Adaptive Backstepping Control of Nonlinear Systems with Unmatched Uncertainty

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Abstract

This paper considers control design using an adaptive backstepping algorithm for a class of nonlinear continuous uncertain processes with disturbances which can be converted to a parametric semi-strict feedback form. Sliding mode control using a combined adaptive backstepping sliding mode control (SMC) algorithm is also studied. The algorithm follows a systematic procedure for the design of adaptive control laws for the output of observable minimum phase nonlinear systems with matched and unmatched uncertainty. An existing sufficient condition for sliding is not needed by the new algorithm.

1. Introduction

The backstepping procedure is a systematic design technique for globally stable and asymptotically adaptive tracking controllers for a class of nonlinear systems. Adaptive backstepping algorithms have been applied to systems which can be transformed into a triangular form, in particular, the parametric pure feedback (PPF) form and the parametric strict feedback (PSF) form [4]. This method has been studied widely in recent years [3], [4], [8]-[11]. When plants include uncertainty with lack of information about the bounds of unknown parameters, adaptive control is more convenient; whilst, if some information about the uncertainty, e.g. bounds, is available, robust control is usually employed. Sliding mode control (SMC) is a robust control method, and backstepping can be considered to be a method of adaptive control. The combination of these methods yields benefits from both approaches.

A systematic design procedure has been proposed to combine adaptive control and SMC for nonlinear systems with relative degree one [13]. The sliding mode backstepping approach has been extended to some classes of nonlinear systems which need not be in the PPF or PSF forms [8]-[11]. A symbolic algebra toolbox allows straightforward design [7] of *dynamical* backstepping control.

The adaptive sliding backstepping control of semi-strict feedback systems (SSF) has been studied by Koshkouei and Zinober [5]. Their method ensures that the error state trajectories move on a sliding hyperplane.

In this paper we develop the backstepping approach for SSF systems with unmatched uncertainty. We also design a controller based upon sliding backstepping mode techniques so that the state trajectories approach a specified hyperplane. We remove the sufficient condition (for the existence of the sliding mode) given by Rios-Bolívar and Zinober [7]-[9]. We consider nonlinear systems which can be converted to a particular form, the so-called parametric semi-strict feedback form. We extend the classical backstepping method to this class of systems in Section 2 to achieve the output tracking of a dynamical reference signal. The sliding mode control design based upon the backstepping techniques is presented in Section 3. We consider an example to illustrate the results in Section 4 with some conclusions in Section 5.

2. Adaptive Backstepping Control

Consider the semi-strict feedback form (SSF) [5], [6], [12]

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \varphi_i^T(x_1, x_2, \ldots, x_i)\theta + \eta_i(x, w, t), \\ &\qquad\qquad 1 \leqslant i \leqslant n-1 \\ \dot{x}_n &= f(x) + g(x)u + \varphi_n^T(x)\theta + \eta_n(x, w, t) \qquad (1) \\ y &= x_1 \end{aligned}$$

where $x = [x_1\ x_2\ \ldots\ x_n]^T$ is the state, y the output, u the scalar control and $\varphi_i(x_1, \ldots, x_i) \in \mathbb{R}^p,\ i = 1, \ldots, n,$ are known functions which are assumed to be sufficiently smooth. $\theta \in \mathbb{R}^p$ is the vector of constant unknown parameters and $\eta_i(x, w, t)$, $i = 1, \ldots, n,$ are unknown nonlinear scalar functions including all the disturbances. w is an uncertain time-varying parameter.

Assumption 1 *The functions* $\eta_i(x,w,t)$, $i=1,\dots,n$ *are bounded by known positive functions* $h_i(x_1,\dots x_i) \in \mathbb{R}^p$, *i.e.*

$$|\eta_i(x,w,t)| \leqslant h_i(x_1,\dots x_i), \qquad i=1,\dots,n \tag{2}$$

The output y should track a specified bounded reference signal $y_r(t)$ with bounded derivatives up to n-th order.

If a plant has unmatched uncertainty, the system may be stabilized via state feedback control [1]. Some techniques have been proposed for the case of plants containing unmatched uncertainty [2]. The plant may contain unmodelled terms and unmeasurable external disturbances, bounded by known functions.

First, a classical backstepping method will be extended to this class of systems to achieve the output tracking of a dynamical reference signal. The sliding mode control design based upon the backstepping techniques is then presented in Section 3.

