Hierarchical Optimization in the Presence of an Intelligent Adversary - an H_∞ Approach

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Abstract

In this paper, we consider the H_∞ optimization problem for systems with a hierarchical structure. We propose an iterative algorithm to treat the problem. In each iteration of our algorithm, we need to solve a problem of H_∞ optimization feature, but with a much smaller dimension. We also show that the solution of the optimization problem in each iteration can be obtained by solving a set of linear matrix inequalities.

Keywords: hierarchical structure, H_∞ control synthesis

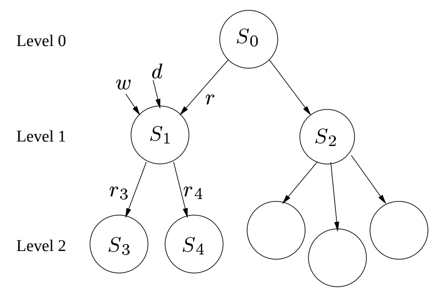

Figure 1: Hierarchical structure in the military operations model

1 Motivation

The chain of command in military operations gives a natural hierarchy and decomposition strategies. The notion of hierarchical optimization is not a new one and has been studied for over thirty years. The large scale nature of the military operations' decision making problem in the presence of uncertain data and an intelligent adversary makes the problem more difficult. This class of problems have not been previously dealt with. The recent advances in robust control theory and the associated H_∞ optimization problems have given us new tools and techniques unlike those considered in the past. Some of the relevant issues that we consider here include the presence of a large number of variables and constraints, dynamic nature of the optimization problem, presence of an intelligent adversary. We begin by formulating the mathematical model.

2 Mathematical Model

In this section, we will formulate a mathematical model of an "hierarchical H_∞ optimization problem". We assume that there is a natural decomposition of the system which has been already chosen, as shown in Figure 1. Each node in the figure represents a "sector" that is a representative of a certain level of granularity of the model. Typically, this could represent a specific geographical region where a person is in command. As we move from bottom to top in the figure, we assume that we are moving higher in the chain of command. The topmost node represents the top level of the command hierarchy. At each sector, there are several types of resources present. Examples include number of tanks, aircrafts, bombers, radars, men, food supplies etc. Our objective is to distribute these resources among the various sectors in order to suppress the enemy activities.

This problem is combinatorial in nature and the complexity increases dramatically even with a few choices of resources and sectors. The dynamics of the problem introduces even larger number of variables that makes the problem intractable. We would therefore like to approximate the problem and also use the hierarchical structure to design a suboptimal control strategy. There are two different ways of approximating the problem: one can formulate a detailed and fairly accurate nonlinear model of the system and use a generic optimization method to obtain a suboptimal solution for this highly nonconvex problem, or on the other hand, one can approximate the mathematical model instead and obtain the optimal solution to the optimization problem using the interior point methods. We will use the second approach and pose the problem as an H_∞ optimization problem. This would allow us to include the enemy as an intelligent adversary as opposed to a disturbance that we would like to suppress.

The mathematical model used is a linearized model of the flow of resources. We use a symmetrical model for both friendly and enemy resources in any given sector and they are represented in any given sector as

$$x_s(n+1) = x_s(n) + r_s(n) - u_s(n) - \alpha_s\ y_s(n) \quad (1)$$
$$y_s(n+1) = y_s(n) + d_s(n) - w_s(n) - \beta_s\ x_s(n) \quad (2)$$

where x_s represents friendly resources in sector s, y_s the enemy resources in the same sector, r_s are friendly resources coming from upper level, u_s are friendly resources leaving sector s, d_s are enemy resources from his upper level and w_s the enemy resources that leave the sector. The term $\alpha_s\ y_s(n)$ represent a linearized conflict loss model for friendly resources and similarly there is a corresponding term in enemy state evolution equations. The parameter α_s represents a percentage of loss the enemy could inflict in this sector. The conflict loss term is nonlinear in general, but by choosing variables that are variations about a nominal value, we could linearize this term and represent them as above. Our objective then is to regulate about these nominal values. One way to see the importance of this approach is by assuming that an initial strategy has been designed that brings the state of the system to a desired value. We then use the method that will be described here to maintain this desired state even in the presence of an active intelligent adversary.

