On Robust Non-Fragile Static State-Feedback Controller Synthesis

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Abstract

This paper addresses the design of robust non-fragile state feedback controllers, with controller uncertain parameters described as polytope-type convex-bounded uncertainties. The linearizing change of variables $Z = KW$ that is the basis for a large class of LMI-based robust control design algorithms is slightly modified. The modified change of variables proposed here leads to LMI-based robust non-fragile control design algorithms that are directly derived from any former algorithm formulated in terms of the traditional change of variables. In this way, the "fragility" issue is solved inside the mainstream of the LMI-based robust control synthesis algorithms that have been developed up to now, in a unified basis.

1 Introduction

In the paper [9], some examples have been presented to show that small perturbations in the coefficients of the controller designed by using robust H_2, H_∞, ℓ_1 and μ approaches can destabilize the closed-loop control system. The authors of that paper have suggested to take into account both uncertainties in the controller structure and in the system structure, performing a trade-off between optimality and fragility. However, no technique has been presented in that reference to handle this kind of controller sensitivity.

In fact, the fragility problem is not new. The reference [13], for instance, addresses the "fragility" of linear quadratic controllers with digital implementation. In [3] a review on the subject "fragility" is presented, including references that point out the necessity of non-fragile controllers so early as in 1927. That renewed interest and the polemic that followed the publication of [9] (see [10, 14]) is due to the identification of the fragility problem precisely in the robust control design techniques, which were supposed to handle, in a systematic way, the system uncertainties.

After the publication of [9], the subject of fragility, i.e., the performance debasement of the closed loop system due to inaccuracies in controller implementation, has deserved more attention [2, 3, 4, 6, 7, 8, 11]. In particular, the references [4, 7] specifically address the fragility issue for the class of static state feedback controllers.

The results in [4, 5] lead to general non-convex optimization problems that, in some particular cases, can be cast in terms of convex problems with Linear Matrix Inequalities (LMI's) constraints. The particularizations include, for instance, that the size of perturbation in all controller parameters must be the same. The paper [7], although leading to a convex optimization problem in the LMI framework, suffers from the drawback that the controller uncertainty can be known only after the design procedure, since it is a variable of the optimization problem. This means that the controller uncertainty, differently from the system uncertainty, cannot be a design input, and cannot be previously defined in an arbitrary way.

In this paper, the fragility problem is addressed in the context of methods that use polytopic models for system uncertainties. A simple modification is proposed in the change of variables commonly used in the design of robust (and guaranteed-cost) controllers. This modified change of variables allows the inclusion of the "non-fragility" feature in a large class of existing LMI-based robust control design methods. The resulting formulation is still a linear optimization problem with LMI constraints (that can be solved through convex optimization methods).

As an instance of the proposed methodology, a conventional mixed H_2/H_∞ design is transformed in a "non-fragile" one. Numerical simulations are performed to illustrate the effectiveness of the design algorithm.

The conceptual advantage of the method proposed here is that the controller uncertainty is taken into account exactly as the system uncertainty, leading to design methods that are simple variations of some established methods of robust control design. This allows the "fragility problem" to follow a unified logic with the robust control theory that has been developed up to now.

2 Problem Statement

Consider the linear uncertain dynamic system:

$$\dot{x} = (A+\Delta_A)x + (B+\Delta_B)u + Ew \tag{1}$$

in which $x \in \mathbb{R}^n$ is the system state vector, $u \in \mathbb{R}^m$ is the control input vector and $w \in \mathbb{R}^p$ is an exogenous disturbance vector. Matrices A, B and E are known nominal values for the system matrices, and matrices Δ_A and Δ_B are unknown uncertainties belonging to some bounded sets. Note that since E is the disturbance input matrix, there is no need of considering it uncertain too. In the standard setting of the static state feedback robust control design problem, the state vector x is considered to be available for control law synthesis:

$$u = Kx \tag{2}$$

It is assumed that the control gain K is subject to some uncertainty Δ_K:

