A Worst-Case Estimate of Stability Probability for Polynomials with Multilinear Uncertainty Structure

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Abstract

probability of stability for an uncertain polynomial which has as coefficients multilinear functions of real, random, independent parameters q_i. The result requires little apriori information about the probability distributions of these uncertain parameters. We only require that the distributions are symmetric about zero, non-increasing as $|q_i|$ increases, and supported on a given interval $[-r_i, r_i]$. The probability estimate is sharp in the sense that the estimated probability of stability is $\hat{\mathrm{p}}^* = 1$ when the uncertainty bounds r_i are below the deterministic robustness radius r_{map} obtained with the Mapping Theorem. To obtain the probabilistic estimate, we recast the problem so that the following characterization of stability is applicable: If the Nyquist curve for a proper plant lies to the right of a frequency-dependent separating line through $-1+j0$ at each frequency, then stability is guaranteed. The result is applied in a numerical example, illustrating a common amplification phenomenon: Even when the magnitude of uncertainty is significantly greater than the deterministic robustness bound, the risk of instability is small.

1. Introduction

The main result of this paper applies to robust stability problems involving an uncertain polynomial whose coefficients depend multilinearly on a real random m-tuple of parameters

$$q \doteq (q_1,\ q_2,\ \ldots\ ,\ q_m)$$

with given bounds $|q_i| \leq r_i$ for $i = 1, 2, \ldots, m$. The stability problem is analyzed from a probabilistic point of view. If the bound on an uncertain parameter exceeds the stability margin given by deterministic robustness analysis, there is a risk of instability. This paper provides a worst-case estimate of the probability of stability associated with a given set of uncertainty bounds.

1.1 Motivating Factors

In many applications, some risk of instability may be acceptable, and one can use the estimate provided here to calculate the magnitude of uncertainty allowed at an acceptable risk level. Increasing the amount of uncertainty allowed is important from an applications standpoint; increased parameter tolerance can improve design feasibility, manufacturing cost and other aspects of system design. Often, the risk of instability is small even when the magnitude of the uncertainty far exceeds the deterministic stability margin. This is the case for the numerical example, presented in Section 3.

1.2 Properties of the Estimate

The estimate of stability probability provided in this paper has three important qualities: First, the computational requirement associated with estimate computation increases only modestly with the number of uncertain parameters. Second, the estimate is sharp, in the sense that the probability of stability is correctly estimated as $\hat{\mathrm{p}}^* = 1$ when the radius of uncertainty for q is below the deterministic radius provided by the Mapping Theorem; e.g, see [1]. Third, the estimate requires little apriori information concerning the probability distribution of q. That is, we specify a class of admissible distributions $\mathcal{F}$ for q and prove that the probability of stability associated with any distribution $f \in \mathcal{F}$ is no lower than our worst-case estimate. In this sense, the worst-case estimate is said to be *distributionally robust*; see [2] and [3] for more details motivating this approach.

1.3 Admissible Probability Distributions

In this paper, consistent with the paradigm in [2], it is assumed that each independent uncertain parameter perturbation q_i has an associated probability density function f_i, which is unknown except for the fact that it has the following properties: Each f_i is non-increasing with respect to $|q_i|$, is symmetric about zero, and has a

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support interval $[-r_i, r_i]$. The joint probability density function for the random vector q is thus

$$f(q) \doteq \prod_{i=1}^{m} f_i(q_i).$$

We let $\mathcal{F}$ denote the class of joint probability density functions which can be expressed as such a product of marginal density functions $f_i(q_i)$ with each f_i satisfying the aforementioned conditions. The uncertainty bounding set

$$Q \doteq [-r_1, r_1] \times [-r_2, r_2] \times \cdots \times [-r_m, r_m]$$

specifies the support intervals of the f_i. For further explanation of the paradigm underlying this type of probabilistic setup, see [2] and [3]. We view the uncertainty in the polynomial coefficients as deviations q_i from their nominal values. The non-increasing property essentially means that large errors in q_i are less likely than small errors and symmetry can be interpreted as equal likelihood of positive and negative errors.

2. Formulation, Approach and Main Result

We now provide the problem formulation and summary of our approach and main result.

