ADAPTIVE CONTROL ON MANIFOLDS WITH RBF NEURAL NETWORKS

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Abstract

We propose a new method of adaptive control on manifolds for non-linear plants in the full-state feedback case using radial basis function (RBF) neural networks. We introduce a procedure for synthesis of adaptation algorithms based on associated performance criteria. We analyze applicability of the algorithms developed for a quadratic performance criterion.

1 Introduction

This paper deals with design of adaptive control systems for non-linear plants in the full-state feedback case using RBF neural networks. We utilize main concepts of the synergetic control theory [1, 2]. The synergetic control theory is developed to harness processes of self-organization in certain dissipative structures. Such structures can be constructed for many control systems in the form of desired dynamic invariants or manifolds in an extended state space of the closed-loop system. Goals of adaptive control are defined in that extended state space, which allows us to account for complex, possibly even chaotic, nature of the plant's dynamics. Adaptive control on manifolds is discussed in details in [3].

Methods described in [1] are currently unavailable to English speaking readers, but they are somewhat similar to a popular method of backstepping [6]. Methods of synergetic control theory feature a synthesis procedure for controllers based on macrovariables. These macrovariables and their associated performance criteria are constructed in the extended space, which reflects goals of the adaptive control system. Dissipative structures in the original state of the closed-loop system can be related to the macrovariables. Goals of adaptive control in terms of meeting the specified performance criteria can be directly linked to providing an adequate control of the macrovariables. Macrovariables act as parameters of order of the plant, and control as a function of macrovariables may be efficient and consistent with the plant's dynamics. In general, distinct plants described by different mathematical models are controlled the best with different controllers. This makes the controller synthesis procedure rather unique for each non-linear plant. It is possible to make the synthesis procedure less dependent on properties of the given plant by representing the controller in the form of a general function approximation system, e.g., an RBF neural network, as it is done here. We would like to motivate the need for using adaptive control on manifolds. In conventional control theory the goal of control is almost directly a definition of the plant's desirable behavior. Control goal is specified either in the form of a performance criterion or via a reference model. It is not linked with properties of internal structure of the plant. Such methodology is well developed in modern control theory. However, it is frequently desirable to construct a control system consistent with natural properties of the plant. This is done in synergetic control theory [1]. Control goal is chosen such that natural properties of the plant are exploited to our advantage. For instance, this may permit to minimize energy of the control efforts. According to synergetic control theory, it is useful to separate original goals of control specified in terms of a performance criterion from internal or local goals which, if attained, guarantee that the adaptive system achieves its original goals. We illustrate this with a simple example. Let the plant be described by the following system

$$\begin{aligned}\dot{x}_1 &= -x_2 - x_3;\\ \dot{x}_2 &= x_1 + ax_2 + u_1;\\ \dot{x}_3 &= bx_1 + x_1x_2 - cx_3 + u_2\end{aligned}$$

For $a = 0.2$, $c = 4.32$, $b = 0$, these equations describe the Rossler system whose unperturbed dynamics is a strange attractor. We suppose that our control goal is to move plant from its initial state at time t_0 to the origin $x_1 = x_2 = x_3 = 0$. This is our global goal. As it can be seen from the plant equations, it is sufficient to ensure that the plant moves along the intersection of manifolds $x_1 = 0, x_2 = 0$ in order to reach the global goal. Indeed, movement along the coordinate x_3 is asymptotically stable provided $u_2(0, 0, x_3) = 0$. Intersection of manifolds $\psi_1 = x_1 = 0$ and $\psi_2 = x_2 = 0$ may be interpreted as our local control goal. Here macrovariables $\psi_1(x_1)$ and $\psi_2(x_2)$ coincide with the first two components of the state vector. But in general they can be

much more complicated functions of the state vector. It is these functions for which a suitable control law must be developed, and it is convenient to think about ψ_1 and ψ_2 as extensions of the state space.
This paper is organized as follows. Section 2 describes the proposed adaptive control method on manifolds with RBF network. Section 2.1 formulates the problem and introduces the notions of macrovariables and associated functionals. Section 2.2 introduces adaptive algorithms for an important special case of quadratic associated functionals and discusses various issues pertaining to their applicability. Section 3 provides an illustration of the adaptive control method proposed. Section 4 concludes the paper.