2.1. Backstepping Algorithm

We first follow a backstepping approach which differs from Koshkouei and Zinober [5], [6]. The functions that compensate the system disturbances, are continuous.

Step 1. Define the error variable $z_1 = x_1 - y_r$ then

$$\dot z_1 = x_2 + \varphi_1^T(x_1)\theta + \eta_1(x,w,t) - \dot y_r \tag{3}$$

From (3)

$$\dot z_1 = x_2 + \omega_1^T\hat\theta + \eta_1(x,w,t) - \dot y_r + \omega_1^T\tilde\theta \tag{4}$$

with $\omega_1(x_1) = \varphi_1(x_1)$ and $\tilde\theta = \theta - \hat\theta$ where $\hat\theta(t)$ is an estimate of the unknown parameter θ.

Consider the stabilization of the subsystem (3) and the Lyapunov function

$$V_1(z_1,\hat\theta) = \frac{1}{2}z_1^2 + \frac{1}{2}\tilde\theta^T\Gamma^{-1}\tilde\theta \tag{5}$$

where Γ is a positive definite matrix. The derivative V_1 is

$$\begin{aligned}\dot V_1(z_1,\hat\theta) &= z_1\left(x_2 + \omega_1^T\hat\theta + \eta_1(x,w,t) - \dot y_r\right)\\ &\quad + \tilde\theta^T\Gamma^{-1}\left(\Gamma\omega_1 z_1 - \dot{\hat\theta}\right)\end{aligned} \tag{6}$$

Define $\tau_1 = \Gamma\omega_1 z_1$. Let

$$\begin{aligned}\beta_1 &= \alpha_1(x_1,\hat\theta,t) + \dot y_r\\ &= -\omega_1^T\hat\theta + \dot y_r - c_1 z_1 - \frac{h_1^2 z_1}{h_1|z_1| + \frac{\epsilon}{n}e^{-at}}\end{aligned} \tag{7}$$

with c_1, a and ϵ positive numbers. Define the error variable

$$\begin{aligned}z_2 &= x_2 - \alpha_1(x_1,\hat\theta,t) - \dot y_r\\ &= x_2 + \omega_1^T\hat\theta + c_1 z_1 - \dot y_r + \frac{h_1^2 z_1}{h_1|z_1| + \frac{\epsilon}{n}e^{-at}}\end{aligned} \tag{8}$$

Then

$$\dot z_1 = -c_1 z_1 + z_2 + \omega_1^T\tilde\theta + \eta_1(x,w,t) - \frac{h_1^2 z_1}{h_1|z_1| + \frac{\epsilon}{n}e^{-at}} \tag{9}$$

and $\dot V_1$ is converted to

$$\dot V_1(z_1,\hat\theta) \leqslant -c_1 z_1^2 + z_1 z_2 + \frac{\epsilon}{n}e^{-at} + \tilde\theta^T\Gamma^{-1}\left(\tau_1 - \dot{\hat\theta}\right)$$

Step k $(1 \leqslant k \leqslant n-1)$. The time derivative of the error variable z_k is

$$\begin{aligned}\dot z_k &= x_{k+1} + \omega_k^T\hat\theta - \sum_{i=1}^{k-1}\frac{\partial\alpha_{k-1}}{\partial x_i}x_{i+1} - \frac{\partial\alpha_{k-1}}{\partial\hat\theta}\dot{\hat\theta}\\ &\quad + \xi_k - y_r^{(k)}(t) + \omega_k^T\tilde\theta - \frac{\partial\alpha_{k-1}}{\partial t}\end{aligned} \tag{10}$$

where

$$\begin{aligned}\omega_k &= \varphi_k(x_1,\dots,x_k) - \sum_{i=1}^{k-1}\frac{\partial\alpha_{k-1}}{\partial x_i}\varphi_i(x_1,\dots,x_i)\\ \zeta_k &= \frac{h_k^2}{h_k|z_k| + \frac{\epsilon}{n}e^{-at}} + \sum_{i=1}^{k-1}\left(\frac{\partial\alpha_{k-1}}{\partial x_i}\right)^2 \times\\ &\qquad \frac{h_i^2}{h_i\left|\frac{\partial\alpha_{k-1}}{\partial x_i}z_k\right| + \frac{\epsilon}{n}e^{-at}} \qquad\qquad (11)\\ \xi_k &= \eta_k - \sum_{i=1}^{k-1}\frac{\partial\alpha_{k-1}}{\partial x_i}\eta_i\end{aligned}$$

