On Passivity Based Control for Partial Stabilization of Underactuated Systems

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Abstract.

The stabilization problem for a class of an underactuated Lagrangian systems with respect to part of state variables is considered. It is assumed that this part of the state variables is directly affected by control. Moreover, on the rest of variables we impose additional constraints, based on the form of the total energy of the system, which should be valid after the transition time. Intuitively this can be interpreted as a decoupling, by suggested control strategy, the state variables into parts. To motivate this investigation, the problem of swinging the Pendubot is considered.

1 Introduction

Suppose that the system has a configuration space $Q = \Theta \times X$, $q = (\theta, x)$, and the equations of motion are

$$\frac{d}{dt}\nabla_{\dot\theta}\mathcal{L} - \nabla_\theta \mathcal{L} = 0 \qquad (1)$$

$$\frac{d}{dt}\nabla_{\dot x}\mathcal{L} - \nabla_x \mathcal{L} = u \qquad (2)$$

Here $\mathcal{L}$ is the Lagrangian of the unforced system

$$\mathcal{L}(q,\dot q) = \frac{1}{2}\langle \dot q, \dot q\rangle - \Pi(q)$$

with $\langle \cdot,\cdot\rangle$ being a Riemannian metric on Q and Π being a potential energy; u is a control action. Introduce the total energy $E(q,\dot q)$ of the unforced system (1)–(2)

$$E(q,\dot q) = \dot q^i \nabla_{\dot q^i}\mathcal{L}(q,\dot q) - \mathcal{L}.$$

Given a constant E_0 and a vector a, the problem is to define the feedback control such that along the closed-loop system solutions $(q(t), \dot q(t))$ the limit relations

$$\lim_{t\to+\infty} E(q(t)\dot q(t)) = E_0 \qquad (3)$$

$$\lim_{t\to+\infty} x(t) = a \qquad (4)$$

$$\lim_{t\to+\infty} \dot x(t) = 0 \qquad (5)$$

are valid.

This particular problem could be considered as a partial stabilization with respect to the state variables x, $\dot x$ with additional constraint to the rest of the state variables θ, $\dot\theta$ expressed by the relation (3). Intuitively, the problem (3)–(5) reflects an intention to change (or even exclude) the dynamics of the system related to the x-variables by appropriate control strategy in such way, that the motions of the directly uncontrolled θ-variables behave as uncoupled with the x-variables.

For example for the inverted pendulum the relation (3)–(5) correspond to the intention to stabilize a rotation of the pendulum with a given value E_0 of the total energy provided the cart tends to a prescribed position a. Even more, the solution of this problem for the inverted pendulum could help in global stabilization of the upright and downward positions of the inverted pendulum at given position of the cart. For the downward position one should put E_0 being equal to minimum value of the total energy. For the upright position E_0 should be equal to the value of the total energy at this point, see for details [5].

Another examples, where the relations (3)–(5) have sense, are a stabilization of the Furuta pendulum, a stabilization of the spherical pendulum on the cart moving in the horizontal or inclined plane, a swinging of the Acrobot and the Pendubot, and others underactuated mechanical systems.

The paper is organized as follows. Section 2 contains the main results of the paper. An illustrative example, the problem of swinging up for the Pendubot, is considered in Section 3. Some conclusions are drawn in Section 4.

2 The Main Result

Due to standard assumptions (the compatibility of the Riemannian metric and the geometric connection) the total energy satisfies the passivity relation

$$\frac{d}{dt}E(q(t),\dot{q}(t)) = \dot{x}(t)^T u(t).$$

Consider the storage function

$$V(q,\dot{q}) = \frac{k_E}{2}\left[E - E_0\right]^2 + \frac{k_v}{2}|\dot{x}|^2 + \frac{k_x}{2}|x-a|^2,$$

its derivative along the solutions of (1)–(2) has the form

$$\frac{d}{dt}V = \dot{x}^T\left[k_E\left[E-E_0\right]u + k_v\ddot{x} + k_x[x-a]\right].$$

Using the expression for $\ddot{x}$ from (2), one has

$$\begin{aligned}\frac{d}{dt}V \quad = \quad & \dot{x}^T \times \Bigg[H(q,\dot{q}) + \\ & + \left(k_E\left[E-E_0\right] + k_v\left[0\ \mathbf{1}\right]M(q)^{-1}\begin{bmatrix}0\\ \mathbf{1}\end{bmatrix}\right)u\Bigg]\end{aligned}$$

where $M(q)$ is a metric tensor associated with the Riemannian metric and $H(q,\dot{q})$ is some function, which depends on $\mathcal{L}$ and parameters k_E, k_v, k_x, a.