2.1 Centralized H_∞ Optimization Approach

The combined state-space equations (written in continuous time) of all the sectors can be written in the following standard form

$$\dot{x} = Ax + B_1 w + B_2 u \quad (3)$$

where x represents both the friendly states and the enemy states. The control variables that we need to design are given by u and the enemy disturbances are represented as w. Here, not all states are measurable (clearly not all enemy states are known) and the measurements do have some errors. Let us represent all the measured variables by y. In addition, we have some performance variables z. These variables are chosen such that they represent the cost in moving friendly resources, superiority in each sector, etc. This model is shown in Figure 2.

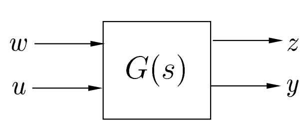

Figure 2: Centralized model

The objective of the centralized H_∞ optimization problem is to design the control u as a function of y such that the worst case 2-norm of the performance variables z over all disturbances w with $||w||_2 \leq 1$ is minimized.

This centralized H_∞ optimization problem has been extensively studied in the last two decades and the solution can be obtained by either solving the Riccati equations associated with the system [1], or solving a system of Linear Matrix Inequalities (LMIs) [2]. Recently, very efficient interior point methods that have been developed to solve these LMI's in particular. However the large dimensionality of our problems make the size of the LMI's sufficiently large that the interior point methods become intractable. So our approach is to decompose the problem by using the hierarchical structure and rewrite them in such a way that we still preserve the H_∞ structure at each level. Further, the aim is to have a small dimensions at each level enabling us to use the efficient interior point methods on these lower dimensional LMI's. We describe this in greater detail in the following section.

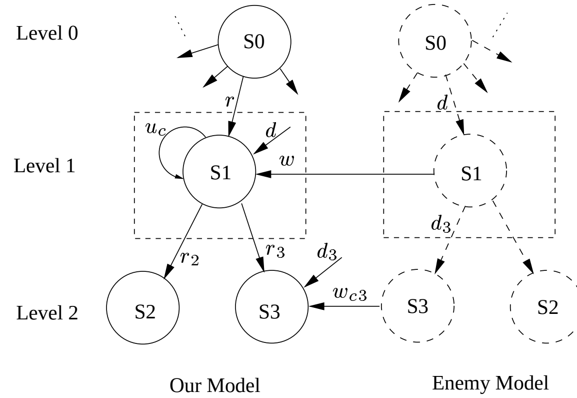

Figure 3: Model with hierarchical structure

2.2 Hierarchical H_∞ Optimization Approach

We formulate a hierarchical model by dividing the problem into several levels as shown in Figure 3. In this formulation, the performance objectives are chosen from the top level to the bottom level whereas the control design problem is solved from bottom to the top level. The nodes in the figure are represented as sectors (shown as S_0, S_1 etc).

As we have mentioned earlier, one of the main drawback in the centralized approach is that the combined state-space has a very large dimension. Therefore, our primary objective in formulating a hierarchical approach is to reduce the size of the state-space at each sector. The second important feature is to have the same structure of equations at each level in the hierarchy. The third feature we need in order to claim that it is a hierarchical H_∞ problem is that the objective that is passed from one level to the higher level is in such a way that we still have a suitable generalized H_∞ problem at the higher level. In other words, we would like to formulate the hierarchical problem with the above three features by satisfying the same global performance objective as that of the centralized problem. We further need to ensure internal stability of the combined system.

One of the main difficulties that we encounter in trying to formulate a hierarchical problem is that the difficulty of the control design problem as we go higher in the hierarchy, should not be increased due to an unreasonable choice of objective at the lower levels. This is reflected in the choice of objectives as well as the choice of the reduced order model that is passed to the upper level. In the rest of this section, we will proposed an iterative algorithm which has the three feasures mentioned above to solve the hierarchical H_∞ optimization problem.

Let us consider sector S_1 shown in Figure 3. A brief description of the signals involved are (see Figure 3):

- w: current sector enemy resource allocations.
- d: upper level enemy resource allocations.
- r: upper level friendly resource allocations that are passed onto the current sector.
- k: index of all sectors of the lower level connected to the current sector.
- w_k: enemy disturbances of all sectors in the lower level (represented by index k) connected to the current sector. This is similar to the variable w described above, but that corresponding to the sector k of the lower level.
- d_k: similar to the variable d corresponding to the sector in lower level k.
- u_c: current sector friendly resource allocations that need to be designed.
- r_k: current sector friendly resource allocations that are passed onto the lower level sector k. This signal corresponds to the signal r described above of the sector k in the lower level.
- y: all the measurement signals.
- z: current sector performance variables.