Define $z_{k+1} = x_{k+1} - \beta_k = x_{k+1} - \alpha_k - y_r^{(k)}$, where

$$\begin{aligned}\alpha_k(x_1,x_2,\dots,x_k,\hat\theta,t) &= -z_{k-1} - c_k z_k - \omega_k^T\hat\theta\\ &+ \sum_{i=1}^{k-1}\frac{\partial\alpha_{k-1}}{\partial x_i}x_{i+1} + \frac{\partial\alpha_{k-1}}{\partial t} - \zeta_k z_k + \frac{\partial\alpha_{k-1}}{\partial\hat\theta}\tau_k\\ &+ \left(\sum_{i=1}^{k-2} z_{i+1}\frac{\partial\alpha_i}{\partial\hat\theta}\right)\Gamma w_k\end{aligned} \tag{12}$$

with $c_k > 0$. Then the time derivative of the error variable z_k is

$$\begin{aligned}\dot z_k &= -z_{k-1} - c_k z_k + z_{k+1} + \omega_k^T\tilde\theta + \xi_k - \zeta_k z_k\\ &\quad - \frac{\partial\alpha_{k-1}}{\partial\hat\theta}\left(\dot{\hat\theta} - \tau_k\right) + \left(\sum_{i=1}^{k-2} z_{i+1}\frac{\partial\alpha_i}{\partial\hat\theta}\right)\Gamma w_k\end{aligned} \tag{13}$$

The time derivative of V_k is

$$\begin{aligned}\dot V_k &\leqslant -\sum_{i=1}^{k} c_i z_i^2 + z_k z_{k+1} + \frac{k(k+1)\epsilon}{2n}e^{-at}\\ &+ \tilde\theta^T\Gamma^{-1}\left(\tau_k - \dot{\hat\theta}\right) + \left(\sum_{i=1}^{k-2}\frac{\partial\alpha_i}{\partial\hat\theta}z_{i+1}\right)\left(\tau_k - \dot{\hat\theta}\right)\end{aligned} \tag{14}$$

since

$$\tau_k = \tau_{k-1} + \Gamma\omega_k z_k = \Gamma \sum_{i=1}^{k} \omega_i z_i \qquad (15)$$

Step n. Define

$$z_n = x_n - \beta_{n-1} = x_n - \alpha_{n-1} - y_r^{(n)}$$

with α_{n-1} obtained from (12) for $k = n$. Then the time derivative of the error variable z_n is

$$\begin{aligned} \dot{z}_n &= f(x) + g(x)u + \omega_n^T(x,t)\hat{\theta} - \sum_{i=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_i} x_{i+1} \\ &\quad -\frac{\partial \alpha_{n-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} - \frac{\partial \alpha_{n-1}}{\partial t} + \omega_n^T(x,t)\tilde{\theta} + \xi_n - y_r^{(n)} \quad (16) \end{aligned}$$

where $\omega_n(x,\hat{\theta})$ is defined in (11) for $k = n$. Extend the Lyapunov function to be

$$V_n = V_{n-1} + \frac{1}{2} z_n^2 \qquad (17)$$

The time derivative

$$\begin{aligned} \dot{V}_n &= \dot{V}_{n-1} + z_n \dot{z}_n \\ &\leqslant -\sum_{i=1}^{n} c_i z_i^2 + \frac{(n+1)\epsilon}{2} e^{-at} - \sum_{i=1}^{n-2} \left(\frac{\partial \alpha_i}{\partial \hat{\theta}} z_{i+1} \right) \times \\ &\quad \left(\dot{\hat{\theta}} - \tau_n \right) + \tilde{\theta}^T \Gamma^{-1} \left(\tau_n - \dot{\hat{\theta}} \right) \qquad (18) \end{aligned}$$

where

$$\tau_n = \tau_{n-1} + \Gamma \omega_n^T z_n \qquad (19)$$

if we select the control

$$\begin{aligned} u &= \frac{1}{g(x)} \Big[-z_{n-1} - c_n z_n - f(x) - \omega_n^T \hat{\theta} \\ &\quad + \sum_{i=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_i} x_{i+1} + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \tau_n + \frac{\partial \alpha_{n-1}}{\partial t} \\ &\quad - \left(\sum_{i=1}^{n-2} z_{i+1} \frac{\partial \alpha_i}{\partial \hat{\theta}} \right) \Gamma w_n + y_r^{(n)} - \zeta_n z_n \Big] \qquad (20) \end{aligned}$$