$$u = (K+\Delta_K)x \tag{3}$$

In this way, matrix K assumes the connotation of a nominal value matrix that is resulting from the robust (and non-fragile) controller design procedure, while matrix Δ_K is unknown but belongs to a known bounded set. This uncertainty is associated to the unavoidable implementation errors, such as round-off errors in digital implementations and component tolerance errors in analog implementations. The sets of possible values of the unknowns Δ_A and Δ_B are inputs for the standard robust control design problem. In this paper, the set of possible values of Δ_K is also considered as an input, leading to a robust non-fragile control design problem.

A common approach for defining bounded sets for the uncertainty matrices is the polytope-type set formalism. This formalism will be employed here. The polytope-type sets are defined by:

$$\begin{aligned} &\Delta_{(\cdot)} = \sum_{i=1}^{v} \alpha_i \Delta_{(\cdot)}^i \\ &\sum_{i=1}^{v} \alpha_i = 1 \\ &\alpha_i \geq 0 \;\; \forall i = 1,\ldots,v \end{aligned} \tag{4}$$

The methodology presented here will make usage of the polytope-type description for the uncertainties in the controller implementation. For notation uniformity reason, the system uncertainties are considered to be of polytope-type too. However, this last assumption is not necessary, and other types of system uncertainty models could be employed within the approach proposed here.

In this paper, the robust performance problem is considered, which means that the design requirement includes the minimization of some performance index of the closed-loop over the whole set of system uncertainties. Note that this already includes the robust stability problem. For the purpose of robust performance design, an additional controlled output vector $z \in \mathbb{R}^r$ is defined:

$$z = Cx + Du \tag{5}$$

With this equation, the transfer matrix $H_{zw}(s)$ is given by:

$$H_{zw}(s) = C_{cl}(s\mathbf{I} - A_{cl})^{-1}E \tag{6}$$

in which:

$$\begin{aligned} A_{cl} &= A + \Delta_A + (B+\Delta_B)(K+\Delta_K) \\ C_{cl} &= C + D(K+\Delta_K) \end{aligned} \tag{7}$$

Several different performance criteria could be defined for the closed-loop transfer matrix $H_{zw}(s)$. In the example presented bellow, some upper bounds on the H_2 and H_∞ norms are used.

3 A Modified Change of Variables

There are several possible formulations of the design problem of robust static state feedback controllers via LMI's with different optimization criteria. To assure closed-loop stability, the LMI constraints associated to the robust control design always guarantee that the following Lyapunov equation holds [16, 17]:

$$\begin{aligned} &A_{cl}W + WA_{cl}' < 0 \\ &W > 0 \end{aligned} \tag{8}$$

A key idea behind the LMI approach for robust control design is the use of a single Lyapunov matrix W for guaranteeing several different constraints (H_2 and H_∞ norms in different system channels, for instance). This allows the explicit inclusion of a set of relevant design objectives in the design procedure.

In the case of precisely known systems, the closed-loop dynamic matrix is given by

$$A_{cl} = A + BK \tag{9}$$

which leads to the following Lyapunov equation:

$$\begin{aligned} &AW + BKW + WA' + WK'B' < 0 \\ &W > 0 \end{aligned} \tag{10}$$

The above Lyapunov inequality is not linear, since there appears the product of variables KW. Reference [17] divides the LMI-based formulations for robust control design in two classes: the *EV-type* (or eliminated-variable type) LMI's and the *CV-type* (or change-of-variable type) LMI's, that employ different strategies for the linearization of the design equations. The EV-type approach eliminates the product terms using the annihilator $B^{\perp}$ which satisfies $B^{\perp}B = 0$.

In the CV-type approaches, an auxiliary optimization variable Z is defined as [1]:

$$Z = KW \tag{11}$$

This change of variables linearizes the Lyapunov design inequality (10) (and other derived objective LMI's), leading to design LMI's in variables Z and W. The optimal robust controller is recovered, from the optimization variables, after the optimization procedure.

