_\tau\|_{\mathcal{L}_p} + \beta \tag{2}$$

  for all $u \in \mathcal{L}_{p\,e}^m$ and $\tau \in [0,\infty)$.

  If $p = q = \infty$, the definition of $\mathcal{L}$ stability is the well-known notion of bounded input-bounded output (BIBO) stability for dynamic systems.

- A mapping $F : \mathcal{L}_{p\,e}^m \to \mathcal{L}_{q\,e}^n$ is *small-signal $\mathcal{L}$-stable* (*small-signal finite-gain $\mathcal{L}$-stable*) if inequality (1), ((2), respectively) is satisfied for $u \in \mathcal{L}_{p\,e}^m$ with $\sup_{0\leq t\leq\tau} \|u(t)\| \leq r$. When dealing with dynamic systems, these concepts allow to consider input-output relationships defined only for a subset of the input space.
- Hölder's inequality: if $f,\, g \in \mathcal{L}_{p\,e}^1$, where $p \in [1,\infty]$ and $1/p + 1/q = 1$, then $f\,g \in \mathcal{L}_{1\,e}^1$, and

$$\int_0^\tau \|f(t)g(t)\|\, dt \leq \|f\|_p \|g\|_q = \left(\int_0^\tau \|f(t)\|^p\, dt\right)^{1/p} \left(\int_0^\tau \|g(t)\|^q\, dt\right)^{1/q} \tag{3}$$

  for every $\tau \in [0,\infty)$.

3 $\mathcal{L}$ stability of the filtering mapping

In the following we consider an unforced dynamic system described by

$$\begin{cases} \dot{x} = f\,(t,x,w) \\ y = h\,(t,x,v)\ . \end{cases} \tag{4}$$

For every $t \geq t_0$, $x(t) \in X \triangleq \{x \in \mathbb{R}^n \mid \|x\| < \bar{x}\,,\, \bar{x} > 0\}$ is the state vector, $y(t) \in Y \triangleq \{y \in \mathbb{R}^m \mid \|y\| < \bar{y}\,,\, \bar{y} > 0\}$ is the output vector, $w(t) \in W \triangleq \{w \in \mathbb{R}^r \mid \|w\| < \bar{w}\,,\, \bar{w} > 0\}$ is the system noise, and $v(t) \in V \triangleq \{v \in \mathbb{R}^s \mid \|v\| < \bar{v}\,,\, \bar{v} > 0\}$ is the measurement noise. We now make the following assumption.

Assumption 3.1

(i) *$f : [t_0,\infty) \times X \times W \to \mathbb{R}^n$ is piecewise-continuous in t and locally Lipschitz in (x,w); more specifically, there exist $L_f^x, L_f^w \in \mathbb{R}^+$ such that $\|f(t,x_1,w_1) - f(t,x_2,w_2)\| \leq L_f^x\,\|x_1 - x_2\| + L_f^w\,\|w_1 - w_2\|$, for all $x_1,x_2 \in X\,, w_1,w_2 \in W$, and $t \in [t_0,\infty)$.*

(ii) *$h : [t_0,\infty) \times X \times V \to \mathbb{R}^m$ is piecewise-continuous in t and locally Lipschitz in (x,v); more specifically, there exist $L_h^x, L_h^v \in \mathbb{R}^+$ such that $\|h(t,x_1,v_1) - h(t,x_2,v_2)\| \leq L_h^x\,\|x_1 - x_2\| + L_h^v\,\|v_1 - v_2\|$, for all $x_1,x_2 \in X\,,\ v_1,v_2 \in V$, and $t \in [t_0,\infty)$.*

If Assumption 3.1 (i) is satisfied, the existence and uniqueness of a local solution of the differential equation describing the dynamics of the system (4) is guaranteed (see, for example, [10]).

A quite general form of an *estimator* for system (4) is

$$\dot{\hat{x}} = f(t,\hat{x},0) + g\,(t,\hat{x},y - h\,(t,\hat{x},0)) \tag{5}$$

where $\hat{x}(t) \in X$, $t \geq t_0$, is the state estimate. The estimator dynamics is the summation of the system dynamics and an innovation term, represented by the function g; in the following, $y - h(t,\hat{x},0)$ and g will be called, respectively, *innovation* and *innovation function*. Such a function is required to satisfy the following assumption.