with $c_n > 0$. Selecting $\dot{\hat{\theta}} = \tau_n$, $\tilde{\theta}$ is eliminated from the right-hand side of (18). Then

$$\dot{V}_n \leqslant -W_n + \epsilon\,(n+1)\, e^{-at}/2 \qquad (21)$$

with $W_n = \frac{1}{2}\sum_{i=1}^{n} c_i z_i^2$. Then

$$V_n - V_n(0) \leqslant -\int_0^t W_n + \epsilon(n+1)\left(1 - e^{-at}\right)/(2a) \quad (22)$$

Therefore,

$$0 \leqslant \int_0^t W_n \leqslant V_n(0) + \epsilon(n+1)\left(1 - e^{-at}\right)/(2a)$$

and

$$\lim_{t\to\infty} \int_0^t W_n \leqslant V_n(0) + \epsilon(n+1)/(2a) < \infty$$

Since W_n is a uniformly continuous function, according to the Barbalat Lemma, $\lim_{t\to\infty} W_n = 0$. This implies that $\lim_{t\to\infty} z_i = 0$, $i = 1, 2, \ldots, n$. Particularly, $\lim_{t\to\infty} (x_1 - y_r) = 0$.

Remark 1 *When there is no unknown parameter θ in the system equation, one can attain the tracking performance directly. Then (21) becomes*

$$\dot{V}_n \leqslant -cV_n + \epsilon(n+1)e^{-at}/2 \qquad (23)$$

with $0 < c \leqslant \min_{1\leqslant i \leqslant n} c_i$ and $c \neq a$. Then

$$0 \leqslant V_n \leqslant \frac{(n+1)\epsilon}{2(a-c)}\left(e^{-ct} - e^{-at}\right) + V_n(0)e^{-ct} \qquad (24)$$

$\lim_{t\to\infty} V_n = 0$ *implies that* $\lim_{t\to\infty} z_i = 0$, $i = 1, 2, \ldots, n$. *This guarantees* $\lim_{t\to\infty} (y - y_r) = 0$.

3. Sliding Backstepping Control

Sliding mode techniques, yielding robust control, and adaptive control techniques are both popular when there is uncertainty in the plant. The combination of these methods has been studied in recent years [8]-[11]. In general, at each step of the backstepping method, the new update tuning function and the defined error variables (and virtual control law) take the system to the equilibrium position. At the final step, the system is stabilized by suitable selection of the control.

The adaptive sliding backstepping control of SSF systems has been studied by Koshkouei and Zinober [5], [6]. The controller is based upon sliding backstepping mode techniques so that the state trajectories approach a specified hyperplane. The sufficient condition (for the existence of the sliding mode) given by Rios-Bolívar and Zinober [7]-[9], is no longer needed.

To provide robustness, the adaptive backstepping algorithm can be modified to yield adaptive *sliding* output tracking controllers. The modification is carried out at the final step of the algorithm by incorporating the following *sliding surface* defined in terms of the error coordinates

$$\sigma = k_1 z_1 + \ldots + k_{n-1} z_{n-1} + z_n = 0 \qquad (25)$$

where $k_i > 0$, $i = 1, \ldots, n-1$, are real numbers. Additionally, the Lyapunov function is modified as follows

$$V_n = \frac{1}{2}\sum_{i=1}^{n-1} z_i^2 + \frac{1}{2}\sigma^2 + \frac{1}{2}(\theta - \hat{\theta})^T \Gamma^{-1} (\theta - \hat{\theta}) \quad (26)$$