2 Method

2.1 Problem formulation

Let the plant be described by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \theta) + \mathbf{b}(\mathbf{u}, \mathbf{x}), \tag{1}$$

where $\mathbf{f}(\cdot)$ and $\mathbf{b}(\cdot)$ are smooth and differentiable functions, $\mathbf{x} \in \Omega_{\mathbf{x}} \subseteq \mathbf{R}^n$ is plant state vector, $\Omega_{\mathbf{x}}$ is the domain of plant states, $\mathbf{u} \in \mathbf{R}^m$ is a control vector, $\mathbf{x_0} = \mathbf{x}(\mathrm{t}_0)$ or $\mathbf{x}(\mathbf{0})$ is a vector of initial conditions; $\mathbf{w} \in \mathbf{R}^K$ is a vector of controller parameters; $\theta \in \mathbf{R}^d$ is a vector of unknown parameters of the plant. In this paper, control vector $\mathbf{u}$ is an output of an RBF network, and parameters $\mathbf{w}$ are weights of the RBF network. Activation functions (kernels) of the network are denoted as $q_i(\mathbf{x} - \mathbf{a}_i) : \mathbf{R}^n \longrightarrow \mathbf{R}$, where $\mathbf{a}_i$ are centers of the kernels. We can write

$$\mathbf{u} = \mathbf{q}(\mathbf{w}, \mathbf{x}) = col(\textstyle\sum_{i=1}^{l} w_{1,i} q_i(\mathbf{x} - \mathbf{a}_i), \ldots, \textstyle\sum_{i=1}^{l} w_{m,i} q_i(\mathbf{x} - \mathbf{a}_i)), \quad K = l\,m.$$

Then each component of vector $\mathbf{u}$ is

$$u_j(\mathbf{x}) = \textstyle\sum_{i=1}^{l} w_{j,i} q_i(\mathbf{x} - \mathbf{a}_i), \quad q_i(\mathbf{x} - \mathbf{a}_i) = \mathbf{e}^{-\|\mathbf{x} - \mathbf{a}_i\|^2/\sigma^2} \tag{2}$$

where $j = 1, 2, ..., m$ and σ is the kernel width. Both width σ and centers of kernels $\mathbf{a}_i$ are assumed fixed.

We introduce our general notation. The manifold is denoted as $\psi_s(\mathbf{x}, \mathbf{c}_s) = \mathbf{0}$, $s \leq n$, and $\mathbf{c}_s \in \mathbf{R}^h$ is its parameter vector. As in [2], functions $\psi_s(\mathbf{x}, \mathbf{c}_s)$ are called *macrovariables* in the adaptive system's state space. Macrovariables $\psi_s(\mathbf{x}, \mathbf{c}_s)$ are assumed to be smooth functions of their arguments. Adaptive control law $\mathbf{u}(\psi_s, \mathbf{x}, \mathbf{w})$ should ensure that a *global control goal* is achieved. The global control goal is to move the state vector of system (1) from its initial value $\mathbf{x}_0$ to the given *goal manifold* $\psi_s(\mathbf{x}, \mathbf{c}_s) = \mathbf{0}$ while assuring stability of movement to the chosen manifold. We require global asymptotic stability of the movement, i.e., $\psi(\mathbf{x}(t), \mathbf{c}) \to \mathbf{0}$ for $t \to \infty$. The state vector of the system (1) moves along intersections of *local manifolds* $\psi_s(\mathbf{x}, \mathbf{c}_s) = \mathbf{0}$ to the goal manifold $\psi^*(\mathbf{x}, \mathbf{c}^*) = \mathbf{0}$ during controller tuning. Its movement should minimize an *associated performance functional* $J(\psi_s^{(i)}(\mathbf{x}, \mathbf{c}^*))$ on trajectories of (1), where $\psi_s^{(i)}(\mathbf{x}, \mathbf{c}^*)$ is the i-th derivative of the macrovariable. We note that the state vector trajectory characterizes the quality of adaptive system functioning and reflects *local control goals*. For local control goals, we also require asymptotic stability of the movement, i.e., $Q(\psi_s(\mathbf{x}(t), \mathbf{c}_s)) \to 0$ for $t \to 0$, where $Q(\cdot) > 0$, and $Q(\cdot) = 0 \iff \psi_s(\mathbf{x}, \mathbf{c}_s = \mathbf{0})$. Issues of choosing manifolds ψ_s and their parameters are not considered here, so we omit $\mathbf{c}_s$ from the notation:

$$Q(\psi_s(\mathbf{x}(t))) \to 0, \quad t \to 0. \tag{3}$$

Thus, the associated functional $J(\psi_s^{(i)}(\mathbf{x}, \mathbf{c}_s))$ below must be minimized during movement of the state vector $\mathbf{x}$ of (1) from its initial condition $\mathbf{x}_0$ to the target manifold $\psi_s(\cdot) = 0$:

$$J = \int_{t_0}^{\infty} F(\psi_s^{(n)}, \ldots, \psi_s^{(1)}, \psi_s, T) dt, \tag{4}$$

where $T = const$ is a vector of parameters of the functional; $F(\psi_s^{(n)}, \ldots, \psi_s^{(1)}, \psi_s, T) \geq 0$ is a smooth function, and $F(\cdot)$ is equal to zero only at $\psi_s = \psi_s^{(1)} = \ldots = \psi_s^{(n)} = 0$. We assume that the minimum of the functional (4) is achieved on its stable extremal

$$g(\psi_s^{(n)}, \ldots, \psi_s^{(1)}, \psi_s, T) = 0. \tag{5}$$

Adaptive control law satisfying (3) and (5) is a function of state space vector $\mathbf{x}$ and controller parameters vector $\mathbf{w} \in \mathbf{R}^K$. It can be written as

$$\dot{w}_i = -\gamma A_i(\mathbf{w}, \mathbf{x}, \psi_s(\mathbf{x})); \quad \mathbf{u} = \mathbf{q}(\mathbf{w}, \mathbf{x}), \tag{6}$$

where γ is a positive gain, and $A(\cdot)$ is unknown operator to be defined. We assume existence of the vector $\mathbf{w}^*$ such that for all solutions $\mathbf{x}(t)$ of the system (1) the vector - function $\mathbf{u}^* = \mathbf{q}(\mathbf{w}^*, \mathbf{x})$ ensures that we reach the goal (3) and the minimum of (4) (symbol $\mathbf{u}^*$ further denotes the optimal control). The following relevant result about approximation capabilities of RBF networks is due to Park and Sandberg (see [5], p.291).

Theorem 1 (Park and Sandberg) *For any continuous input-output mapping function $\mathbf{u}^*(\mathbf{x})$ there is an RBF network with a set of centers $\{\mathbf{a}_i\}_{i=1}^K$ and a common width $\sigma > 0$ such that input-output mapping function $\mathbf{q}(\mathbf{w}, \mathbf{x})$ realized by the RBF network is close to $\mathbf{u}^*(\mathbf{x})$ in the L_p norm, $p \in [1, \infty]$.*

The theorem above points out that only an approximation is possible. For the moment, we assume that the control $\mathbf{u}^*$ can be represented exactly (without approximation errors), which in practice amounts to using unrealistically large networks. In Section 2.2 we will relax this requirement and discuss how to ensure that our adaptive algorithms are applicable in spite of various error sources.

2.2 Design of adaptive algorithms for quadratic associated functionals

Adaptive algorithms for quadratic associated functionals are of great interest in practice. This is due to the fact that most physical processes are described in terms of state variables and their first derivatives. Let the associated functional (4) be given by equation:

$$J = \int_{t_0}^{\infty} (T^2 \dot{\psi}^2 + \phi^2(\psi))dt, \qquad (7)$$

where $T > 0$ is a quality parameter and function $\phi(\psi)$ is chosen from the following conditions:

$$\phi(\psi) = 0 \Longleftrightarrow \psi = 0;\ \phi(\psi) \in C^1 \text{ and } \phi(\psi)\psi > 0. \quad (8)$$

A stable extremal (5) of functional (7), together with (8), is $g(\dot{\psi}, \psi) = T\dot{\psi} + \phi(\psi) = 0$. Now we can write the adaptive algorithm for $\mathbf{w}$:

$$\dot{\mathbf{w}} = -\Gamma(t) T \frac{\partial \tilde{Q}(g)}{\partial g} \frac{\partial \dot{\psi}}{\partial \mathbf{w}}, \qquad (9)$$

where gain $\Gamma(t) > 0$, and $\tilde{Q}(g)$ is a tuning criterion: $\tilde{Q}(g) \in C^1$, $\tilde{Q}(g) > 0$, $\tilde{Q}(g) = 0 \Longleftrightarrow g = 0$. Thus, the adaptive control system can be written as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \theta) + \mathbf{b}(\mathbf{q}(\mathbf{w}, \mathbf{x}), \mathbf{x}); \\ \dot{\mathbf{w}} &= -\Gamma(t) T \tfrac{\partial \tilde{Q}(g)}{\partial g} \tfrac{\partial \dot{\psi}}{\partial \mathbf{w}}. \end{aligned} \qquad (10)$$