Assumption 3.2

(i) $g\,(t,\hat{x},0) = 0\,,\ \forall t \geq t_0$ *;*

(ii) $g : [t_0,\infty) \times X \times Z \to \mathbb{R}^n$ *is piecewise-continuous in* t *and locally Lipschitz in* $(\hat{x},z)$ *, where* $z \triangleq y - h\,(t,\hat{x},0)$ *and* $Z \triangleq \{z \in \mathbb{R}^m \mid \|z\| < \bar{z}\,,\ \bar{z} > 0\}$. *More specifically, there exist* $L_g^x, L_g^z \in \mathbb{R}^+$ *such that* $\|g(t,x_1,z_1) - g(t,x_2,z_2)\| \leq L_g^x\,\|x_1 - x_2\| + L_g^z\,\|z_1 - z_2\|$ *for all* $x_1, x_2 \in X$ *,* $z_1, z_2 \in Z$ *, and* $t \in [t_0,\infty)$ *.*

We call functions $g : X \times Z \to X$ satisfying Assumption 3.2 *admissible innovation functions.*

Assumption 3.2 (ii) is a sufficient requirement for having a unique local solution of the differential equation (5) describing the estimator. In addition, Assumption 3.2 (i) is necessary to ensure convergence. The innovation function has been considered of this form for sake of simplicity, although it may be taken of a more general type [11], i.e., $\tilde{g}\,(t,\hat{x},h(t,x,v),h(t,\hat{x},0))$, where $\tilde{g}\,(t,\hat{x},h(t,x,v),h(t,\hat{x},0)) = 0$ if $h(t,x,v) = h(t,\hat{x},0)$.

Depending on the presence or not of disturbances on the system and channel equations, the estimator (5) is referred to as *filter* or *observer*, respectively. Thus, an observer is an estimator for the system where the disturbances w and v are identically zero, namely

$$\begin{cases} \dot{x} = f\,(t,x,0) \\ y = h\,(t,x,0)\,. \end{cases} \tag{6}$$

If the estimation error is defined as $e(t) \triangleq x(t) - \hat{x}(t)$, $t \geq t_0$ the error dynamics for the observer and for the filter are given, respectively, by

$$\begin{cases} \dot{e} = f(t,x,0) - f(t,\hat{x},0) - g\,(t,\hat{x},y - h\,(t,\hat{x},0)) \\ y = h\,(t,x,0) \end{cases} \tag{7}$$

and

$$\begin{cases} \dot{e} = f(t,x,w) - f(t,\hat{x},0) - g\,(t,\hat{x},y - h\,(t,\hat{x},0)) \\ y = h\,(t,x,v)\ . \end{cases} \tag{8}$$

In the following we denote $E \triangleq \{e \in \mathbb{R}^n \mid \|e\| < \bar{e}\,,\ \bar{e} > 0\}$. We now recall the following definitions.

Definition 3.3 (Exponential observer) *(5) is a (global) exponential observer for the system (6) if the estimation error with dynamics described by (7) has* 0 *as a (global) exponentially stable equilibrium point.*

Definition 3.4 (Finite-gain $\mathcal{L}$-stable filter) *(5) is a (small-signal) finite-gain $\mathcal{L}$-stable filter for the system (4) if the estimation error with dynamics described by (8) has bounded (small-signal) $\mathcal{L}_p$ norm.*

If we introduce the mapping

$$\mathcal{E} : \mathcal{L}^r_{p\,e} \times \mathcal{L}^s_{p\,e} \to \mathcal{L}^n_{p\,e} \tag{9}$$

such that $e = \mathcal{E}(w,v)$, where $\dot{e} = f(t,x,w) - f(t,x-e,0) - g\,(t,x-e,y-h\,(t,x-e,0))$, $y = h\,(t,x,v)$, then Definition 3.4 is equivalent to requiring that $\mathcal{E}(w,v)$ is a (small-signal) finite-gain $\mathcal{L}$-stable mapping.

The problem of finding suitable innovation functions g is difficult to solve for general nonlinear dynamic systems. Consequently, we introduce the following assumption, that restricts the class of admissible innovation functions for a given system to those admitting a Lyapunov function for the corresponding error dynamics. For the sake of notational simplicity and without any loss of generality, in the following we let $t_0 = 0$

Assumption 3.5
There exist a Lyapunov function $V : [0,\infty) \times E \to [0,\infty)$ *and positive constants* c_1 *,* c_2 *,* c_3 *, and* c_4 *such that*