The approach taken to solve the hierarchical optimization problem can be outlined conceptually as (taking as a specific example from the Figure 3)

1. the reduced-order states and the objectives of sectors S_2 and S_3 are passed to S_1.
2. current objective and objectives from S_2 and S_3 are combined to form a generalized H_∞ problem.
3. solve the H_∞ problem for the controls and simultaneously the reduced-order states and objective of S_1 to pass to S_0.
4. the same process is repeated at all the other sectors.

We now describe the above steps in greater detail. Figure 4 shows the plant dynamics expressed in the standard form for a current sector level. Some of the sectors may not have all the signals shown in the figure (for example, the topmost level will not have w, d and r, while the lowest level will not have w_k, d_k and r_k). We have split the disturbance and control signals so that the hierarchy becomes apparent. We begin by first writing the state-space equations of the current sector level as

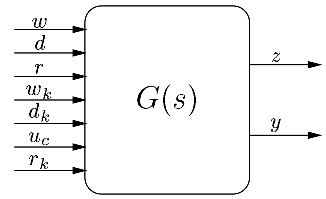

Figure 4: Plant dynamics of a current sector expressed in standard form

given in equations (1) and (2) in a more general form (and in continuous time) as

$$\begin{aligned}
\dot{x} &= Ax + B_1 w + B_2 d + B_3 r + B_4 u_c + B_{2k} d_k + B_{3k} r_k \\
z &= C_1 x + D_{11} w + D_{12} d + D_{13} r + D_{14} u_c + D_{13k} r_k \\
y &= C_2 x + D_{21} w + D_{22} d + D_{23} r, \qquad (4)
\end{aligned}$$

where y is the measurements in the current sector, and z is the current sector performance variables. We have used the notation that repeated index k as in, for example, $B_{2k} d_k$ is actually a summation $\sum_{k \in K} B_{2k} d_k$ where K is the set of all lower level sectors connected to the current sector. The states of a suitable "reduced-order" system passed from the lower level k are represented by x_{Π_k} and the corresponding state-space equations are given by

$$\dot{x}_{\Pi_k} = A_{\Pi_k} x_{\Pi_k} + B_{w_k} w_k + B_{d_k} d_k + B_{r_k} r_k \qquad (5)$$

and the objectives from the lower level sectors are given by

$$\int_{-\infty}^{+\infty} \tilde{d}_k' \Pi^k(\omega) \tilde{d}_k d\omega, \quad \tilde{d}_k' = \begin{pmatrix} w_k' & d_k' & r_k' \end{pmatrix}'. \qquad (6)$$

We now represent the current sector generalized H_∞ optimization problem as minimizing the maximum eigenvalue over all frequencies of $\Pi(\omega)$ of the form

$$\Pi(\omega) = \begin{pmatrix} 0 & \beta_1(j\omega) & \beta_2(j\omega) \\ \beta_1(j\omega)' & \gamma_{11}(\omega) & \gamma_{12}(j\omega) \\ \beta_2(j\omega)' & \gamma_{12}(j\omega)' & \gamma_{22}(\omega) \end{pmatrix} \qquad (7)$$

in other words, we want to solve

$$\begin{aligned}
&\max_{\omega} \lambda_{\max}(\Pi(\omega)) \to \min, \quad \text{subj to} \\
&||z||^2 + \sum_{k=1}^{n_l} \int_{-\infty}^{+\infty} \tilde{d}_k' \Pi^k(\omega) \tilde{d}_k d\omega \le \lambda(||w||^2 + \sum_{k=1}^{n_l} \|d_k\|^2) \\
&+ \int_{-\infty}^{+\infty} \tilde{d}' \Pi(\omega) \tilde{d}\, d\omega \quad \forall\, w, d, r, w_k \text{ and } d_k, \qquad (8)
\end{aligned}$$

for a given value of λ. Here, $\tilde{d}' = \begin{pmatrix} w' & d' & r' \end{pmatrix}'$. Note that in (8), n_l denotes the number of sectors that is under control of the current one. For example, $n_l = 2$ for the sector $S1$ of the system in Figure 3. The design variables are $u_c, r_k, \beta_1(j\omega), \beta_2(j\omega), \gamma_{11}(\omega), \gamma_{12}(j\omega)$ and $\gamma_{22}(\omega)$. Note that this optimization problem is a convex optimization problem jointly in all the design variables if we rewrite equation (8) in terms of the Youla parameter.