The time derivative is

$$
\begin{aligned}
\dot{V}_n &= \dot{V}_{n-1} + \sigma\dot{\sigma} \\
&\leqslant -\sum_{i=1}^{n-1} c_i z_i^2 + \epsilon(n-1)e^{-at}/2 + z_{n-1}z_n \\
&\quad +\sigma\Bigg[f(x) + g(x)u + \omega_n^T\hat{\theta} - \sum_{i=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_i}x_{i+1} \\
&\quad -\frac{\partial\alpha_{n-1}}{\partial\hat{\theta}}\dot{\hat{\theta}} - \frac{\partial\alpha_{n-1}}{\partial t} + \xi_n - y_r^{(n)} \\
&\quad +k_1\left(z_2 - c_1 z_1 + \eta_1 - \frac{h_1^2 z_1}{h_1|z_1| + \frac{\epsilon}{n}e^{-at}}\right) \\
&\quad +\sum_{i=2}^{n-1} k_i\Bigg(- z_{i-1} - c_i z_i + z_{i+1} + \xi_i - \zeta_i z_i \\
&\quad -\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\left(\dot{\hat{\theta}} - \tau_i\right) + \Gamma w_i \sum_{l=1}^{i-2} z_{l+1}\frac{\partial\alpha_l}{\partial\hat{\theta}}\Bigg) \\
&\quad -\left(\sum_{i=1}^{n-2} z_{i+1}\frac{\partial\alpha_i}{\partial\hat{\theta}}\right)\Gamma\left(\omega_n + \sum_{i=1}^{n-1}k_i\omega_i\right)\Bigg] \\
&\quad -\sum_{i=1}^{n-2} z_{i+1}\frac{\partial\alpha_i}{\partial\hat{\theta}}\left(\dot{\hat{\theta}} - \tau_n\right) + \tilde{\theta}^T\Gamma^{-1}\left(\tau_n - \dot{\hat{\theta}}\right) \quad (27)
\end{aligned}
$$

with

$$
\begin{aligned}
\tau_n &= \tau_{n-1} + \Gamma\sigma\left(\omega_n + \sum_{i=1}^{n-1}k_i\omega_i\right) \\
&= \Gamma\left[\sum_{i=1}^{n-1}\omega_i z_i + \sigma\left(\omega_n + \sum_{i=1}^{n-1}k_i\omega_i\right)\right] \quad (28)
\end{aligned}
$$

Setting $\dot{\hat{\theta}} = \tau_n$, $\tilde{\theta}$ is eliminated from the right-hand side of (27). From (25) we obtain

$$
z_n = \sigma - k_1 z_1 - k_2 z_2 - \ldots - k_{n-1}z_{n-1} \quad (29)
$$

Substituting (29) in (27) removes the need for the sufficient condition for the existence of the sliding mode [8], [9] and

$$
\begin{aligned}
\dot{V}_n &\leqslant -\sum_{i=1}^{n-1} c_i z_i^2 - z_{n-1}\left(k_1 z_1 + k_2 z_2 + \ldots + k_{n-1}z_{n-1}\right) \\
&\quad + \epsilon(n-1)e^{-at}/2 + \sigma\Bigg[z_{n-1} + f(x) + g(x)u + \omega_n^T\hat{\theta} \\
&\quad -\sum_{i=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_i}x_{i+1} - \frac{\partial\alpha_{n-1}}{\partial\hat{\theta}}\dot{\hat{\theta}} - \frac{\partial\alpha_{n-1}}{\partial t} + \xi_n \\
&\quad - y_r^{(n)} + k_1\left(z_2 - c_1 z_1 - \frac{h_1^2 z_1}{h_1|z_1| + \frac{\epsilon}{n}e^{-at}} + \eta_1\right) \\
&\quad +\sum_{i=2}^{n-1} k_i\Bigg(- z_{i-1} - c_i z_i + z_{i+1} + \xi_i - \zeta_i z_i \\
&\quad -\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\left(\dot{\hat{\theta}} - \tau_i\right) + \Gamma w_i \sum_{l=1}^{i-2} z_{l+1}\frac{\partial\alpha_l}{\partial\hat{\theta}}\Bigg) \\
&\quad -\left(\sum_{i=1}^{n-2} z_{i+1}\frac{\partial\alpha_i}{\partial\hat{\theta}}\right)\Gamma\left(\omega_n + \sum_{i=1}^{n-1}k_i\omega_i\right)\Bigg] - \\
&\quad \sum_{i=1}^{n-2} z_{i+1}\frac{\partial\alpha_i}{\partial\hat{\theta}}\left(\dot{\hat{\theta}} - \tau_n\right) + \tilde{\theta}^T\Gamma^{-1}\left(\tau_n - \dot{\hat{\theta}}\right) \quad (30)
\end{aligned}
$$