It is desirable to analyze properties for a family of adaptive algorithms (9) and specify their applicability conditions. We assume that our goal functional Q is such that

$$Q(\psi) > 0,\ Q(\psi) \in C^1,\ Q(\psi) \to \infty \text{ at } \psi \to \infty. \quad (11)$$

We specify the following conditions:

Condition C. 1 (Attainability condition) *Functions $\phi(\psi)$ are chosen such that the conditions (8) are satisfied, and there is a vector $\mathbf{w}^* \in \Omega_{\mathbf{w}}$ such that for all trajectories $\mathbf{x}(t)$ of system (1) the equality $\dot{\psi}(\mathbf{x}, \mathbf{w}^*) = -\phi(\psi)T^{-1}$ holds.*

Attainability condition (C.1) is a form of the certainty equivalence condition (9). It stipulates existence of a controller that minimizes the quality functional (7).

Condition C. 2 (Convexity condition) *For all vectors $\mathbf{w} \in \Omega_{\mathbf{w}}$ and for all trajectories $\mathbf{x}(t)$ the following inequality holds:*

$$\frac{\partial \tilde{Q}}{\partial g} \frac{\partial \dot{\psi}}{\partial \mathbf{w}} (\mathbf{w}^* - \mathbf{w}) \leq \frac{\partial \tilde{Q}}{\partial g} (\dot{\psi}(\mathbf{w}^* - \mathbf{x}) - \dot{\psi}(\mathbf{w} - \mathbf{x})).$$

Condition C. 3 (Non-stationary convexity cond.) *There exist $\Gamma(t) > 0$ and $\gamma(t) > 0$ such that for any vectors $\mathbf{w} \in \Omega_{\mathbf{w}}$, $\mathbf{x}$ the following inequality holds:*

$$\frac{\partial \tilde{Q}}{\partial g} \frac{\partial \dot{\psi}}{\partial \mathbf{w}} \Gamma(t)(\mathbf{w}^* - \mathbf{w}) \leq \gamma(t) \frac{\partial \tilde{Q}}{\partial g} (\dot{\psi}(\mathbf{w}^* - \mathbf{x}) - \dot{\psi}(\mathbf{w} - \mathbf{x})).$$

Convexity condition (C.2) is a restriction on the right-hand side of the equation (1). It requires that the right-hand side is linear with respect to the vector of controller parameters $\mathbf{w}$. For instance, this condition holds for plants affine in control, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \theta) + \mathbf{b}(\mathbf{x})\mathbf{q}(\mathbf{w}, \mathbf{x})$, and it naturally holds for the RBF network. Conditions C.1 and C.2 for the algorithm (9) are similar to applicability conditions for the speed-gradient algorithms [4]. These algorithms can be obtained asymptotically from the algorithms (9) for $T \to 0$ [3].

A theorem about fulfilling the control objective is formulated below:

Theorem 2 *Assume that conditions C.1, C.2 hold, and there exists functional (11) such that equality $\partial Q/\partial \psi = \phi(\psi)T$ holds. Then for any $\Gamma > 0$ the control goal is reached for the system (10) if*

$$a)\left(\tfrac{\partial \tilde{Q}}{\partial g} - \phi(\psi)\right)\dot{\psi} > 0,\ \tfrac{\partial \tilde{Q}}{\partial g} - \phi(\psi) = 0 \Longleftrightarrow \dot{\psi} = 0;$$
$$b)\left(\tfrac{\partial \tilde{Q}}{\partial g} - T\dot{\psi}\right)\phi(\psi) > 0,\ \tfrac{\partial \tilde{Q}}{\partial g} - T\dot{(\psi)} = 0 \Longleftrightarrow \phi(\psi) = 0.$$

The proof of this theorem (and others in this paper) is analogous to proofs of theorems for speed-gradient adaptation algorithms [4], and it is provided in the appendix. We note that Theorem 2 does not require convergence to optimal parameters $\mathbf{w}^*$ in controller parameters space $\Omega_{\mathbf{w}}$. Furthermore, restrictions a) and b) on the class of tuning functionals $\tilde{Q}$ can be removed. This follows from the theorem below:

Theorem 3 *Assume that conditions C.1, C.3 hold, $g \cdot \partial \tilde{Q}/\partial g > 0$, and $\partial \tilde{Q}/\partial g$ is equal to zero only at $g = 0$. Furthermore, let the functions $\partial \tilde{Q}/\partial g$, g, $\gamma(t)$ are uniformly continuous for $t > 0$. Then the control goal (11) is reached in the system (10) for any $\Gamma(t) > 0$ satisfying C.3.*

As mentioned above, the algorithm (9) does not require that $\mathbf{w} \to \mathbf{w}^*$ for $t \to \infty$, i.e., the control goal (3) or (10) can be achieved even for $\mathbf{w} \neq \mathbf{w}^*$. However, sometimes the algorithms (9) have identifying properties.

Definition 1 *Assume that there is a unique vector $\mathbf{w}^* \in \Omega_{\mathbf{w}}$ for which the control goal 11 is reached for any $\mathbf{x}(t_0) \in \Omega_{\mathbf{x}}$. An adaptive algorithm is called identifying if $\mathbf{w}(t) \to \mathbf{w}^*$ for $t \to \infty$.*

To prove convergence of $\mathbf{w}$, we need to make additional assumptions about the adaptive system. Let us introduce the following system:

$$\begin{array}{l}\dot{\psi} = f_0(\psi(\mathbf{x}), \mathbf{r}(t)) + (\mathbf{w} - \mathbf{w}^*)^T \mathbf{f}_1(\psi(\mathbf{x}), \mathbf{r}(t)); \\ \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \theta) + \mathbf{b}(\mathbf{u}, \mathbf{x}),\end{array} \quad (12)$$

where $\mathbf{r}(t)$ is a piece-wise smooth function. We use the following definition from [4] about strong limiting non-degeneracy of a function in the theorem below:

Definition 2 *Function $\mathbf{f}(t) : \mathbf{R} \rightarrow \mathbf{R}^n$ measured and limited for $t \geq 0$ is a strong limiting non-degenerate (persistently exciting) function if there are such $L > 0$, $\alpha > 0$, t_0 that for any $t > t_0$ the inequality holds $\int_t^{t+L} \mathbf{f}(s)\mathbf{f}^T(s)ds \geq \alpha I_n$ (I_n is identity matrix).*

Theorem 4 *Let conditions of Theorem 2 or Theorem 3 hold, and the adaptive control system is given in the form (12), and functions $\mathbf{f}$, $\mathbf{f}_1$, $f_0 \in C^1$. If $f_0(0, \mathbf{r}(t)) = 0$ at $\psi = 0$ and the function $\mathbf{f}_1(0, \mathbf{r}(t))$ is a strong limiting non-degenerate function, then the algorithm (9) is identifying for all initial conditions.*

Theorem 4 assumes that the unique vector $\mathbf{w}^*$ of ideal parameters of the adaptive controller exists (condition C.1). It is required that the conditions of attainability C.1 and convexity C.2 hold in order to achieve the goal (3) with the algorithm (9). However, in practice it is not always possible to design a neurocontroller satisfying C.1 and C.2. For instance, there may exist natural constraints on $\mathbf{u}^*$ which can not permit implementation of the ideal control required to move the state vector $\mathbf{x}$ along the extremals of the chosen associated functional with no errors. In addition, an approximation error is always present since the RBF network only approximates the unknown ideal control. Furthermore, the derivative $\dot{\psi}$ often can only be approximated. Due to the abovementioned sources of errors, we recommend that the adaptation algorithms based on (9) be implemented with an insensitivity ("dead") zone. Let it be known that $| T\dot{\psi}(\mathbf{w}^*, \mathbf{x}) + \phi(\psi) | < \delta^*$. One possible modification of the algorithm (9) is

$$\dot{\mathbf{w}} = \begin{cases} -\Gamma T \frac{\partial \tilde{Q}}{\partial g} \frac{\partial \dot{\psi}}{\partial \mathbf{w}}, & \text{if } |g| > \delta \\ 0, & \text{if } |g| \geq \delta \end{cases} \quad (13)$$

where the parameter δ is such that $\delta \geq \delta^*$. The value of δ^* can be chosen during the design stage from some prior knowledge about properties of the closed-loop system or based on preliminary experimental studies. We also need to change the condition C.1:

Condition C. 4 *The functions $\phi(\psi)$ are unbounded, monotone and satisfy the conditions (8), and there exists a vector $\mathbf{w}^* \in \Omega_{\mathbf{w}}$ such that for any solution $\mathbf{x}(t)$ of the system (12) the following inequality holds:*

$$|T\dot{\psi}(\mathbf{x}, \mathbf{w}^*) + \phi(\psi)| \leq \delta^*, \ \delta^* > 0.$$

Instead of (3), we can then write the goal as

$$|\psi| \leq |\phi^{-1}(\delta)|. \quad (14)$$

The following theorem establishes applicability conditions of the algorithm (13):

Theorem 5 *Let the conditions C.2 and C.4 hold, and the adaptation algorithm is defined as in (13); derivatives $\dot{\mathbf{w}}$ and $\dot{\psi}$ are bounded. Then the system (12) attains the goal (14) for any $\Gamma > 0$, and there exists $\limsup_{t\rightarrow\infty} \mathbf{w}(t) < \infty$.*

Theorem 5 may be proved via analysis of a positive definite function $V(\mathbf{w}) = (\mathbf{w} - \mathbf{w}^*)^T \Gamma^{-1} (\mathbf{w} - \mathbf{w}^*)$.

It is important to keep in mind that any RBF network can not approximate unknown functions well enough outside a compact set. If the state vector $\mathbf{x}$ may (even occasionally) move from its domain $\Omega_{\mathbf{x}}$ where such an approximation is adequate to a region where it is not, then the control system designer must resort to countermeasures. The well-known technique is to employ an additional control component which takes over from the RBF network as its approximation ability begins to degrade, and which forces the state vector back into $\Omega_{\mathbf{x}}$ [7]. It is easy to avoid discontinuity when switching between two control schemes by allowing a simple modulation function to determine the contribution of each control component into the total control signal.

Concluding this section, we wish to summarize our approach to adaptive control on manifolds with RBF networks. The main idea of the approach is that adaptive control for a complex nonlinear plant (1) with (possibly) parametric uncertainties θ is carried out in the extended state space with the help of macrovariables. Applicability conditions of the approach are the attainability conditions (C.1), (C.4) and convexity conditions (C.2), (C.3). Convexity conditions hold almost always for plants affine in control. But conditions (C.1) or (C.4) should be checked. Clearly, the proposed approach can be used not only with the RBF neural networks but also with any controller whose parameters $\mathbf{w}$ enter its equations linearly.

3 Example

Let us consider application of our algorithms to spacecraft orbital movement stabilization. The mathematical model of spacecraft is given in [1]:

$$\begin{array}{l}\dot{x}_1 = k_1 x_4 x_5 + u_1; \\ \dot{x}_2 = -x_1 x_3 + x_5 \sqrt{1 - x_2^2 - x_3^2} \\ \dot{x}_3 = x_1 x_2 - x_4 \sqrt{1 - x_2^2 - x_3^2}; \\ \dot{x}_4 = k_2 x_1 x_5 + u_4; \\ \dot{x}_5 = k_3 x_1 x_4 + u_5.\end{array} \quad (15)$$

System (15) describes angular movement of the spacecraft around its center of mass. Parameters k_1, k_2, k_3 depend on main moments of inertia of the spacecraft (like θ in (1)); x_1, x_4, x_5 are projections of the angular velocity on their respective main inertial axes; x_2, x_3 are spacecraft deviations from the vertical axis; u_1, u_4, u_5 are control torques. It is necessary to design an adaptive controller that ensures spacecraft movement from the state in a ball $\Omega_{\mathbf{w}} = \{\mathbf{x}|x_2^2 + x_3^2 < 1\}$ to a desired state. It is important to note that parameters k_1, k_2, k_3 are not measured, and they can abruptly jump from one level to the other.

We first choose the global control goal:

$$x_1 = x_2 = x_3 = x_4 = x_5 = 0. \tag{16}$$

The global goal can be represented by an intersection of the following manifolds:

$$\psi_1 = x_1 = 0;\ \psi_2 = x_2 = 0;\ \psi_3 = x_3 = 0. \tag{17}$$

Indeed, if $x_1 \to 0$, $x_2 \to 0$ and $x_3 \to 0$, then $\dot{x}_2 \to 0$ and $\dot{x}_3 \to 0$ according to Barbalat's lemma. Therefore, using (15) and assuming $u_4 \to 0$, $u_5 \to 0$ we can write [1]

$$\begin{array}{l} \dot{x}_2 = -x_1 x_3 + x_5\sqrt{1 - x_2^2 - x_3^2} \to x_5 \to 0 \\ \dot{x}_3 = x_1 x_2 - x_4\sqrt{1 - x_2^2 - x_3^2} \to -x_4 \to 0 \end{array}$$