Let us represent the states of the system $\Pi(\omega)$ as x_Π with state-space representation given by

$$\dot{x}_\Pi = A_\Pi x_\Pi + B_w w + B_d d + B_r r. \tag{9}$$

These are the reduced-order states of the current sector which, together with the objective

$$\int_{-\infty}^{+\infty} \tilde{d}' \Pi(\omega) \tilde{d} \; d\omega$$

will be passed to the upper level. We use the same procedure for all the sectors for the same value of λ starting from the lowest level. The global objective in the hierarchical optimization problem is to obtain the smallest value of λ by using a binary search. We illustrate the algorithm by the following simple example.

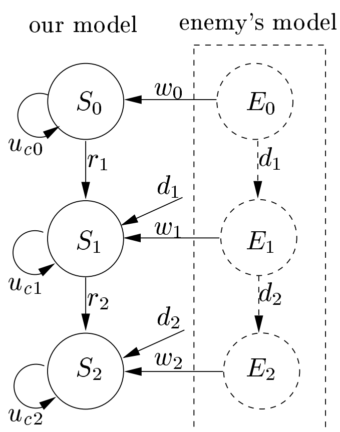

Figure 5: Example for hierarchical H_∞ design algorithm

Example 1. Consider the hierarchical system shown in Figure 5. The friendly resources are divided into three sectors and the sector located in the upper position has control of the resources in the lower sectors. We assume that our enemy has the same command hierarchy. Each w_k represents the influence on the friendly resources at the sector from the enemy at the same level. The enemy's resources movement should also have some influence on our resources. These influences are captured by d_k, another disturbance to the corresponding sector, as shown in the figure. Let z_k denote the performance measurement of each sector, and our goal is to design u_{ck} and r_k such that

$$\sum_{k=0}^{2} ||z_k||^2 \le \lambda (\sum_{k=0}^{2} ||w_k||^2 + \sum_{k=1}^{2} ||d_k||^2) \tag{10}$$

for some λ. Let $\tilde{d}_1' = \begin{pmatrix} w_1' & d_1' & r_1' \end{pmatrix}$, and $\tilde{d}_2' = \begin{pmatrix} w_2' & d_2' & r_2' \end{pmatrix}$. By the proposed algorithm, we first select a λ_1, say 1. Then we try to solve the optimization problem

$$\max_\omega \lambda_{\max}(\Pi_2(\omega)) \to \min, \quad \text{subject to,}$$
$$||z_2||^2 \le \lambda_1 ||w_2||^2 + \int_{-\infty}^{+\infty} \tilde{d}_2' \Pi_2(\omega) \tilde{d}_2 \; d\omega, \;\; \forall \; \tilde{d}_2 \tag{11}$$

to obtain the control strategy u_{c2} and a reduced-order model $\Pi_2(\omega)$ for sector S_2. Supposed (11) is solvable. We then pass $\Pi_2(\omega)$ to the upper sector, move up to the next level and solve

$$\max_\omega \lambda_{\max}(\Pi_1(\omega)) \to \min, \quad \text{subject to,}$$
$$||z_1||^2 + \int_{-\infty}^{+\infty} \tilde{d}_2' \Pi_2(\omega) \tilde{d}_2 \; d\omega \le \lambda_1 (||w_1||^2 + ||d_2||^2)$$
$$+ \int_{-\infty}^{+\infty} \tilde{d}_1' \Pi_1(\omega) \tilde{d}_1 \; d\omega, \quad \forall \; w_2, \; d_2, \; \tilde{d}_1, \tag{12}$$

to obtain u_{c1}, r_2 and $\Pi_1(\omega)$ for the sector S_1. Finally, we move up to toppest level and solve

$$\text{Find } u_{c0} \text{ and } r_1, \quad \text{such that,} \tag{13}$$
$$||z_0||^2 + \int_{-\infty}^{+\infty} \tilde{d}_1' \Pi_1(\omega) \tilde{d}_1 \; d\omega \le \lambda_1 (||w_0||^2 + ||d_1||^2),$$

for all w_0, w_1, d_0 and d_1. It is obvious that if we do solve (11) (12) and (13), then we indeed obtain a control strategy $(u_{ck}, \; r_k)$ such that (10) holds for the λ_1 we select. We then take $\lambda_2 = 0.5\lambda_1$ and repeat the procedure. If any one of (11) (12) (13) is unsolvable, we increase the value of λ and restart the procedure from the buttom level, i.e., (11). We stop the precedure when the λ we obtain matches a certain pre-set precision.