(i) $c_1\,\|e\|^2 \leq V(t,e) \leq c_2\,\|e\|^2$ *;*

(ii) $\dfrac{\partial V}{\partial t} + \dfrac{\partial V}{\partial e}\,[\,f(t,x,0) - f(t,\hat{x},0) - g\,(t,\hat{x},h\,(t,x,0) - h\,(t,\hat{x},0))\,] \leq -c_3\,\|e\|^2$ *;*

(iii) $\left\|\dfrac{\partial V}{\partial e}\right\| \leq c_4\,\|e\|$ *.*

Assumptions 3.1, 3.2 and 3.5 guarantee that the estimator (5) is an exponential observer for the system (6). When system and measurement noises affect the dynamics and channel equations, it is not possible to obtain the convergence to zero of the estimation error, but one can try to design a filter that is non-divergent [12]. The following theorem guarantees that, under suitable regularity hypotheses on the functions f, g, and h, and for $\mathcal{L}_{\infty\,e}$ system and channel noises, the error of the filter is an $\mathcal{L}_{p\,e}$ signal.

Theorem 3.6 *Suppose that Assumptions 3.1, 3.2, and 3.5 are verified. Then for each* $e(0)$ *such that* $\|e(0)\| < \bar{e}\sqrt{\frac{c_1}{c_2}}$, *the mapping (9) is small-signal, finite-gain* $\mathcal{L}$*-stable for each* $p \in [1,\infty]$. *More specifically, for each* $w \in \mathcal{L}^r_{pe}$ *and* $v \in \mathcal{L}^s_{pe}$ *such that* $\sup_{0\le\sigma\le t}\|w(\sigma)\| < \min\left(\bar{w}, \frac{c_1 c_3 \bar{e}}{2 c_2 c_4 L^x_f}\right)$, *and* $\sup_{0\le\sigma\le t}\|v(\sigma)\| < \min\left(\bar{v}, \frac{c_1 c_3 \bar{e}}{2 c_2 c_4 L^v_h L^z_g}\right)$, *there exist non-negative constants* γ, λ, *and* β *such that*

$$\|e_\tau\|_{\mathcal{L}_p} \le \gamma\,\|w_\tau\|_{\mathcal{L}_p} + \lambda\,\|v_\tau\|_{\mathcal{L}_p} + \beta \tag{10}$$

for all $\tau \in [0,\infty)$ *with* $\gamma = \frac{c_2\, c_4 L^x_f}{c_1\, c_3}$, $\lambda = \frac{c_2\, c_4\, L^v_h\, L^z_g}{c_1\, c_3}$ *and* $\beta = \sqrt{\frac{c_1}{c_2}}\,\|e(0)\|\,\rho$, *where*

$$\rho = \begin{cases} 1 & , \quad p = \infty \\ \left(\frac{2\,c_2}{c_3\,p}\right)^{1/p} & , \quad p \in [1,\infty)\,. \end{cases}$$

If, in addition, $w \in \mathcal{L}^r_p$, $v \in \mathcal{L}^s_p$, $\sup_{\sigma\in[0,\infty)}\|w(\sigma)\| < \min\left(\bar{w}, \frac{c_1 c_3 \bar{e}}{2 c_2 c_4 L^x_f}\right)$, *and* $\sup_{\sigma\in[0,\infty)}\|v(\sigma)\| < \min\left(\bar{v}, \frac{c_1 c_3 \bar{e}}{2 c_2 c_4 L^v_h L^z_g}\right)$, *(10) holds with the subscript* τ *dropped.*

□

Proof. The evaluation of the derivative of $V(t,e)$ gives

$$\begin{aligned}
\dot{V}\,(t,e) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial e}\,[\,f(t,x,w) - f(t,\hat{x},0) \\
&\quad -g\,(t,\hat{x},h\,(t,x,v) - h\,(t,\hat{x},0))\,] = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial e}\,[\\
&\quad f(t,x,0) - f(t,\hat{x},0) + g\,(t,\hat{x},h\,(t,x,0) - h\,(t,\hat{x},0))\,] \\
&\quad + \frac{\partial V}{\partial e}\,[\,f(t,x,w) - f(t,x,0) - g(t,\hat{x},h\,(t,x,v) \\
&\quad -h\,(t,\hat{x},0)) + g\,(t,\hat{x},h\,(t,x,0) - h\,(t,\hat{x},0))\,] \qquad (11) \\
&\le -c_3\|e(t)\|^2 + c_4\|e(t)\|\,\left(L^x_f\|w(t)\| + L^v_h\,L^z_g\|v(t)\|\right)
\end{aligned}$$