We now obtain the update law

$$
\begin{aligned}
\dot{\hat{\theta}} &= \tau_n = \tau_{n-1} + \Gamma\sigma\big(\omega_n + \sum_{i=1}^{n-1}k_i\omega_i\big) \\
&= \Gamma\Big(\sum_{i=1}^{n-1} z_i\omega_i + \sigma\big(\omega_n + \sum_{i=1}^{n-1}k_i\omega_i\big)\Big) \quad (31)
\end{aligned}
$$

and the adaptive sliding mode output tracking controller

$$
\begin{aligned}
u &= \frac{1}{g(x)}\Bigg[- z_{n-1} - f(x) - \omega_n^T\hat{\theta} + \frac{\partial\alpha_{n-1}}{\partial\hat{\theta}}\tau_n \\
&\quad +\sum_{i=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_i}x_{i+1} + y_r^{(n)} + \frac{\partial\alpha_{n-1}}{\partial t} \\
&\quad -k_1\left(-c_1 z_1 + z_2 - \frac{h_1^2 z_1}{h_1|z_1| + \frac{\epsilon}{n}e^{-at}}\right) \\
&\quad -\sum_{i=2}^{n-1} k_i\Bigg(- z_{i-1} - c_i z_i + z_{i+1} - \zeta_i z_i \\
&\quad -\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\left(\tau_n - \tau_i\right) + \left(\sum_{l=1}^{i-2} z_{l+1}\frac{\partial\alpha_l}{\partial\hat{\theta}}\right)\Gamma w_i\Bigg) \\
&\quad +\left(\sum_{i=1}^{n-2} z_{i+1}\frac{\partial\alpha_i}{\partial\hat{\theta}}\right)\Gamma\left(\omega_n + \sum_{i=1}^{n-1}k_i\omega_i\right) \\
&\quad -W\sigma - \left(K + \sum_{i=1}^{n}k_i\nu_i\right)\operatorname{sgn}(\sigma)\Bigg] \quad (32)
\end{aligned}
$$

where $k_n = 1$, $K > 0$ and $W \geqslant 0$ are arbitrary real numbers and

$$
\nu_i = h_i + \sum_{j=1}^{i-1}\left|\frac{\partial\alpha_{k-1}}{\partial x_j}\right| h_j, \qquad 1 \leqslant i \leqslant n \quad (33)
$$

Then substituting (32) in (27) yields

$$
\begin{aligned}
\dot{V}_n &= \dot{V}_{n-1} + \sigma\dot{\sigma} \\
&\leqslant -\left[z_1 z_2 \ldots z_{n-1}\right] Q \left[z_1 z_2 \ldots z_{n-1}\right]^T - K|\sigma| \\
&\quad - W\sigma^2 + \epsilon(n-1)e^{-at}/2 \quad (34)
\end{aligned}
$$

with Q as

$$
Q = \begin{bmatrix} c_1 & 0 & \ldots & 0 \\ 0 & c_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ k_1 & k_2 & \ldots & k_{n-1} + c_{n-1} \end{bmatrix} \quad (35)
$$

which is a positive definite matrix.

Let

$$\tilde{W}_n = [z_1 z_2 \dots z_{n-1}]\, Q\, [z_1 z_2 \dots z_{n-1}]^T + K\,|\sigma| + W\sigma^2$$

Then, similarly to (21), we have

$$\dot{V}_n \leqslant -\tilde{W}_n + \epsilon(n+1)e^{-at}/2 \tag{36}$$

which yields

$$V_n - V_n(0) \leqslant -\int_0^t \tilde{W}_n ds + \epsilon(n+1)\left(1 - e^{-at}\right)/(2a) \tag{37}$$

Therefore,

$$0 \leqslant \int_0^t \tilde{W}_n ds \leqslant V_n(0) + \epsilon(n+1)\left(1 - e^{-at}\right)/(2a)$$

and

$$\lim_{t\to\infty} \int_0^t \tilde{W}_n ds \leqslant V_n(0) + \epsilon(n+1)/(2a) < \infty$$

From the Barbalat Lemma, $\lim_{t\to\infty} \tilde{W}_n = 0$. This implies that $\lim_{t\to\infty} z_i = 0$, $i = 1, 2, \dots, n$ and $\lim_{t\to\infty} \sigma = 0$. Particularly, $\lim_{t\to\infty} (x_1 - y_r) = 0$. Therefore, the stability of the system along the sliding surface $\sigma = 0$ is guaranteed. Note that the sufficient condition for the existence of the sliding mode in [10] has been removed by using (25).