3 Solution for Hierarchical H_∞ Control Synthesis

In the previous section, we proposed an iterative algorithm for H_∞ control synthesis of hierarchical systems. In each iteration of the algorithm, we have to solve an optimization problem: For a given λ,

$$\min_{u_{tot}=Ky, \; \Pi(\omega)} \quad \max_\omega \lambda_{\max}(\Pi(\omega)), \quad \text{subj to}$$
$$||z||^2 + \sum_{k=1}^{n_l} \int_{-\infty}^{+\infty} \tilde{d}_k' \Pi^k(\omega) \tilde{d}_k \; d\omega \le \lambda ||w||^2 + \tag{14}$$
$$\lambda \sum_{k=1}^{n_l} ||d_k||^2 + \int_{-\infty}^{+\infty} \tilde{d}' \Pi(\omega) \tilde{d} \; d\omega \;\; \forall \; \tilde{d}, w_k \text{ and } d_k,$$

where $\tilde{d}_k' = \begin{pmatrix} w_k' & d_k' & r_k' \end{pmatrix}$, $\tilde{d}' = \begin{pmatrix} w' & d' & r' \end{pmatrix}$, $u_{tot}' = \begin{pmatrix} u_c', r_1', \cdots, r_{nl}' \end{pmatrix}$, and K denotes the dynamic controller that we would like to design. In the followings, we will show that solving optimization problem (14) is equivalent to solving a set of Linear Matrix Inequalities (LMIs). The advantage of having LMI formulation is that the optimization can be done by efficient numerical algorithms [5, 4].

We first observe that each $\Pi^k(\omega)$ in (14) has the same structure as $\Pi(\omega)$ and satisfies

$$\Pi_I^k(\omega) := \Pi^k(\omega) + \begin{pmatrix} \lambda I & & \\ & 0 & \\ & & 0 \end{pmatrix} > 0, \quad \forall\ \omega. \tag{15}$$

The reason for (15) is that each $\Pi^k(\omega)$ was obtained by the same type optimization problem as (14) in the lower level sectors, in which $\Pi^k(\omega)$ plays the role of $\Pi(\omega)$. In problem (14), since the inequality has to be satisfied for all $\tilde{d}$, w_k d_k, we conclude that, by setting $w_k = d_k = 0$,

$$\begin{aligned} &\|z\|^2 + \sum_{k=1}^{n_l} \int_{-\infty}^{+\infty} {r_k}' \gamma_{22}^{(k)}(\omega) r_k \ d\omega \leq \lambda \|w\|^2 \\ &+ \int_{-\infty}^{+\infty} \tilde{d}' \Pi(\omega) \tilde{d} \ d\omega, \quad \forall\ \tilde{d}. \end{aligned} \tag{16}$$

Inequality (15) follows the fact that the left-hand size of (16) is positive-definite.

Since $\Pi_I^k(\omega)$ is positive definite, it can be factorized as $\Psi^k(j\omega)^* \Psi^k(j\omega)$. Let ${w_{\Pi_k}}' = \begin{pmatrix} {w_k}' & {d_k}' \end{pmatrix}$ and the state-space representation of $\Psi^k(j\omega)$ be

$$\Psi^k: \begin{cases} \dot{x}_{\Pi_k} = A_k x_{\Pi_k} + B_{k1} w_{\Pi_k} + B_{k2} r_k \\ z_k = C_k x_{\Pi_k} + D_{k1} w_{\Pi_k} + D_{k2} r_k \end{cases}.$$

Incorporate the open-loop system in (4) with all Ψ^k, $k = 1, 2, \cdots, nl$, we obtain a combined system

$$G_{tot}: \begin{cases} \dot{x}_{tot} = A_{tot} x_{tot} + B_{tot1} w_{tot} + B_{tot2} u_{tot} \\ z_{tot} = C_{tot1} x_{tot} + D_{tot11} w_{tot} + D_{tot12} u_{tot} \\ y = C_{tot2} x_{tot} + D_{tot21} w_{tot} \end{cases},$$

where $x'_{tot} = \left(x', x'_{\Pi_1}, \cdots, x'_{\Pi_{nl}}\right)$ is the vector of all state variables, $z'_{tot} = \left(z', z'_1, \cdots, z'_{nl}\right)$ is the total performance measurement, and $w'_{tot} = \left(w', d', r', w'_{\Pi_1}, \cdots, w'_{\Pi_{nl}}\right)$ is the collection of all disturbances. Integral inequality in (14) can be equvalently written as

$$\|z_{tot}\|^2 - \lambda \|w_{tot}\|^2 \leq \lambda \|r\|^2 + \int_{-\infty}^{+\infty} \tilde{d}' \Pi(\omega) \tilde{d} \ d\omega.$$