Hence, the state vector of (15) moving to the intersection of manifolds (17) guarantees that the global control goal is achieved. The RBF network should ensure our system's movement to (17) and stabilization of its trajectories along it. If the associated functionals are chosen as

$$J_j = \int_{t_0}^{\infty} (T^2\dot{\psi}_j^2 + \psi_j^2)dt, \tag{18}$$

then the extremal equations can be written in the form

$$T\dot{\psi}_j + \psi_j = 0. \tag{19}$$

It is clear that the extremals (19) for $j = 2, 3$ do not depend explicitly on control inputs u_4 and u_5 . Therefore it is necessary to choose controllable macrovariables, i.e., those macrovariables whose first derivatives depend on control explicitly. It was shown [1] that the zeroes of the following macrovariables

$$\begin{array}{l} \psi_1 = x_1; \\ \psi_2 = x_5 + T_2 x_2, \quad T_2 > 0; \\ \psi_3 = -x_4 + T_3 x_3, \quad T_3 > 0 \end{array}$$

are sufficient conditions for achieving the global control goal (16). If the tuning criterion is quadratic, $\tilde{Q}_j = 0.5\, g^2(\psi_j, \dot{\psi}_j)$, then, according to (9), the network training algorithm is

$$\dot{w}_{j,i} = -\gamma\,(\psi_j + T_j\dot{\psi}_j)\, q_i(\mathbf{x} - \mathbf{a}_i). \tag{20}$$

It is easy to see that both the attainability condition (C.1) and the convexity condition (C.2) hold (it is required to adjust linear parameters $\mathbf{w}$ only, as follows from Theorem 1). Hence, the algorithm (20) can be used due to Theorems 2 and 3. In practice the algorithm (20) must be modified to estimate derivatives and take into account approximation errors of the unknown control function $\mathbf{u}^*$. The modified algorithm is

$$\dot{w}_{j,i} = \begin{cases} -\gamma\,(\psi_j + T_j\dot{\psi}_j)\, q_i(\mathbf{x} - \mathbf{a}_i), & |\psi_j + T_j\dot{\psi}_j| \geq \delta; \\ 0, & |\psi_j + T_j\dot{\psi}_j| < \delta; \end{cases} \tag{21}$$

where $\delta \geq \delta^*$, δ^* is a generalized error of measurements and control realization. Applicability conditions of the algorithm (21) follow from Theorem 5. When it is unrealistic to estimate derivatives $\dot{\psi}_j$ accurately enough, we recommend using an asymptotic form of the algorithm (20) ($T \to 0$), which is equivalent to the speed-gradient algorithm: $\dot{w}_{j,i} = -\gamma\,\psi_j\, q_i(\mathbf{x} - \mathbf{a}_i)$.

4 Conclusion

We proposed a new method for adaptive control on manifolds for non-linear plants in the full-state feedback case using RBF networks as controllers. We introduced a family of adaptive algorithms for adjusting controller parameters based on associated performance criteria and discussed conditions such algorithms should satisfy to. Our discussion concentrated around adaptive algorithms for quadratic performance functionals.

One open problem remains in the area of choosing an appropriate structure of the manifolds as well as their parameters. This process should be as independent of the problem at hand as possible. Another area of future work is studies of adaptive control on manifolds in the partial-state feedback case.

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A Appendix

Theorem 2 proof. Let us define function $V(\psi, \mathbf{w})$:

$$V = 2Q + (\mathbf{w} - \mathbf{w}^*)^T \Gamma^{-1} (\mathbf{w} - \mathbf{w}^*). \quad \text{(A.1)}$$

If conditions (11) hold for the functional Q, then the function $V \to \infty$ if $\psi \to \infty$. The time-derivative of function (A.1) is:

$$\dot{V} = 2\,\partial Q/\partial\psi\,\dot{\psi} - (\mathbf{w} - \mathbf{w}^*)^T\,\partial\tilde{Q}/\partial g\,\partial\dot{\psi}/\partial\mathbf{w} = \\ = 2\,\partial Q/\partial\psi\,\dot{\psi} + T(\mathbf{w}^* - \mathbf{w})^T\,\partial\tilde{Q}/\partial g\,\partial\dot{\psi}/\partial\mathbf{w}. \quad \text{(A.2)}$$