Take any smooth function $\phi(x)$, such that $x^T\phi(x) > 0$ for any $x \neq 0$, and consider the equation

$$\begin{aligned}\left(k_E\left[E-E_0\right]\cdot\mathbf{1} + k_v\left[0\,\mathbf{1}\right]M(q)^{-1}\begin{bmatrix}0\\ \mathbf{1}\end{bmatrix}\right)u+ \\ +H(q,\dot{q}) = -\phi(\dot{x}).\end{aligned} \qquad (6)$$

To solve this equation with respect to u one should invert the matrix

$$k_E\left[E-E_0\right]\cdot\mathbf{1} + k_v\left[0\,\mathbf{1}\right]M(q)^{-1}\begin{bmatrix}0\\ \mathbf{1}\end{bmatrix} \qquad (7)$$

The next statement is rather straightforward and simple, but could be applied for many mechanical systems.

Theorem 1 *If the total energy E of the unforced system is bounded from below, then there exist positive parameters k_E, k_v such that this matrix (7) is globally strictly positive definite, i.e. it is globally invertible.*

Thus, for properly chosen parameters k_E, k_v, the equation (6) can be solved and for such defined control variable u one has

$$\frac{d}{dt}V(q(t),\dot{q}(t)) = -\dot{x}(t)^T\phi(\dot{x}(t)) \leq 0, \qquad (8)$$

i. e. the non-negative function V does not increase along the closed loop system solutions. The last inequality also shows that the system is passive with the storage function V from the input ϕ to the output $\dot{x}$. In other words Theorem 1 guarantees the existence a state feedback transformation, which passifies the system with output $\dot{x}$ with respect to a nonnegative function V.

To determine the range of the positive parameters k_E, k_v, mentioned in Theorem 1, is an additional problem. To find such parameters one could proceed as follows: Denote the vectors

$$s_1 := q, \quad s_2 := \dot{q}.$$

Substituting them to the matrix (7), consider the algebraic inequality

$$k_E\left[E(s_1,s_2) - E_0\right]\cdot\mathbf{1} + k_v\left[0\,\mathbf{1}\right]M(s_1)^{-1}\begin{bmatrix}0\\ \mathbf{1}\end{bmatrix} > 0 \qquad (9)$$

with respect to the parameters k_E, k_v, which should be valid for any vectors s_1, s_2. Putting the expression for $E(s_1,s_2)$ into (9), one has

$$\begin{aligned}k_E\left[\frac{1}{2}\langle s_2,s_2\rangle + \Pi(s_1) - E_0\right]\cdot\mathbf{1} + \\ k_v\left[0\,\mathbf{1}\right]M(s_1)^{-1}\begin{bmatrix}0\\ \mathbf{1}\end{bmatrix} > 0.\end{aligned} \qquad (10)$$

This inequality will be valid if the next one

$$k_E\left[\Pi(s_1) - E_0\right]\cdot\mathbf{1} + k_v\left[0\,\mathbf{1}\right]M(s_1)^{-1}\begin{bmatrix}0\\ \mathbf{1}\end{bmatrix} > 0 \qquad (11)$$

holds. The next criterion is again straightforward, but could provide the rough estimation of the range of k_E, k_v.