However, if ϵ is a sufficiently small positive number and a suitably large, $\dot{V} < 0$. Also, the design parameters K, W, c_i and k_i, $i = 1, \dots, n-1$, can be selected to ensure that $\dot{V} < 0$.

Remark 2 *Alternatively, one can apply a different procedure at the n-th step which yields*

$$\begin{aligned} u \;=\; & \frac{1}{g(x)}\Bigg[- z_{n-1} - f(x) - \omega_n^T \hat{\theta} + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \tau_n \\ & + \sum_{i=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_i} x_{i+1} + y_r^{(n)} + \frac{\partial \alpha_{n-1}}{\partial t} \\ & - k_1 \left(-c_1 z_1 + z_2 - \frac{h_1^2 z_1}{h_1|z_1| + \frac{\epsilon}{n} e^{-at}} \right) \\ & - \sum_{i=2}^{n-1} k_i \Bigg(- z_{i-1} - c_i z_i + z_{i+1} - \zeta_i z_i \\ & - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} (\tau_n - \tau_i) + \left(\sum_{l=1}^{i-2} z_{l+1} \frac{\partial \alpha_l}{\partial \hat{\theta}} \right) \Gamma w_i \Bigg) \\ & + \left(\sum_{i=1}^{n-2} z_{i+1} \frac{\partial \alpha_i}{\partial \hat{\theta}} \right) \Gamma \left(\omega_n + \sum_{i=1}^{n-1} k_i \omega_i \right) \\ & - K \operatorname{sgn}(\sigma) - \left(W + \sum_{i=1}^{n} k_i \nu_i \right) \sigma \Bigg] \end{aligned} \tag{38}$$

with $k_n = 1$, $K > 0$ and $W \geqslant 0$ arbitrary real numbers and for all i, $1 \leqslant i \leqslant n$

$$\nu_i = \frac{h_i^2}{h_i|\sigma| + \frac{\epsilon}{n} e^{-at}} + \sum_{j=1}^{i-1} \left| \frac{\partial \alpha_{i-1}}{\partial x_j} \right| \frac{h_j^2}{h_j|\sigma| + \frac{\epsilon}{n} e^{-at}} \tag{39}$$

4. Example

Consider the second-order system in SSF form

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1\theta + A x_1^2 \cos(B x_1 x_2) \\ \dot{x}_2 &= u \end{aligned} \tag{40}$$

where A and B are considered unknown but it is known that $|A| \leqslant 2$ and $|B| \leqslant 3$. We have

$$\begin{aligned} h_1 &= 2x_1^2 \\ z_1 &= x_1 - y_r \\ z_2 &= x_2 + x_1\hat{\theta} + c_1 z_1 + \frac{4x_1^4 z_1}{h_1|z_1| + \frac{\epsilon}{2} e^{-at}} - \dot{y}_r \\ \alpha_1 &= -x_1\hat{\theta} - c_1 z_1 - \frac{4x_1^4 z_1}{2x_1^2|z_1| + \frac{\epsilon}{2} e^{-at}} \\ \omega_1 &= x_1 \\ \omega_2 &= -\frac{\partial \alpha_1}{\partial x_1} x_1 \\ \tau_2 &= \Gamma\left(\omega_1 z_1 + \omega_2 z_2\right) \\ \zeta_2 &= \frac{4x_1^4}{2x_1^2 \left| \frac{\partial \alpha_1}{\partial x_1} z_2 \right| + \frac{\epsilon}{2} e^{-at}} \left(\frac{\partial \alpha_1}{\partial x_1} \right)^2 \end{aligned}$$

Then the control law (20) becomes

$$\begin{aligned} u \;=\; & -z_1 - c_2 z_2 - \omega_2^T \hat{\theta} + \frac{\partial \alpha_1}{\partial x_1} x_2 + \frac{\partial \alpha_1}{\partial \hat{\theta}} \tau_2 \\ & + \frac{\partial \alpha_1}{\partial t} + y_r^{(2)} - \zeta_2 z_2 \end{aligned} \tag{41}$$

Simulation results showing desirable transient responses are shown in Fig. 1 with $y_r = 0.4$, $a = 1$, $\epsilon = 0.06$, $\theta = 1$, $\Gamma = 1$, $A = 2$, $B = 3$ and $c_1 = c_2 = 0.01$.