If we let $W(t) \triangleq \sqrt{V(t,e(t))}$, in the lines of [10, Theorem 6.1] it is possible to verify that

$$D^+W(t) \le -\frac{c_3}{2c_2}W(t) + \frac{c_4}{2\sqrt{c_1}}\,\left(L^x_f\,\|w(t)\| + L^v_h\,L^z_g\|v(t)\|\right)$$

where $D^+W(t)$ denotes the upper right-hand derivative of $W(t)$ (see [10, p. 651]). Then the comparison lemma (see Lemma 2.5 of [10]) gives

$$W(t) \le e^{-\frac{c_3}{2c_2}t}\,W(0) + \frac{c_4}{2\sqrt{c_1}}\Big\{ L^x_f \int_0^t e^{-\frac{c_3}{2c_2}(t-\tau)}\,\|w(\tau)\|\,d\tau + L^v_h\,L^z_g \int_0^t e^{-\frac{c_3}{2c_2}(t-\tau)}\,\|v(\tau)\|\,d\tau\Big\}$$

Using Assumption 3.5(i), the above-written inequality gives

$$\|e(t)\| \le \sqrt{\frac{c_2}{c_1}}\,\|e(0)\|\,e^{-\frac{c_3}{2c_2}t} + \frac{c_4}{c_1}\Big\{ L^x_f \int_0^t e^{-\frac{c_3}{2c_2}(t-\tau)}\,\|w(\tau)\|\,d\tau + L^v_h\,L^z_g \int_0^t e^{-\frac{c_3}{2c_2}(t-\tau)}\,\|v(\tau)\|\,d\tau\Big\}$$

It is easy to check that, if $\|e(0)\| < \bar{e}\sqrt{\frac{c_1}{c_2}}$, $\sup_{0\le\sigma\le t}\|w(\sigma)\| < \frac{c_1 c_3 \bar{e}}{2c_2 c_4 L^x_f}$, and $\sup_{0\le\sigma\le t}\|v(\sigma)\| < \frac{c_1 c_3 \bar{e}}{2c_2 c_4 L^v_h L^z_g}$, then $e(t)$ is inside the domain of validity of the assumptions.

By letting $k_1 \triangleq \sqrt{\frac{c_1}{c_2}}\|e(0)\|$, $k_2 \triangleq \frac{c_4}{2c_1}L^x_f$, $k_3 \triangleq \frac{c_4}{2c_1}L^v_h L^z_g$, and $\alpha \triangleq \frac{c_3}{2c_2}$, the previous inequality can be rewritten as

$$\|e(t)\| \le k_1\,e_1(t) + k_2\,e_2(t) + k_3\,e_3(t)\,,$$

where $e_1(t) \triangleq e^{-\alpha\,t}$, $e_2(t) \triangleq \int_0^t e^{-\alpha\,(t-\tau)}\,\|w(\tau)\|\,d\tau$, $e_3(t) \triangleq \int_0^t e^{-\alpha\,(t-\tau)}\,\|v(\tau)\|\,d\tau$. As regards $e_1(t)$, it is easy to verify that $\|e_{1\,\tau}\|_{\mathcal{L}_p} \le \rho$, where ρ is defined in the statement of the theorem. As regards $e_2(t)$ and $e_3(t)$, in the lines of [10, Example 6.2] we get

$$\|e_{2\,\tau}\|_{\mathcal{L}_p} \le \frac{\|w_\tau\|_{\mathcal{L}_p}}{\alpha} \quad , \quad \|e_{3\,\tau}\|_{\mathcal{L}_p} \le \frac{\|v_\tau\|_{\mathcal{L}_p}}{\alpha}$$

Putting all together, we obtain (10).

■

According to Theorem 3.6, if suitable Lipschitz conditions are satisfied by the system, channel and innovation nonlinearities, and if the noises belong to $\mathcal{L}_{\infty\,e}$, then the $\mathcal{L}_p$ norm of the estimation error e_τ is bounded by a linear combination of the $\mathcal{L}_p$ norms of the noises w_τ and v_τ, plus a term due to the initial uncertainty on the state value. In other words, by means of suitable Lipschitz regularity conditions for f, g, and h, the belonging of the noises to $\mathcal{L}_{\infty\,e}$ implies the belonging of the estimation error to $\mathcal{L}_{p\,e}$. Obviously, when the assumptions of Theorem 3.6 are satisfied globally, it is not necessary to restrict $\|e(0)\|$, $\|w(t)\|$ and $\|v(t)\|$: equation (10) holds true for every $e(0) \in \mathbb{R}^n$, $w \in \mathcal{L}^r_{p\,e}$, $v \in \mathcal{L}^s_{p\,e}$.