As oppose to $\Pi^k(\omega)$, $\Pi(\omega)$ in (14) plays the role of a reduced-order model of the current sector and is one of the decision variables that will be determined by the optimization process. In order to make a finite dimensional optimization problem, we select a set of bases

$$\{ \ \frac{1}{s + a_k} \ : \ a_k > 0, \ k = 1, 2, \cdots, n_p \ \}$$

and let $\Pi(\omega) = \tilde{\Pi}(j\omega) + \tilde{\Pi}(j\omega)^*$, where

$$\begin{aligned} \tilde{\Pi}(s) &= \tilde{\Pi}_0 + \sum_{k=1}^{n_p} \tilde{\Pi}_k \frac{1}{s + a_k}, \\ \tilde{\Pi}_k &= \begin{pmatrix} 0 & \tilde{\Pi}_k^{(1,2)} & \tilde{\Pi}_k^{(1,3)} \\ \tilde{\Pi}_k^{(2,1)} & \tilde{\Pi}_k^{(2,2)} & \tilde{\Pi}_k^{(2,3)} \\ \tilde{\Pi}_k^{(3,1)} & \tilde{\Pi}_k^{(3,2)} & \tilde{\Pi}_k^{(3,3)} \end{pmatrix}, \quad k = 0, 1, \cdots, n_p, \end{aligned} \tag{17}$$

and every one of $\tilde{\Pi}_k^{(i,j)}$ is a matrix variable that will be determined by the optimizer. By (17), we can express

$$\lambda ||r||^2 + \int_{-\infty}^{+\infty} \tilde{d}' \Pi(\omega) \tilde{d} \ d\omega \tag{18}$$

in the time-domain as

$$\begin{aligned} &\sigma_1(x_{\Pi_1}, w_{tot}) := \int_0^\infty \begin{pmatrix} x_{\Pi_1} \\ w_{tot} \end{pmatrix}' \begin{pmatrix} 0 & \mathbf{F}_1 \\ \mathbf{F}'_1 & \mathbf{R}_1 \end{pmatrix} \begin{pmatrix} x_{\Pi_1} \\ w_{tot} \end{pmatrix} dt, \\ &\dot{x}_{\Pi_1} = A_{\Pi_1} x_{\Pi_1} + B_{\Pi_1} w_{tot}, \end{aligned} \tag{19}$$

where A_{Π_1}, B_{Π_1} are constant matrices, and $\mathbf{F}_1$, $\mathbf{R}_1 = \mathbf{R}'_1$ are matrix variables which consist of $\tilde{\Pi}_k^{(i,j)}$. Since their structures are irrelevant to the LMI formulation, we will neglect the detailed construction of A_{Π_1}, B_{Π_1}, $\mathbf{F}_1$, and $\mathbf{R}_1$.

As we mentioned before, $\Pi(\omega)$ serves as a reduced-order model of the current sector and will be passed to the next level of the hierarchy as a part of the control design objective. Therefore, it is important that the bases $1/(s + a_k)$ we choose are able to reflect the frequency-domain features of the current sector. It is also important that the order of $\Pi(\omega)$ is significantly smaller than the order of the current sector, otherwise we miss the main point of introducing the hierarchical structure.