Theorem 2 *Suppose $\Pi(s_1) \geq 0$ for any s_1 and*

$$\min_{s_1}\left\|\left[0\,\mathbf{1}\right]M(s_1)^{-1}\begin{bmatrix}0\\ \mathbf{1}\end{bmatrix}\right\| \geq \varepsilon.$$

Then the inequality (11) holds (and therefore the matrix (7) is invertible) if

$$k_v > k_E\cdot\frac{E_0}{\varepsilon}. \qquad (12)$$

Taking into account the form of the potential energy $\Pi(q)$ and the metric tensor $M(q)$ could result in much better estimation than (12). For example, for the inverted pendulum (with the mass of the cart, the mass of the bob and the length of the rob being equal to 1) the value of ε is

$$\varepsilon \leq \min_{s_1} \left| \frac{1}{1+\sin^2 s_1} \right| = \frac{1}{2}.$$

Therefore the inequality (12) takes the form

$$k_v > k_E \cdot 2 \cdot E_0$$

At the same time the necessary and sufficient condition for the positivity of the matrix (11) for the inverted pendulum with aforementioned parameters is

$$k_v > \rho \cdot k_E, \tag{13}$$

where

$$\rho = \begin{cases} E_0, & E_0 \leq \frac{1}{2}g \\ \frac{g}{27}\left[2\frac{E_0-g}{g} + \sqrt{\left(\frac{E_0-g}{g}\right)^2+6}\right] \times \\ \left[18 - \left(\frac{E_0-g}{g} - \sqrt{\left(\frac{E_0-g}{g}\right)^2+6}\right)^2\right], & E_0 > \frac{1}{2}g \end{cases}$$

see [3] for details.

Coming back to the inequality (8), one could easily verify that the value of the function $V(t)$ tends to a constant $V_\infty \geq 0$ as $t \to +\infty$. Taking advantage of Barbalat's lemma, one concludes that $\dot{x}(t)$ tends to zero. But, in general, it is possible that the constant V_∞ is different from zero and $x(t)$ does not tend to a desired position a.

Theorem 3 *Consider the controlled system (1), (2). Take the control law u as a solution of the equation (6), where the parameters k_E, k_v are positive and satisfy the inequality (12). Then the limit relations (3)–(5) are valid for any solution of the closed loop system if and only if the system (1), (2) with the output $y = \dot{x}$ is V-detectable.*

The notion of V-detectability of the set is a generalization of the property of the system to be *zero-state* detectable, see [4] where definition of this property and several tests were suggested. The proof of Theorem 3 could be found in [4] and based on standard passivity arguments related to a invariant set (rather than equilibrium point) stabilization.

For some systems the property to be V-detectable could fail, for example this situation takes place for any system having periodic coordinates. For these systems Theorem 3 should be replaced by its local analog, where *local* variant of V-detectability becomes crucial, see [4].

Here we would like briefly to mention one important marginal case. which does not fit Theorem 3, but it is of interest in some examples, like the inverted pendulum and the Pendubot.

Theorem 4 *Consider the controlled system (1), (2). Take the control law u as a solution of the equation (6), where the parameters k_E, k_v are positive and satisfy the inequality (12). Suppose that the union Ω of all the ω-limit sets of the closed loop system solutions consists of the set*

$$V_0 = \{(q,\dot{q}) : V(q,\dot{q}) = 0\}$$

and a finite number of equilibria of the closed loop system. If one could find the positive parameters k_E, k_v, k_x of the control law, which makes simultaneously these equilibria hyperbolic, then the set V_0 is a generic attractive set of the closed loop system, i. e. for almost all solution the limit relations (3)–(4) are valid.

One of the possible ways to make use of Theorem 4 is to find these equilibria, then to make linear approximations of the closed loop system around these additional equilibria, and then to find ranges of the positive parameters k_E, k_v, k_x, for which the appropriate characteristic equations have roots with positive real parts.

3 Example: Swinging Up the Pendubot

Here we are going to show how the ideas made in the previous section could be applied to the problem of swinging up of an underactuated two-link robot with an actuator at the shoulder (link 1) and no actuator at the elbow. It called the Pendubot.

This problem was considered by Prof. Lozano and coworkers in [2], where to solve the problem the authors proposed two stage (hybrid) control law: first, using the passivity based control bring the system in small neighborhood of the upper unstable equilibrium point, and, second, switch the control to any locally stabilizing (or balancing) controller, which already available in literature, see for example [6].