2.1 Formulation of Problem and Main Result

We consider the uncertain polynomial

$$p(s, q) \doteq p_0(s) + \Delta p(s, q)$$

where $p_0(s)$ is a fixed polynomial, the so-called *nominal*, and $\Delta p(s, q)$ is a polynomial with coefficients depending multilinearly on q with $\Delta p(s, 0) \equiv 0$. Given only the polynomial $p(s, q)$ and the uncertainty bounding set Q, we wish to find the probability that $p(s, q)$ is stable. We assume that the q_i are independent, and that q has a joint probability density function $f \in \mathcal{F}$. We make the following additional assumptions:

Assumption 1: $\deg \Delta p(s, q) \leq \deg p_0(s)$
Assumption 2: $\deg p(s, q) = \deg p_0(s)$ for all $q \in Q$
Assumption 3: $p_0(s)$ is stable; i.e., $p(s, 0)$ is stable with strict left half plane roots.

The main objective of this paper is to obtain a *distributionally robust worst-case* estimate $\hat{\mathrm{p}}^*$ of the probability of stability. Namely, we seek to estimate the quantity

$$\mathrm{p}^* \doteq \inf_{f \in \mathcal{F}} \mathrm{Prob}\{p(s, q^f) \text{ is stable}\}$$

where q^f is the random vector with probability density function $f \in \mathcal{F}$.

2.2 Characterization of Stability

Assumption 1 indicates that the *fictitious plant*

$$G(s, q) \doteq \frac{\Delta p(s, q)}{p_0(s)}$$

is proper. Hence, stability of $p(s, q)$ for a particular q can be analyzed by applying the Nyquist criterion to the plant $G(s, q)$. Specifically, if the Nyquist path for $G(j\omega, q)$ does not cross the ray $(-\infty, -1]$, then encirclement of $-1 + j0$ is impossible, indicating that $p(s, q)$ is stable. Assumption 2 indicates that $\lim_{\omega \to \infty} \mathrm{Re}\, G(j\omega, q) > -1$; if the degrees of Δp and p_0 were equal and the leading coefficients of Δp and p_0 could be made equal in magnitude and opposite in sign, then $\deg p(s, q) < \deg p_0(s)$ for some q. Therefore, $G(j\omega, q)$ cannot reach a limit on the ray $(-\infty, -1]$ as $\omega \to \infty$. Thus, $p(s, q)$ is stable if

$$G(j\omega, q) \cap (-\infty, -1] = \phi$$

for all $\omega \geq 0$. In turn, this condition is equivalent to the existence of a real-valued function $x(\omega)$ such that

$$x(\omega)\mathrm{Im}\ G(j\omega, q) - \mathrm{Re}\ G(j\omega, q) < 1$$

for all $\omega \geq 0$. This condition can be interpreted geometrically as follows: If $G(j\omega, q)$ lies strictly to the right of the line through $-1 + j0$ with slope $\frac{1}{x(\omega)}$ for all $\omega \geq 0$, then $G(j\omega, q)$ is separated from the ray $(-\infty, -1]$ at each $\omega \geq 0$. To illustrate this condition, Figure 2.2.1 depicts a Nyquist curve separated from $(-\infty, -1]$ by a frequency-dependent line: $x(\omega) = x_1$ for $\omega < \omega_1$ and $\omega \geq \omega_2$, $x(\omega) = x_2$ for $\omega \in [\omega_1, \omega_2)$ accomplishes the separation.

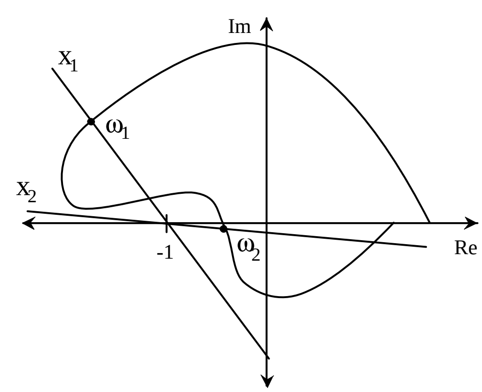

Figure 2.2.1: Separation of $G(j\omega, q)$ From $(-\infty, -1]$

This stability condition is also equivalent to the condition provided by the Mapping Theorem, namely

$$-1 + j0 \notin \mathrm{conv}\ G(j\omega, \{q^i\})$$

for all $\omega \geq 0$ where $\{q^i\}$ is the set of vertices of the uncertainty bounding set Q. For, if a line separates each $G(j\omega, \{q^i\})$ from the ray $(-\infty, -1]$ at some ω, it also separates $\mathrm{conv}\ G(j\omega, \{q^i\})$ from the ray at that ω. Conversely, it is known that $G(j\omega, q) \in \mathrm{conv}\ G(j\omega, \{q^i\})$ for all $q \in Q$.