In this paper, it is proposed the simple re-utilization of any already known design formulation using the above mentioned change of variables replaced by the following modified one:

$$Z = (K + \Delta_K)W \tag{12}$$

The linearizing product of variables KW is given by:

$$KW = Z - \Delta_K W \tag{13}$$

Note that the right hand of equation (13) is affine in both Z and W, and therefore the new variables still render affine all equations that could become affine by the application of the change of variables (11). It is easy to verify that any constraint on the controller $K + \Delta_K$ is still guaranteed by the newly defined variables. In this way, the definition of a set of LMI's for all controller uncertainty polytope vertices Δ_K^i will lead to a design that guarantees the closed-loop performance in all vertices, and also in the polytope "nominal" center, in which $\Delta_K = 0$. The resulting controller, as before, is given by:

$$K = ZW^{-1} \tag{14}$$

The difference from this controller to the controller obtained by the change of variables (11) is that the new one is explicitly designed in order to handle with controller implementation errors, since any possible instance of the controller implementation $K + \Delta_K$ can be taken into account in the design LMI's.

Notice that even other LMI-based robust control design methods that consider system model uncertainties other than the polytope-type can be transformed in non-fragile methods through the line depicted above. Systems with norm-bounded uncertainties or with sector-bounded uncertainties, for instance, can be dealt with by the proposed method, provided that: (i) the controller uncertainty is of polytope-type, and (ii) the design LMI is of the CV-type.

4 Application to a Mixed H_2/H_∞ Design

To illustrate the proposed approach, the following well-known formulation of a mixed H_2/H_∞ guaranteed cost controller design is used (see, for instance, [12]):

$$(\theta^*)^2 = \min_{Z,W} \mathbf{tr}\,\Theta$$

subject to:

$$\begin{cases} \begin{bmatrix} \Theta & CW + DZ \\ (CW + DZ)' & W \end{bmatrix} \geq 0 \\ \begin{bmatrix} \Gamma_i & (CW + DZ)' \\ CW + DZ & \gamma^2 \mathbf{I} \end{bmatrix} \geq 0 \;\; \forall\, i = 1, \ldots, v \end{cases} \tag{15}$$

where

$$\Gamma_i \triangleq -(W(A + \Delta_A^i)' + (A + \Delta_A^i)W + Z'(B + \Delta_B^i)' + (B + \Delta_B^i)Z + EE') \tag{16}$$

The first LMI and the objective function define an upper bound for the system H_2 norm, and the second LMI defines the constraint γ for the system H_∞ norm. For simplicity, a single-channel problem has been chosen. In this formulation, the second LMI constraint is defined for each uncertainty polytope vertex.

The proposed change of variables leads to the modified formulation for robust and non-fragile controller design:

$$(\theta^*)^2 = \min_{Z,W} \mathbf{tr}\,\Theta$$

subject to:

$$\begin{cases} \begin{bmatrix} \Theta & CW + D(Z - \Delta_K^i W) \\ (CW + D(Z - \Delta_K^i W))' & W \end{bmatrix} \geq 0 \\ \begin{bmatrix} \Upsilon_i & (CW + D(Z - \Delta_K^i W))' \\ CW + D(Z - \Delta_K^i W) & \gamma^2 \mathbf{I} \end{bmatrix} \geq 0 \end{cases}$$

$$\forall\, i = 1, \ldots, v \tag{17}$$

with

$$\Upsilon_i \triangleq -(W(A + \Delta_A^i)' + (A + \Delta_A^i)W + (Z - \Delta_K^i W)'(B + \Delta_B^i)' + (B + \Delta_B^i)(Z - \Delta_K^i W) + EE') \tag{18}$$

In this formulation, derived directly from the former one, there is a pair of LMI constraints for each uncertainty polytope vertex.