Remark 3.7 From inspection of the proof of Theorem 3.6, it results that only the following Lipschitz conditions of f and h with respect to the noises and of g with respect to the innovation are necessary: $\|f(t,x,w) - f(t,x,0)\| \leq L^w_f \|w\|$, $\|h(t,x,v) - h(t,x,0)\| \leq L^v_h \|v\|$, and $\|g(t,\hat{x},z_1) - g(t,\hat{x},z_2)\| \leq L^z_g \|z_1 - z_2\|$. However, to guarantee the existence and uniqueness of a local solution to the system (6) and estimator (7) equations, f and h have to be Lipschitz in x and g has to be Lipschitz in $\hat{x}$, respectively. For this reason, in the hypotheses of Theorem 3.6 we have included Assumptions 3.1 and 3.2 in their entirety.

4 An illustrative example

The following example illustrates two main aspects of the proposed approach.

- The existence of a Lyapunov function in the noiseles framework ensures non-divergence for the estimation error in the noisy case.
- Once a Lyapunov function is considered to guarantee the stability of the error dynamics for the system without noises, the class of admissible innovation functions turns out to be restricted, by imposing an upper bound on their Lipschitz constant.

Let us consider a dynamic system whose dynamics is the summation of a linear term and a nonlinear one, with a linear measurement channel:

$$\begin{cases} \dot{x} = A\,x + f(x) \\ y = C\,x\,, \end{cases} \tag{12}$$

where for every $t \geq 0$, $x(t) \in X \triangleq \{x \in \mathbb{R}^n \mid \|x\| < \bar{x}\,,\, \bar{x} > 0\}$ and $y(t) \in Y \triangleq \{y \in \mathbb{R}^m \mid \|y\| < \bar{y}\,,\, \bar{y} > 0\}$. Let $f : X \to \mathbb{R}^n$ be piecewise-continuous and locally Lipschitz in x, with Lipschitz constant L^x_f, and suppose that (A, C) is detectable. An observer for such a system can be written as:

$$\dot{\hat{x}} = A\,\hat{x} + f(\hat{x}) + g\,(y - C\,\hat{x})\ , \tag{13}$$

with an innovation function $g : Z \to \mathbb{R}^n$ locally Lipschitz in z, where $z \triangleq y - C\,\hat{x}$. Let g be chosen as the summation of two contributions, i.e.,

$$g\,(y - C\,\hat{x}) = D\ (y - C\,\hat{x}) + \gamma\ (y - C\,\hat{x})\ , \tag{14}$$

where D is a $n \times m$ matrix and $\gamma : Z \to \mathbb{R}^n$ is locally Lipschitz in z, with Lipschitz constant L^z_γ.

4.1 Convergence in the noiseless framework

We first study under which hypotheses the convergence of the estimation error can be guaranteed; this evidences how Assumption 3.5 about the existence of a Lyapunov function restricts the class of admissible innovation functions.

The estimation error $e(t) \triangleq x(t) - \hat{x}(t)$ is given by

$$\dot{e} = (A - D\,C\,)\ e + f(x) - f(\hat{x}) - \gamma\,(y - C\,\hat{x})\,. \tag{15}$$

We now introduce the Lyapunov function $V \triangleq e^T\,P\,e$, where the matrix P is positive definite and symmetric. The derivative of V is given by

$$\begin{aligned} \dot{V} = {} & e^T\ \Big[(A - DC)^T\,P + P\ (A - DC)\Big]\,e \\ & +2\,[f(x) - f(\hat{x})]^T\,P\,e - 2\,\gamma^T\,(C\,e)\ P\,e\,. \end{aligned} \tag{16}$$

As it is usually done in the design of observers for dynamic systems with Lipschitz nonlinearities [13, 14], by a simple algebra we can compute upper bounds to the last two terms of (16):

$$\begin{aligned} & 2\,[f(x) - f(\hat{x})]^T\,P\ (x - \hat{x}) \leq 2\,L^x_f\,\|x - \hat{x}\|\ \|P\ (x - \hat{x})\| \\ & \quad \leq (L^x_f)^2\ (x - \hat{x})^T\ P\,P\ (x - \hat{x}) + (x - \hat{x})^T\ (x - \hat{x}) \end{aligned}$$

and

$$\begin{aligned} & -2\gamma^T\,[C\,(x - \hat{x})]\ P\ (x - \hat{x}) \leq 2\,L^z_\gamma\,\|C\ (x - \hat{x})\,\| \times \\ & \quad \|P\,(x - \hat{x})\| \leq (L^z_\gamma)^2\,\|C\|^2\ (x - \hat{x})^T\ P\,P\ (x - \hat{x}) \\ & \quad + (x - \hat{x})^T\ (x - \hat{x}) \end{aligned}$$