Under the standard assumptions the equations of the Pendubot are

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau, \tag{14}$$

where $q = [q_1, q_2]^T \in S^1 \times S^1$, q_1 is the angle that link 1 makes with horizontal, q_2 is the angle that the link 2 makes with link 1;

$$M(q) = \begin{bmatrix} \theta_1 + \theta_2 + 2\theta_3 \cos q_2 & \theta_2 + \theta_3 \cos q_2 \\ \theta_2 + \theta_3 \cos q_2 & \theta_2 \end{bmatrix}, \tag{15}$$

where

$$\begin{aligned}\theta_1 &= m_1 l_{c_1}^2 + m_2 l_1^2 + I_1\\ \theta_2 &= m_2 l_{c_2}^2 + I_2\\ \theta_3 &= m_2 l_1 l_{c_2}\end{aligned}$$

m_1, m_2 are the masses of the link 1 and the link 2; l_1, l_2 are the lengths of the link 1 and the link 2; l_{c_1} is the distance to the center mass of the link 1 from the suspension point, l_{c_2} is the distance to the center mass of the link 2 from the suspension point; I_1, I_2 are the moments of inertia of the link 1 and the link 2 about their centroids;

$$C(q,\dot q) = \theta_3 \sin q_2 \begin{bmatrix} -\dot q_2 & -\dot q_2 - \dot q_1\\ \dot q_1 & 0\end{bmatrix}, \tag{16}$$

$$G(q) = \begin{bmatrix} \theta_4 g \cos q_1 + \theta_5 g \cos(q_1+q_2)\\ \theta_5 g \cos(q_1+q_2)\end{bmatrix}, \tau = \begin{bmatrix}\tau_1\\ 0\end{bmatrix} \tag{17}$$

where τ_1 is the control input to be defined, and

$$\theta_4 = m_1 l_{c_1} + m_2 l_1 \quad \theta_5 = m_2 l_{c_2}.$$

The total energy of the Pendubot is

$$\begin{aligned}E(q,\dot q) &= \frac{1}{2}\dot q^T M(q)\dot q + \Pi(q)\\ &= \frac{1}{2}\dot q^T M(q)\dot q + \theta_4 g(\sin q_1 + 1) +\\ &\quad +\theta_5 g(\sin(q_1+q_2)+1)\end{aligned} \tag{18}$$

and it could attain any values from the interval $[0,+\infty)$. Let E_0 be equal to $2(\theta_4+\theta_5)g$ (the value of the total energy at the upper equilibrium point), introduce the function

$$V(q,\dot q) = \frac{k_E}{2}\left(E(q,\dot q)-E_0\right)^2 + \frac{k_v}{2}\dot q_1^2 + \frac{k_x}{2}\left(q_1 - \frac{\pi}{2}\right)^2, \tag{19}$$

where k_E, k_v, k_x are some positive constants. As it is shown in [2], the set V_0 of the phase space, where $V(q,\dot q)$ is zero, consists of a union of two homoclinic curves with $q_2 = 0$ and the upright equilibrium of the Pendubot. So successful stabilization of the set V_0 leads to the solution of the swinging up problem for the Pendubot. Indeed, in this case any solution will have the upright equilibrium as an ω-limit point and eventually comes to any neighborhood of this point, where the controller could be switched to local one.

Taking the time derivative of this function along the solutions of the system (14), one has

$$\begin{aligned}\dot V &= \dot q_1 \left[\tau_1 \cdot \left(k_E(E-E_0) + k_v [0\,1] M(q)^{-1}\begin{bmatrix}0\\1\end{bmatrix}\right)\right.\\ &\quad \left. + H(q,\dot q)\right],\end{aligned} \tag{20}$$

where

$$[0\ 1]M(q)^{-1}\begin{bmatrix}0\\1\end{bmatrix} = \frac{\theta_2}{\theta_1\theta_2 - \theta_3^2\cos^2 q_2}$$

and

$$H(q,\dot q) = k_x\left(q_1 - \tfrac{\pi}{2}\right) + k_v \times$$
$$\frac{\theta_2\theta_3 \sin q_2 (\dot q_1 + \dot q_2)^2 + \theta_3^2 \cos q_2 \sin q_2 \dot q_1^2 - \theta_2\theta_4 g \cos q_1 + \theta_3\theta_5 g \cos q_2 \cos(q_1+q_2)}{\theta_1\theta_2 - \theta_3^2\cos^2 q_2}.$$