Let $\gamma = \max_\omega \lambda_{\max}(\Pi(\omega))$ and $\tilde{w}' = \left(w', d', r'\right)$. By the definition of γ, we have the integral quadratic constraint

$$\int_{-\infty}^{+\infty} \tilde{w}' (\Pi(\omega) - \gamma I) \tilde{w} \ d\omega \leq 0, \quad \forall\ \tilde{w}. \tag{20}$$

Similarly, we can express the integral form in (20) as

$$\begin{aligned} &\sigma_2(x_{\Pi_2}, \tilde{w}) := \int_0^\infty \begin{pmatrix} x_{\Pi_2} \\ \tilde{w} \end{pmatrix}' \begin{pmatrix} 0 & \mathbf{F}_2 \\ \mathbf{F}'_2 & \mathbf{R}_2 \end{pmatrix} \begin{pmatrix} x_{\Pi_2} \\ \tilde{w} \end{pmatrix} dt, \\ &\dot{x}_{\Pi_2} = A_{\Pi_2} x_{\Pi_1} + B_{\Pi_2} \tilde{w}, \end{aligned} \tag{21}$$

for some constant matrices A_{Π_2} B_{Π_2} and variable matrices $\mathbf{F}_2$ $\mathbf{R}_2$. Notice that $\mathbf{F}_2$ and $\mathbf{R}_2$ are also functions of $\Pi_k^{(i,j)}$ and not independent from $\mathbf{F}_1$ and $\mathbf{R}_1$.

By (19) and (21), we can express problem (14) as

$$\min_{K,\ \gamma,\ \tilde{\Pi}_k} \quad \gamma \quad \text{subj to,} \tag{22}$$

$$\begin{cases} ||z_{tot}||^2 - \lambda ||w_{tot}||^2 \le \sigma_1(x_{\Pi_1}, w_{tot}) \\ \sigma_2(x_{\Pi_2}, \tilde{w}) \le 0 \end{cases}, \ \forall\ w_{tot},$$

where z_{tot}, w_{tot}, $\tilde{w}$, x_{Π_1}, and x_{Π_2} satisfy the following dynamical systems

$$G_{tot}: \begin{cases} \dot{x}_{tot} = A_{tot}x_{tot} + B_{tot1}w_{tot} + B_{tot2}u_{tot} \\ z_{tot} = C_{tot1}x_{tot} + D_{tot11}w_{tot} + D_{tot12}u_{tot} \\ y = C_{tot2}x_{tot} + D_{tot21}w_{tot} \end{cases}$$

$$G_{\Pi_1}: \ \dot{x}_{\Pi_1} = A_{\Pi_1}x_{\Pi_1} + B_{\Pi_1}w_{tot}$$

$$G_{\Pi_2}: \ \dot{x}_{\Pi_2} = A_{\Pi_2}x_{\Pi_1} + B_{\Pi_2}\tilde{w}.$$

It is shown in [3] that problem (22) can be casted as an optimization problem over a set of LMIs.

Theorem 1 : Assume that D_{tot12} is of full column rank and has the structure $D'_{tot12} = \begin{pmatrix} \hat{D}'_{12} & \bar{D}'_{12} \end{pmatrix}$ with an invertable $\hat{D}_{12}$. Let $\tilde{B}_2 = B_{tot2}\hat{D}_{12}^{-1}$ and $\tilde{D}'_{12} = (\hat{D}_{12})^{-1}D'_{tot12}$. Let $\begin{pmatrix} W'_{r_1} & W'_{r_2} \end{pmatrix}'$ be the null space of $\begin{pmatrix} C_{tot2} & D_{tot21} \end{pmatrix}$, and $W'_{l_1} = \begin{pmatrix} -\bar{D}_{12}\hat{D}_{12}^{-1} & I \end{pmatrix}$, $W'_{l_2} = \begin{pmatrix} -\tilde{B}_2 & 0 \end{pmatrix}$ be such that

$$\begin{pmatrix} \tilde{B}'_2 & \tilde{D}'_{12} \end{pmatrix} \begin{pmatrix} 0 & I \\ W_{l_1} & W_{l_2} \end{pmatrix} = 0.$$

Then there exist a controller $K, \tilde{\Pi}_k$ for $k = 1, 2, \cdots, np$, and γ which solve the optimization problem (22) if and only if there exist $\mathbf{P} = \mathbf{P}' > 0$, $\mathbf{E}_1 = \mathbf{E}'_1 > 0$, $\mathbf{E}_2 = \mathbf{E}'_2 > 0$, $\mathbf{E}_3$, γ, $\tilde{\Pi}_k$ for $k = 1, 2, \cdots, np$, and

$$\mathbf{S} = \mathbf{S}' = \begin{pmatrix} \mathbf{S}_1 & \mathbf{S}_3 \\ \mathbf{S}'_3 & \mathbf{S}_2 \end{pmatrix} > 0,$$

which solve the following problem

$$\inf \quad \gamma, \ \text{subj to} \ \begin{cases} L_1 < 0, & L_2 < 0 \\ L_3 > 0, & L_4 < 0 \end{cases} \tag{23}$$