Note: *The conventional design techniques can lead to nomi-*

nal controllers that are already non-fragile (in the sense that no design constraint is violated with the addition of the controller uncertainties). For the polytopic uncertainty case, the "fragility analysis" can be performed through the algorithms presented in [15]. The "non-fragile" algorithms should be employed only in the case of some constraint violation.

5 Numerical Example

In order to evaluate the proposed approach, a simple numerical test is conducted with the mixed H_2/H_∞ methods above.

Consider the following 4th-order linear system

$$A = \begin{bmatrix} 0.7031 & 0 & 0.3946 & -0.3201 \\ -0.0524 & 0 & 0 & -0.1374 \\ 0 & 0 & 0 & 0.6158 \\ 0 & 1.0282 & 1.7524 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.4015 & 0.9778 \\ 0 & 0 \\ -0.2627 & 0.1593 \\ 0 & 0 \end{bmatrix} ; \; D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.6774 & 0.9343 \\ 0.8750 & 0.5457 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.4233 & 0 & 0 & 0.3174 \\ 0 & 0.1374 & 0.5757 & 0.3410 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} 0.6348 & 0 & 0 & 0 \\ 0 & 0.8204 & 0 & 0 \\ 0 & 0 & -0.1760 & 0 \\ 0 & 0 & 0 & 0.5625 \end{bmatrix}$$

satisfying the conditions $C'D = 0$, $D'D > 0$ and $E'E > 0$.

Since the effect of system uncertainties is not under investigation here, these uncertainties were not taken into account (i.e. $\Delta A = \Delta_B = 0$).

The constraint on the H_∞-norm is fixed in $\gamma = 10$. The state feedback controller obtained through the conventional formulation (15) is:

$$K_f = \begin{bmatrix} -6.2218 & 18.3450 & 28.3673 & 17.7932 \\ 1.8928 & -8.1480 & -13.4789 & -8.5242 \end{bmatrix}$$

Now, the uncertainty Δ_K is introduced in the non-fragile control design. The eight controller parameters with independent uncertainties will lead to a 256 vertices polytope. Such uncertainty polytope is defined considering parameter perturbations that are up to 0.2% of the absolute values of the above parameters. Note that this approximately corresponds to the uncertainty associated to the round-off in an 8-bit floating point digital implementation of the controller.

The non-fragile controller obtained through the proposed formulation (17) is:

$$K_n = \begin{bmatrix} -5.7708 & 17.2797 & 26.9063 & 16.9498 \\ 0.9303 & -5.2583 & -10.3092 & -6.6689 \end{bmatrix}$$

At this point, a new uncertainty polytope could be considered, with the size of 0.2% of the new "nominal" parameters, associated to the non-fragile controller. Since all parameters have become smaller than the former ones in absolute values, the polytope that has been considered is already sufficient for guaranteeing the system performance in the desired range of uncertainties. The controller re-calculation would be necessary only if a less conservative design is needed.

For a comparative evaluation of the two approaches, a random set of uncertainties has been generated inside the polytope ΔK (with uniform probability distribution), and the H_∞ and H_2 norms of the closed-loop systems have been computed for each resulting controller. The extreme results obtained from these simulations (the best and worst values of both norms) are presented in table 1.

	H_2 norm		H_∞ norm	
	max	min	max	min
K_f	7.3336	7.2460	10.6639	8.1247
K_n	7.6922	7.5954	8.1569	7.9301

Table 1: Extremal values of the closed-loop norms inside the uncertainty polytope.

The conventional controller has violated the H_∞ constraint when the uncertainties were added to it. The non-fragile controller, on the other hand, has satisfied it, at the price of reaching worse values for the H_2 norm, as would be expected. Notice that, if the H_∞ constraint had the connotation of a stability insurance (through small gain techniques), then the non-fragile controller would have the interpretation of a controller that holds the stability requirement for any allowed implementation error, while the conventional controller would loss the stability for some possible implementation errors.