To define a control law we will use the equation (6), which now is

$$\begin{aligned}\tau_1 &\left(k_E(E-E_0) + k_v \frac{\theta_2}{\theta_1\theta_2 - \theta_3^2\cos^2 q_2}\right)\\ &+H(q,\dot q) = -\phi(\dot q_1),\end{aligned} \tag{21}$$

and $\phi(x)$ is any smooth function, $x^T\phi(x) > 0\ \forall x \neq 0$. Using Theorem 2, one could obtain the values of the positive constants k_E, k_v, for which this equation is solvable with respect to τ. One can easily verify that

$$\begin{aligned}\varepsilon &\leq \min_{s_1}\left|[0\,1]\,M(s_1)^{-1}\begin{bmatrix}0\\1\end{bmatrix}\right| = \min_q \frac{\theta_2}{\theta_1\theta_2 - \theta_3^2\cos^2 q_2}\\ &= \frac{1}{\theta_1}.\end{aligned}$$

This estimation and Theorem 2 lead to following statement

Theorem 5 *If k_E, k_v are positive and satisfy the inequality*

$$k_v > 2\cdot k_E \cdot g \cdot \theta_1 \cdot (\theta_4 + \theta_5), \tag{22}$$

then the value of

$$k_E(E(q,\dot q) - E_0) + k_v \frac{\theta_2}{\theta_1\theta_2 - \theta_3^2\cos^2 q_2} \tag{23}$$

with $E_0 = 2g(\theta_4+\theta_5)$ is strictly positive for any $(q,\dot q)$, and the control variable τ can be always found from (21).

To emphasize the difference with [2], we mention that one of the main results of the paper [2] states the solvability of the equation (21) with respect to τ only for points $(q,\dot q)$, which satisfy the inequality, see [2, the formula (23)],

$$\|E(q,\dot q) - E_0\| \leq \frac{k_v - \delta}{k_E\theta_1},$$

where δ is an arbitrary positive constant. One can easily see that a union of the points which satisfy the last inequality is a bounded subset of the phase space. Moreover, it is worth to mention that the estimation (22) is rough and could be improved. The globalization of the controller defined by (21) makes possible to find all ω-limit points of the closed loop systems.

Theorem 6 *Consider the Pendubot together with the controller defined by (21) with the positive coefficients k_E, k_v satisfying the inequality (22). Then the ω-limit set Γ of the closed loop system consists of the set V_0 and a number equilibria with the coordinates (q_1^*, q_2^*) defined as a solution of the equations*

$$(\theta_4 g)^2 \cos q_1^* \,(1 - \sin q_1^*) = \frac{k_x}{k_E}(q_1^* - \frac{\pi}{2}), \qquad (24)$$

$$q_1^* + q_2^* = \left\{\frac{\pi}{2} \quad or \quad -\frac{\pi}{2}\right\} \bmod 2\pi \qquad (25)$$

which always have at least one solution $(q_1^*, q_2^*) = (\frac{\pi}{2}, \pi)$ outside the set V_0.

Analyzing the equations (24), (25), one can conclude that depending on the parameter of the Pendubot θ_4 and the parameters of the controller k_E, k_x the closed loop system could have other new equilibria than mentioned in (39). We would like to avoid such appearance of these new equilibria so to do this we should answer the question: when the equation (38) has just one solution.

Theorem 7 *If the positive coefficients k_E, k_v, k_x for the controller (21) satisfy (22) and the following inequality*

$$\frac{\sqrt{\pi^2+1}}{2} > (\theta_4 g)^2 \frac{k_E}{k_x}, \qquad (26)$$

then the equation (38) has a unique solution and the closed loop system has just one equilibrium $(q_1, q_2) = (\pi/2, \pi)$ outside the desired attractive set V_0.