where

$$L_1 = \begin{pmatrix} -W'_{l_1}W_{l_1} & \Gamma_1 & W'_{l_1}C_{tot1}\mathbf{E}_3 & W'_{l_1}D_{tot11} \\ \Gamma_1{}' & \Gamma_2 & \Gamma_3 & \Gamma_4 \\ \mathbf{E}_3{}'C'_{tot1}W_{l_1} & \Gamma_3{}' & \Gamma_5 & \mathbf{E}_2B_{\Pi_2} - \mathbf{F}_1 \\ D'_{tot11}W_{l_1} & \Gamma_4{}' & B'_{\Pi_2}\mathbf{E}_2 - \mathbf{F}'_1 & -\mathbf{R}_1 \end{pmatrix}$$

$$L_2 = \begin{pmatrix} \mathbf{S}_2A_{\Pi_1} + A'_{\Pi_1}\mathbf{S}_2 & \Gamma_6 & 0 \\ \Gamma_6{}' & \Gamma_7 & \Gamma_8 \\ 0 & \Gamma_8{}' & -I \end{pmatrix}$$

$$L_3 = \begin{pmatrix} \mathbf{S}_1 & \mathbf{S}_3 & I \\ \mathbf{S}'_3 & \mathbf{S}_2 - \mathbf{E}_2 & -\mathbf{E}'_3 \\ I & -\mathbf{E}_3 & \mathbf{E}_1 \end{pmatrix}$$

$$L_4 = \begin{pmatrix} A_{\Pi_2}\mathbf{P} + \mathbf{P}A'_{\Pi_2} & PB_{\Pi_2} + \mathbf{F}_2 \\ B'_{\Pi_2}\mathbf{P} + \mathbf{F}'_2 & \mathbf{R}_2 \end{pmatrix}$$

$$\Gamma_1 = W_{l_1}{}'(C_{tot1}\mathbf{E}_1 - W_{l_2})$$

$$\Gamma_2 = A_{tot}\mathbf{E}_1 + \mathbf{E}_1A'_{tot} + \mathbf{E}_1C'_{tot1}W_{l_2} + W'_{l_2}C_{tot1}\mathbf{E}_1 - W'_{l_2}W_{l_2}$$

$$\Gamma_3 = (A_{tot} + W'_{l_2}C_{tot1})\mathbf{E}_3 - \mathbf{E}_3A_{\Pi_1}$$

$$\Gamma_4 = B_{tot1} + W_{l_2}{}'D_{tot11} - \mathbf{E}_3B_{\Pi_2}$$

$$\Gamma_5 = \mathbf{E}_2A_{\Pi_1} + A'_{\Pi_1}\mathbf{E}_2$$

$$\Gamma_6 = (\mathbf{S}'_3A_{tot} + A'_{\Pi_1}\mathbf{S}'_3)W_{r_1} + (\mathbf{S}'_3B_{tot1} + \mathbf{S}_2B_{\Pi_2})W_{r_2} - \mathbf{F}_1W_{r_2}$$

$$\Gamma_7 = W'_{r_1}(\mathbf{S}_1A_{tot} + A'_{tot}\mathbf{S}_1)W_{r_1} + W'_{r_1}\mathbf{S}_1B_{tot1}W_{r_2} + W'_{r_1}\mathbf{S}_3B_{\Pi_2}W_{r_2} + W'_{r_2}(B'_{tot1}\mathbf{S}_1 + B'_{\Pi_2}\mathbf{S}'_3)W_{r_1} - W'_{r_2}\mathbf{R}_1W_{r_2}$$

$$\Gamma_8 = W'_{r_1}C'_{tot1} + W'_{r_2}D'_{tot11}$$

Suppose that (23) is solvable. A stabilizing controller K which solves (22) can then be recovered by again solving a set of LMIs. See [3] for the details.

4 Concluding Remarks

We propose an iterative algorithm for H_∞ hierarchical optimization problem. The algorithm has three important feastures: First of all, the size of the optimization problem in each iteration is significantly smaller than the one of the centralized optimization approach. Secondly, the optimization problem in each iteration has the same structure. Finally, in each iteration, the optimization problem is a generalized H_∞ problem which we know how to solve efficiently by numerical algorithm. How well can the algorithm reduce computation complexity while still obtain a satisfactory suboptimal solution is subject to the future research.

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