2.3 First Main Result: A Separator Demonstrating Stability

Let r_{map} denote the maximum value r for which the Mapping Theorem stability condition is satisfied for all $r_i < r$, $i = 1, 2, \ldots, m$. We now sketch the construction of a rotating separator $x^*(\omega)$ which demonstrates stability whenever $Q \subseteq \prod_{i=1}^{m}[-r_{map}, r_{map}]$. We will show that under this assumption on Q, with σ sufficiently small and $1 \geq \sigma > 0$, the function $x^*(\omega)$ which minimizes

$$\Phi(x, \omega) \doteq \int_Q \max\{1-\sigma, x \text{ Im } G(j\omega, q) - \text{Re } G(j\omega, q)\} dq$$

with respect to x at each $\omega \geq 0$, establishes robust stability of $p(s, q)$.

2.4 Proof of First Main Result

Indeed, we assume that $Q \subseteq \prod_{i=1}^{m}[-r_{map}, r_{map}]$. We first claim that there exists an $\epsilon_1 > 0$ and corresponding ω_{max} such that

$$-\text{Re } G(j\omega, q) \leq 1 - \epsilon_1$$

for all $\omega > \omega_{max}$ and all $q \in Q$. This is possible under our assumption on Q: We must have $\lim_{\omega \to \infty} \text{Re } G(j\omega, q) > -1$. Let $x_{min}(\omega)$ and $x_{max}(\omega)$ define a cone of acceptable separators for each $\omega \geq 0$, as shown in Figure 2.4.1.

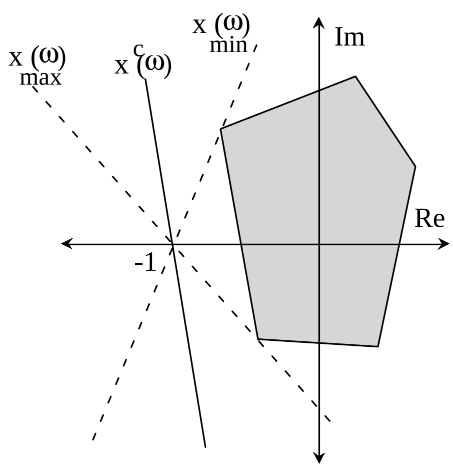

Figure 2.4.1: Cone of Acceptable Separators

Our assumption on Q indicates that, for each $\omega \geq 0$, $x_{min}(\omega) < x_{max}(\omega)$ holds. If this were not true, then conv $G(j\omega, \{q^i\})$ would contain $-1 + j0$. Letting $x^c(\omega) \doteq \frac{1}{2}(x_{max}(\omega) - x_{min}(\omega))$ and letting $\theta(\omega)$ denote the angle between the separator for $x^c(\omega)$ and the real axis, we see that $x^c(\omega) = \cot \theta(\omega)$. Since $\theta(\omega) \in (0, \pi)$ for all ω and the vertices $G(j\omega, q^i)$ vary continuously with ω, $x^c(\omega)$ must be continuous. Thus the function $x^c(\omega)\text{Im } G(j\omega, q) - \text{Re } G(j\omega, q)$ is continuous in ω and attains a maximum value on the interval $[0, \omega_{max}]$. Our assumption on Q indicates that this maximum value is strictly less than one. Thus, for some $\epsilon_2 > 0$,

$$x^c(\omega)\text{Im } G(j\omega, q) - \text{Re } G(j\omega, q) \leq 1 - \epsilon_2$$

for all $\omega \in [0, \omega_{max}]$ and all $q \in Q$. Now with $\sigma \doteq \min\{\epsilon_1, \epsilon_2\}$, $\chi_{[0,\omega_{max}]}(\omega)$ denoting the characteristic function on $[0, \omega_{max}]$, and

$$\tilde{x}^c(\omega) \doteq x^c(\omega)\chi_{[0,\omega_{max}]}(\omega),$$

it follows that

$$\tilde{x}^c(\omega)\text{Im } G(j\omega, q) - \text{Re } G(j\omega, q) \leq 1 - \sigma$$

for all $\omega \geq 0$ and all $q \in Q$. Note that at each ω,

$$\Phi(x^*(\omega), \omega) \leq \Phi(\tilde{x}^c(\omega), \omega) \leq \int_Q (1 - \sigma) dq.$$