The next step in the controller design for the Pendubot is to determine the range of the positive parameters k_E, k_v, k_x which satisfy the constraints (22), (26) and guarantee that the additional equilibrium $(q_1, q_2) = (\pi/2, \pi)$ is hyperbolic. It could be done for example by taking linear approximation of the closed loop system around these equilibrium and then playing with the coefficients of its characteristic equation. But such an analysis is not straightforward. From the other hand the ω-limit set of the closed loop system (14), (21) has very simple structure: it consists of asymptotically stable compact set V_0 and the equilibrium point $(q_1, q_2) = (\pi/2, \pi)$. The topological arguments help us to show the next statement.

Theorem 8 *Suppose that the parameters k_E, k_v, k_x are positive and constraints (22), (26) are valid. Then the set $\mathcal{O}$ of initial conditions for which closed loop system (14), (21) solutions tend to the equilibrium $(q_1, q_2) = (\pi/2, \pi)$ cannot be open subset in the phase space $R^2 \times S^1 \times S^1$ of the Pendubot.*

Proof. Suppose, by contrary, the set $\mathcal{O}$ is open. The area of attraction $\mathcal{V}$ for the asymptotically stable set V_0 is also open, see [1, Theorem V.3.6], and the union of $\mathcal{O}$ and $\mathcal{V}$ should give the whole phase space. So the closed loop system does not have other attractors, then this immediately implies that $\mathcal{O}$ cannot be open. **This particularly implies that the equilibrium $(q_1, q_2) = (\pi/2, \pi)$ cannot be asymptotically stable and even stable in Lyapunov sense.** ■

4 Conclusions

The paper is devoted to a special stabilization problem for an underactuated nonlinear systems. It is assumed that a desired attractor of the closed loop system could be described by an appropriate value of the total energy of the unforced system provided that the directly affected variables attain also prescribed values. This makes the problem nontrivial, the desired attractive set, constructed in this way, is not invariant with respect to motions of the unforced systems. Particularly, swinging up problems for the inverted pendulum, the Furuta pendulum, the spherical pendulum and the Pendubot can be reformulated in such a way.

The solution is based on a successful feedback transformation of the system, which makes the system to be passive with respect to a storage function equals to zero on the desired attractive set and positive outside. The main results are illustrated by the example: the problem of swinging up the Pendubot.

5 Appendix: Proof of Theorem 6

Along any closed loop system solution $[q(t), \dot{q}(t)]$ the passivity inequality (8) holds. The function V is proper on the cylindrical phase space of the Pendubot, therefore any solution of the closed loop system bounded and its ω-limit set Γ is compact and invariant with respect to vector field of the closed loop system. Due to (8) and smoothness of ϕ we have the following: a) the value of $\dot{q}_1$ is zero, i. e. q_1 is constant q_1^*, on the set Γ; b) the value of the function V is constant V^* on Γ.

The set Γ could have several connected components. Particularly, we know one component of Γ, which is V_0 (the desired attractive set of the phase space where the function V equals to zero). But Γ possibly has different components corresponding different attractive sets of the closed loop system and we are going to find these sets. For them the value of constant V^* should differ from zero.

Both the relations a) and b) imply, see (19), that the value of the total energy $E(q, \dot{q})$ is constant E^* on Γ.