From (16), using the above-written inequalities, we obtain

$$\begin{aligned} \dot{V} \leq {} & e^T\,\Big[(A - DC)^T\,P + P\ (A - DC) \\ & + \big((L^x_f)^2 + (L^z_\gamma)^2\,\|C\|^2\big)\ P\,P + 2\,I\Big]e\,. \end{aligned}$$

If there exists a gain matrix D and a symmetric positive definite matrix Q such that $A - DC$ is stable and the algebraic Riccati equation

$$\begin{aligned} & (A - DC)^T\ P + P\ (A - DC) + \big((L^x_f)^2 \\ & \quad + (L^z_\gamma)^2\,\|C\|^2\big)\,P\,P + 2\,I = -Q \end{aligned} \tag{17}$$

has a symmetric, positive definite matrix P as solution, then the observer (13) is asymptotically stable. Note that the detectability hypothesis of (A, C) is a necessary condition for the existence of such a solution. The requirement that (17) has a solution for a given value of L_f^x determines an upper bound $\bar{L}$ on the Lipschitz constant L_γ^z; consequently, the class of admissible innovation functions has to satisfy the condition $L \leq \bar{L}$ (see Assumption 3.2). In other words, such a class contains functions with a Lipschitz constant bounded by the largest value of L_γ^z for which the Riccati equation (17) has a solution.

4.2 Non-divergence in presence of noises

Suppose that additive noises affect the system and channel equations, i.e.,

$$\left\{ \begin{array}{l} \dot{x} = A\,x + f(x) + w \\ y = C\,x + v\,; \end{array} \right. \tag{18}$$

recall that $w \in W \stackrel{\triangle}{=} \{w \in \mathbb{R}^r \mid \|w\| < \bar{w}\,,\, \bar{w} > 0\}$ and $v \in V \stackrel{\triangle}{=} \{w \in \mathbb{R}^s \mid \|w\| < \bar{v}\,,\, \bar{v} > 0\}$. As a consequence, the error dynamics is

$$\begin{aligned} \dot{e} = {} & (A - D\,C)\; e + f(x) - f(\hat{x}) + w - D\,v \\ & -\gamma\,(C\,e + v)\;. \end{aligned} \tag{19}$$

By using the previous inequalities, we get

$$\begin{aligned} \dot{V} \leq {} & -e^T Q\,e + \left[2\,(\|w\| + \|D\|\,\|v\|)\,\|P\| \right. \\ & \left. + L_\gamma^z\,\|v\|\,\|P\|^2\right] \|e\| \leq -e^T Q\,e + c\,\|e\|\,, \end{aligned} \tag{20}$$

where $c \stackrel{\triangle}{=} 2\,(\bar{w} + \|D\|\bar{v})\,\|P\| + L_\gamma^z\,\bar{v}\,\|P\|^2$. It is easy to verify Theorem 3.6, i.e., the satisfaction of the Riccati equation (17) enables one to find a non-divergent filter in the noisy framework. Indeed, in this example, the derivative of the Lyapunov function becomes negative when the estimation error increases over a certain threshold, i.e.,

$$\|e\| \geq \bar{e} \stackrel{\triangle}{=} \frac{c}{\lambda_{min}\,(Q)} \;\Rightarrow \dot{V}\,(e) \leq 0. \tag{21}$$

Thus, the estimation error is bounded and its upper bound is $\bar{e}$; in other words, there exists a closed ball $B(\bar{e})$ of radius $\bar{e}$ and centered in the origin, such that $e(t) \in B(\bar{e})\,,\ \forall t \geq 0\,.$

5 Conclusions

The relationship between the observation and the filtering problems has been investigated within an $\mathcal{L}_p$ framework for continuous-time, nonlinear dynamic systems. An illustrative example has been considered, to show how the proposed method can be applied to dynamic systems with Lipschitz nonlinearities.

Future work will concern the design of estimators for Lipschitz nonlinear dynamic systems in an $\mathcal{L}_p$ framework.

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