The integrand in Φ can never be smaller than $1 - \sigma$. Thus, the above inequality implies that for each $\omega \geq 0$ and each $q \in Q$,

$$\max\{1 - \sigma, x^*(\omega)\text{Im } G(j\omega, q) - \text{Re } G(j\omega, q)\} = 1 - \sigma$$

and thus

$$x^*(\omega)\text{Im } G(j\omega, q) - \text{Re } G(j\omega, q) \leq 1 - \sigma$$

for all $\omega \geq 0$ and all $q \in Q$. Therefore the rotating separator $x^*(\omega)$ proves robust stability of $p(s, q)$ when $Q \subseteq \prod_{i=1}^{m}[-r_{map}, r_{map}]$.

Although both $x^*(\omega)$ and $x^c(\omega)$ prove stability and, as will be shown next, lead to sharp probability estimates, extreme point computations are not needed to calculate $x^*(\omega)$. As a result, the computation required for our probabilistic estimate does not increase quickly with the number of uncertain parameters.

2.5 Second Main Result: The Sharp Worst-Case Probabilistic Estimate

Define the function

$$\varphi(q) \doteq \max\{1-\sigma, \sup_{\omega \geq 0} x^*(\omega)\text{Im } G(j\omega, q) - \text{Re } G(j\omega, q)\}$$

with σ and $x^*(\omega)$ as in Section 2.3. We now claim that

$$\hat{\text{p}}^* \doteq 1 - \inf_{\gamma \geq 1} \frac{1}{\prod_{i=1}^{m} r_i} \int_Q \varphi^\gamma(q) dq$$

is a worst-case probabilistic estimate for p^*. That is, we claim $\text{p}^* \leq \hat{\text{p}}^*$. We also claim that $\hat{\text{p}}^*$ is a sharp estimate. That is, it agrees with classical robustness analysis in that $\hat{\text{p}}^* = 1$ when $Q \subseteq \prod_{i=1}^{m}[-r_{map}, r_{map}]$. We make use of the lemma below from [4] in the proof of our main result.

2.6 Lemma

Suppose that $J : \mathbf{R}^m \rightarrow \mathbf{R}_+$ is a coordinate convex function, and q^f is an m-dimensional vector of independent random variables with uncertainty bounding set Q and joint probability density function $f \in \mathcal{F}$. Then

$$E[J(q^f)] \leq E[J(q^u)]$$

where q^u is the m-dimensional random variable with uniform probability distribution over the box Q.

2.7 Proof of Second Main Result

To form the probability estimate, we consider the set of q which may lead to instability according to the previous characterization, denoted as Q_{bad}. Note that for such q,

$$x^*(\omega)\text{Im } G(j\omega, q) - \text{Re } G(j\omega, q) \geq 1$$

for some $\omega \geq 0$. This means

$$Q_{bad} = \{q \in Q : \varphi(q) \geq 1\}.$$

Let q^f denote the random vector with joint density function $f \in \mathcal{F}$. By Markov's inequality, for any $\gamma \geq 1$, we have

$$\text{Prob}\{\varphi(q^f) \geq 1\} \leq E[\varphi^\gamma(q^f)].$$

One can see that $\varphi(q)$ is a nonnegative, coordinate convex function of q; that is, for each i, $\varphi(q)$ is a convex function of q_i when the other components of q are held constant. The function $\varphi^\gamma(q)$ is then nonnegative and coordinate convex for $\gamma \geq 1$. Applying Lemma 2.6, it follows that for any $f \in \mathcal{F}$ and any $\gamma \geq 1$,