For the set V_0 the value of E^* is E_0. The dynamics of the Pendubot subjected to aforementioned conditions is described by the equations

$$(\theta_2+\theta_3\cos q_2)\ddot{q}_2-\dot{q}_2^2+\theta_4 g\cos q_1^* \\ +\theta_5 g\cos(q_1^*+q_2)=\tau_1, \quad (27)$$

$$\theta_2\ddot{q}_2+\theta_5 g\cos(q_1^*+q_2)=0, \quad (28)$$

where τ_1 is defined by the equation (21), which, in turn, reduces to

$$\tau_1\cdot k_E\cdot(E^*-E_0)+k_x\cdot(q_1^*-\frac{\pi}{2})=0. \quad (29)$$

We would like to identify parts of Γ others than V_0, so we will assume that $E^*-E_0\neq 0$. Then relation (29) immediately implies that the value of τ_1 on the set $\Gamma\setminus V_0$ is constant corresponding to the value of E^*

$$\tau_1=\frac{k_x\cdot(q_1^*-\frac{\pi}{2})}{k_E\cdot(E_0-E^*)}. \quad (30)$$

Substituting the expression for $\ddot{q}_2$ from (28) into the equation (27), we can rewrite (27) in new form without second time derivative of q_2

$$-\frac{\theta_3\theta_5 g\cos q_2\cos(q_1^*+q_2)}{\theta_2}-\dot{q}_2^2+\theta_4 g\cos q_1^*=\tau_1, \quad (31)$$

Differentiating this identity with respect to time, we obtain

$$2\dot{q}_2\left(\ddot{q}_2-\frac{\theta_3\theta_5 g}{2\theta_2}\sin(q_1^*+2q_2)\right)=0 \quad (32)$$

This relation together with (27), (28) and (29) are valid for any trajectory of the closed loop system belonging to $\Gamma\setminus V_0$.

Now we are going to show that these equations imply that $\dot{q}_2=0$. By a contradiction, suppose that $\dot{q}_2$ is not identically equal to zero for some solution of the closed loop system belonging to $\Gamma\setminus V_0$. Then from (32) this solution should satisfy simultaneously the equation

$$\ddot{q}_2-\frac{\theta_3\theta_5 g}{2\theta_2}\sin(q_1^*+2q_2)=0$$

and the equation (28). This could take place only if

$$\frac{\theta_3\theta_5 g}{2\theta_2}\sin(q_1^*+2q_2)+\theta_5 g\cos(q_1^*+q_2)=0. \quad (33)$$

But the last relation could be valid for any function of time q_2 except q_2 is constant. Indeed, differentiating (33) with respect to time and taking into account that $\dot{q}_2\neq 0$, we obtain new identity

$$\frac{\theta_3\theta_5 g}{\theta_2}\cos(q_1^*+2q_2)-\theta_5 g\sin(q_1^*+q_2)=0. \quad (34)$$

Differentiating (34) again, we have

$$2\frac{\theta_3\theta_5 g}{\theta_2}\sin(q_1^*+2q_2)+\theta_5 g\cos(q_1^*+q_2)=0. \quad (35)$$

The equation (33) and (35) do not have solution simultaneously. Thus $\dot{q}_2\equiv 0$ on $\Gamma\setminus V_0$, i.e $q_2(t)=q_2^*$. Taking advantage of this fact, we find from (28) that

$$q_1^*+q_2^*=\frac{\pi}{2}\ \text{or}\ -\frac{\pi}{2}. \quad (36)$$

Using (36) and (30), we can rewrite the second order differential equation (27) as a algebraic one

$$\theta_4 g\cos q_1^*=\tau_1=\frac{k_x\cdot(q_1^*-\frac{\pi}{2})}{k_E\cdot(E_0-E^*)} \quad (37)$$

Furthermore E^* is the Pendubot energy corresponding to the motion with constant values of q_1^* and q_2^* related by (36). Using the formula (18) and the value for E_0, the relation (37) is equivalent to a simple equation on q_1^*

$$(\theta_4 g)^2\cos q_1^*\,(1-\sin q_1^*)=\frac{k_x}{k_E}(q_1^*-\frac{\pi}{2}) \quad (38)$$

The equations (36) and (38) describe an additional parts of the set Γ others than the set V_0. It is worth mentioning that these are just new equilibria of the closed loop system and that the points with coordinates

$$(q_1^*,q_2^*)=(\frac{\pi}{2},0)\ \text{or}\ (\frac{\pi}{2},\pi) \quad (39)$$

are the solutions for (36), (38). The equilibrium $(\frac{\pi}{2},0)$ belongs to the set V_0. Theorem 2 is proven. ∎

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