A further insight on the properties of the proposed non-fragile design can be extracted from the histograms of the closed-loop norms for the two designs. These histograms are shown in figures 1 and 2.

These figures show that the non-fragile design has lead, in this case, to somewhat well-defined H_2 and H_∞ norms inside the uncertainty polytope. The conventional design, on the other hand, has produced a scattered histogram of the H_∞ norm, although its H_2 norm histogram had become well-defined.

The example touches another issue that is important in real controller design problems: in most of the cases, the size of the uncertainty in the controller parameters depends on the size of the parameters themselves. Since the parameters are the design output, the size of the uncertainty polytope cannot

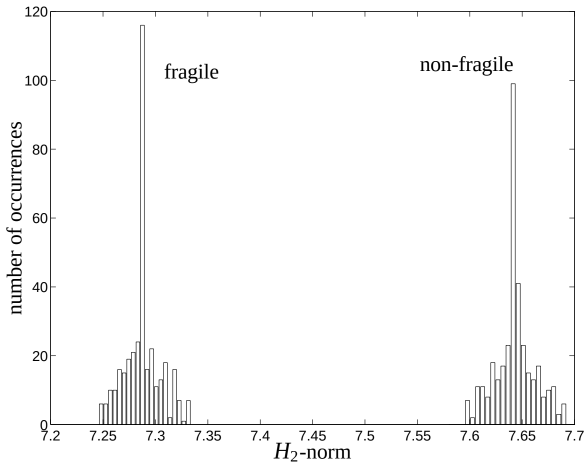

Figure 1: Histogram of the H_2 norm in the set of uncertain controllers. Both the conventional and the non-fragile designs are shown.

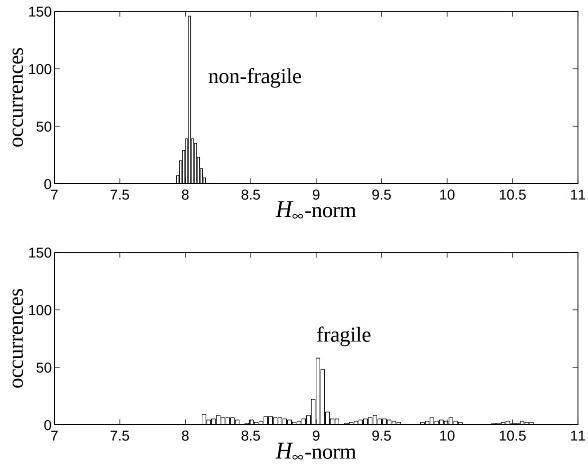

Figure 2: Histogram of the H_∞ norm in the set of uncertain controllers. The conventional design is shown bellow, and the non-fragile design is shown above.

be *a priori* known. From the example, it can be inferred that a conservative design, with the controller uncertainty polytope greater than a pre-determined fraction of the nominal controller parameters can be determined, with a few iterations, using the proposed methodology (in the example, a single iteration was needed). This conservativeness can be reduced with more iterations of this sequence of (i) polytope definition and (ii) nominal controller computation. This issue will not be discussed in more depth here. The key points that must be stressed are:

- If the relative size of the uncertainty is not too big, the parameters of the non-fragile controller are in general of size similar to the ones of the controller designed through the conventional method. This means that the uncertainty polytope does not vary significantly in many cases, and a less-conservative design can be found with low effort.

- Even in the cases in which significant changes in the polytope size could be expected (this could be the case in some design contexts in which the component tolerances are large), the proposed design can lead to solutions that, although conservative, can be found through a systematic design procedure. This systematic procedure, additionally, is a simple modification of any algorithm chosen from a large class of design algorithms that are being currently employed.

6 Conclusion

This paper has presented a modified variable transformation that allows the inclusion of controller uncertainties in a large class of existing LMI-based robust control design methods. In this way, the fragility problem can be solved in the framework of the existing methods of robust control.

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