$$E[\varphi^\gamma(q^f)] \leq E[\varphi^\gamma(q^u)].$$

Thus, the quantity

$$\hat{\text{p}}^* \doteq 1 - \inf_{\gamma \geq 1} \frac{1}{\prod_{i=1}^m r_i} \int_Q \varphi^\gamma(q) dq$$

is a lower bound for the worst-case probability of stability p^*. The analysis of Section 2.4 led to

$$\varphi(q) = 1 - \sigma$$

valid for all $q \in Q$ when $Q \subseteq \prod_{i=1}^m [-r_{map}, r_{map}]$. Then under this assumption on Q,

$$\lim_{\gamma \rightarrow \infty} \frac{1}{\prod_{i=1}^m r_i} \int_Q \varphi^\gamma(q) dq = 0,$$

that is, $\hat{\text{p}}^* = 1$. Our probabilistic estimate of stability thus agrees with deterministic results, and we call our estimate sharp.

3. Numerical Example

We now illustrate the application of our results to an example system.

3.1 Mass-Spring-Damper System

We consider a five-mass version of the mass-spring-damper system described in [5], shown in Figure 3.1.1. Fifteen uncertain parameters are considered. The robust performance specification is that the real parts of the eigenvalues of the system matrix $A(s, q)$ do not exceed -0.01. The system is described by the differential equations

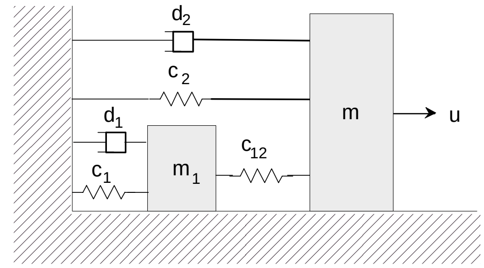

Figure 3.1.1: Mass-Spring-Damper System

$$\begin{aligned}
0 &= m_1\ddot{x}_1 + d_1\dot{x}_1 + c_1 x_1 + c_{12}(x_1 - x_2); \\
0 &= m_2\ddot{x}_2 + d_2\dot{x}_2 + c_2 x_2 + c_{23}(x_2 - x_3) + c_{12}(x_2 - x_1); \\
0 &= m_3\ddot{x}_3 + d_3\dot{x}_3 + c_3 x_3 + c_{34}(x_3 - x_4) + c_{23}(x_3 - x_2); \\
0 &= m_4\ddot{x}_4 + d_4\dot{x}_4 + c_4 x_4 + c_{45}(x_4 - x_5) + c_{34}(x_4 - x_3); \\
u &= m_5\ddot{x}_5 + d_5\dot{x}_5 + c_5 x_5 + c_{45}(x_5 - x_4)
\end{aligned}$$

leading to a system matrix A(s,q) such that

$$A_{ij}(s, q) = \begin{cases} m_i s^2 + d_i s + c_i + c_{i(i+1)} & i = j, \ i < 5; \\ m_i s^2 + d_i s + c_i & i = j = 5; \\ -c_{ij} & i = j - 1; \\ -c_{ji} & i = j + 1; \\ 0 & \text{otherwise} \end{cases}$$

with corresponding uncertain polynomial

$$p(s, q) = \det A(s - 0.01, q)$$

to be analyzed for robust stability. The spring constants for the coupling springs are assumed to be known, with the values $c_{12} = c_{23} = c_{34} = c_{45} = 1$. The masses are uncertain and have the values $m_i = 2 + q_i$ for $i = 1, 3, 5$ and $m_i = 3.5 + 1.5\, q_i$ for $i = 2, 4$. The damping coefficients are uncertain and have the values $d_i = 1.25 + 0.75\ q_{i+5}$ for $i = 1, 2, 3, 4, 5$. The other springs have uncertain spring constants, given by $c_i = 1.5 + 0.5\ q_{i+10}$ for $i = 1, 3, 5$, and $c_i = 2 + q_{i+10}$ for $i = 2, 4$.

3.2 Numerical Results

Table 3.2.1 and Figure 3.2.2 summarize the numerical results obtained in this analysis. The quantity r_{map} is estimated at 0.067, an upper bound. We see the common amplification phenomenon: Namely, the uncertainty level can be increased by almost 50% over r_{map} with a risk of instability of only one in one thousand, and the uncertainty level can be double r_{map} with only a 5% risk of instability.