Alternatively, we can design a sliding mode controller for the system. Assume that the sliding surface is $\sigma = k_1 z_1 + z_2 = 0$ with $k_1 > 0$. The adaptive sliding mode control law (32) is

$$\begin{aligned} u \;=\; & (c_1 k_1 - 1)\, z_1 - k_1 z_2 - \omega_2^T \hat{\theta} + \frac{\partial \alpha_1}{\partial x_1} x_2 + \frac{\partial \alpha_1}{\partial \hat{\theta}} \tau_2 \\ & + \frac{\partial \alpha_1}{\partial t} + y_r^{(2)} + \frac{k_1 h_1^2 z_1}{h_1|z_1| + \frac{\epsilon}{2} e^{-at}} - W\sigma \\ & - \left(K + \left(k_1 + \left|\frac{\partial \alpha_1}{\partial x_1}\right| \right) h_1 \right) \operatorname{sgn}(\sigma) \end{aligned} \tag{42}$$

where $\tau_2 = \Gamma\left(z_1\omega_1 + \sigma(\omega_2 + k_1\omega_1)\right)$. Simulation results showing desirable transient responses are shown in Fig. 2 with $y_r = 0.05\sin(2\pi t)$, $k_1 = 1$, $K = 5$, $W = 0$, $a = 1$, $\epsilon = 0.06$, $\Gamma = 1$, $A = 2$, $B = 3$, $c_1 = 34$, $c_2 = 1$ and $\theta = 1$.

5. Conclusions

Backstepping is a systematic Lyapunov method to design control algorithms which stabilize nonlinear systems. Sliding mode control is a robust control method design and adaptive backstepping is an adaptive control design method. In this paper the control design has benefited from both design approaches. Backstepping control and sliding backstepping control were originally developed for a class of nonlinear systems which can be converted to the parametric semi-strict feedback form. The plant may have unmodelled or external disturbances. The discontinuous control may contain a gain parameter for the designer to select the velocity of the convergence of the state trajectories to the sliding hyperplane. We have extended the previous work of Rios-Bolívar and Zinober [7]-[9] and Koshkouei and Zinober [5], and have removed the sufficient existence condition for the sliding mode to guarantee that the state trajectories converge to a given sliding surface.

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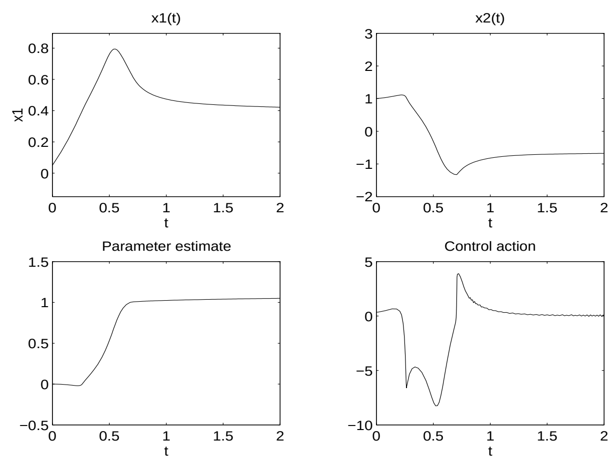

Figure 1: Responses of example with control (41)

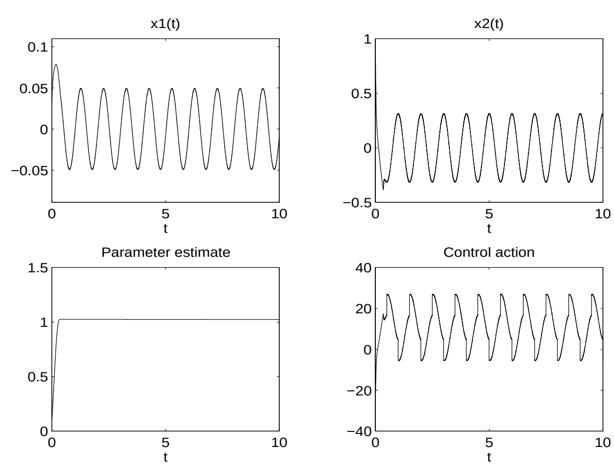

Figure 2: Responses of example with sliding control (42)