Estimate of Stability Probability $\hat{p}^*$	Allowed Uncertainty r_i	Amplification r_i/r_{map}
0.999	0.100	1.49
0.99	0.115	1.71
0.95	0.130	1.94

Table 3.2.1: Confidence vs. Allowable Uncertainty

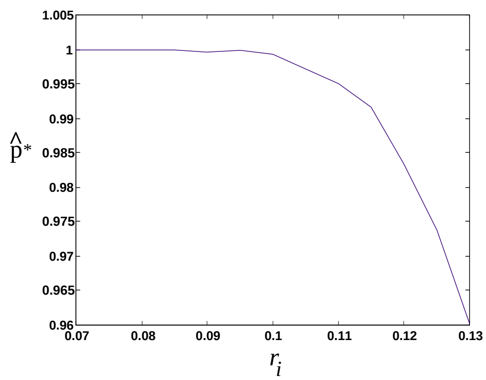

Figure 3.2.2: Confidence Degradation

4. Concluding Remarks

We have demonstrated that our worst-case estimate $\hat{p}^*$ probability of stability p^* is sharp with respect to the classical bound r_{map} obtained via the Mapping Theorem. However, for radii of uncertainty beyond r_{map}, there seems to be room for improvement in the estimation of probability. It is felt that this might be accomplished by further research aimed at obtaining alternative separators in lieu of $x^*(\omega)$. It should also be noted that the bound obtained using uniform distribution is conservative in the sense that it is generated using a Markov inequality; i.e., the probability of stability is bounded via an appropriately constructed expectation. Hence, there may be a way to improve the probabilistic estimate by working directly with the probability of stability.

However, if one proceeds in this manner, a warning is in order: Although the uniform distribution leads to the worst-case expectation with respect to $f \in \mathcal{F}$, this need not be the case when dealing with probability. This point is illustrated via the uncertain polynomial

$$p(s,q) = s^2 + 2s + 1 + (s+1)(q_1 q_2 - 2q_1 + 2q_2).$$

For uncertainty radii $r_i = 3.73$, direct computation using the uniform distribution shows

$$\text{Prob}\{q \in Q_{bad}\} \approx 0.593$$

but as r increases, it can be verified that uniform distribution leads to

$$\lim_{r\to\infty} \text{Prob}\{q \in Q_{bad}\} \approx 0.5.$$

Thus, it follows that if the uncertainty bounding set has sufficiently large radius, the uniform distribution over Q is not the worst-case distribution with respect to $f \in \mathcal{F}$. That is, there is another density function $f^* \in \mathcal{F}$ leading to lower probability of stability.

References

[1] L. A. Zadeh and C. A. Desoer, *Linear Systems Theory – A State Space Approach.* McGraw-Hill, New York, 1963.

[2] Barmish, B. R. and C. M. Lagoa. "The uniform distribution: A rigorous justification for its use in robustness analysis," *Mathematics of Control, Signals and Systems*, vol. 10, no. 3, pp. 203-222, 1997.

[3] Lagoa, C. M. "Contributions to the Theory of Probabilistic Robustness," Ph.D. Dissertation, Department of Electrical and Computer Engineering, University of Wisconsin-Madison, 1998.

[4] B. R. Barmish and P. Shcherbakov, "Distributionally Robust Least Squares," Proceedings of the *International Symposium on Adaptive Systems*, pp. 5-8, St. Petersburg, Russia, 1999.

[5] J. Ackermann, *Robust Control: Systems with Uncertain Physical Parameters.* Springer-Verlag, New York, 1993.

[6] B. R. Barmish, *New Tools for Robustness of Linear Systems.* Macmillan, New York, 1994.

[7] B. R. Barmish, M. Fu and S. Saleh, "Stability of a polytope of matrices; Counterexamples," *International Journal of Control*, vol. AC-33, pp. 569-572, 1988.

[8] R. R. E. de Gaston and M. G. Safonov, "Exact calculation of the multiloop stability margin," *IEEE Transactions on Automatic Control*, vol. AC-33, pp. 156-171, 1988.

[9] A. Sideris and R. S. Sanchez Pena, "Fast computation of the multivariable stability margin for real interrelated uncertain parameters," *IEEE Transactions on Automatic Control*, vol. AC-34, pp. 1272-1276, 1989.