According to condition C.2 in theorem, the expression (A.2) follows from the inequality:

$$\dot{V} \leq 2\,\partial Q/\partial\psi\,\dot{\psi} + T\partial\tilde{Q}/\partial g\,(\dot{\psi}(\mathbf{w}^*, \mathbf{x}) - \dot{\psi}(\mathbf{w}, \mathbf{x})). \quad \text{(A.3)}$$

Based on the attainability condition (C.1) and the equation $\partial Q/\partial\psi = \phi(\psi)\,T$, the inequality (A.3) can be written as

$$\dot{V} \leq 2\,\phi(\psi)\,\dot{\psi}T - T\partial\tilde{Q}/\partial g\,(\phi(\psi)T^{-1} + \dot{\psi}(\mathbf{w}, \mathbf{x})) = \\ = -(\partial\tilde{Q}/\partial g - T\dot{\psi})\,\phi(\psi) - (\partial\tilde{Q}/\partial g - \phi(\psi))\,T\dot{\psi}. \quad \text{(A.4)}$$

According to the theorem's conditions a) and b), the time-derivative $\dot{V}$ is negative definite. Hence, the function (A.1) is a Lyapunov function. Let us consider the increment $\Delta V = V(0) - V(t)$. From (A.4)

$$\Delta V = 2(Q(\psi(t_0)) - Q(\psi(t)) + (\mathbf{w}(\mathbf{t_0}) - \mathbf{w}^*)^T\,\Gamma^{-1} \\ (\mathbf{w}(\mathbf{t_0}) - \mathbf{w}^*) - (\mathbf{w}(\mathbf{t}) - \mathbf{w}^*)^T\,\Gamma^{-1}\,(\mathbf{w}(\mathbf{t}) - \mathbf{w}^*) \geq \\ \geq \int_{t_0}^{t} (\tfrac{\partial\tilde{Q}}{\partial g} - \phi(\psi))\,\dot{\psi} + (\tfrac{\partial\tilde{Q}}{\partial g} - T\dot{\psi})\,\phi(\psi)dt.$$

The increment ΔV is limited because $V(t)$ is a positive definite decreasing function. Therefore, the following inequality holds:

$$\int_{t_0}^{t} (\frac{\partial\tilde{Q}}{\partial g} - \phi(\psi))\,\dot{\psi} + (\frac{\partial\tilde{Q}}{\partial g} - T\dot{\psi})\,\phi(\psi)dt < \infty. \quad \text{(A.5)}$$

Based on a) and b), the expression under the integral of (A.5) is positive. Then $\dot{\psi} \to 0$ and $\phi(\psi) \to$ at $t \to \infty$. If the functions $\phi(\psi)$ satisfies (8) and become zero only at $\psi = 0$, then the control goal (11) is achieved in the system (10), i.e., $Q(\psi) \to 0$ for $\psi \to \infty$. *The theorem is proved.*

Theorem 3 proof. Let us consider a positive definite function $V(\mathbf{w})$:

$$V(\mathbf{w}) = (\mathbf{w} - \mathbf{w}^*)^T(\mathbf{w} - \mathbf{w}^*). \quad \text{(A.6)}$$

Then in (A.6) the time-derivative of $V(\mathbf{w})$ is

$$\dot{V} = T(\mathbf{w}^* - \mathbf{w})^T\Gamma(t)\partial\tilde{Q}/\partial g\,\partial\dot{\psi}/\partial\mathbf{w}. \quad \text{(A.7)}$$

According to the attainability and convexity conditions, the equation (A.7) results in

$$\dot{V} \leq -T\,\gamma(t)\frac{\partial\tilde{Q}}{\partial g}\,(\dot{\psi}(\mathbf{w}, \mathbf{x}) - \dot{\psi}(\mathbf{w}^*, \mathbf{x})) = \\ = -\gamma(t)\frac{\partial\tilde{Q}}{\partial g}\,(T\dot{\psi}(\mathbf{w}, \mathbf{x}) + \phi(\psi))$$

Since $T\dot{\psi}(\mathbf{w}, \mathbf{x}) + \phi(\psi) = g$ and $g\,\partial\tilde{Q}/\partial g > 0$, then the derivative $\dot{V}$ will satisfy inequality $\dot{V} < 0$. Let us consider $\Delta V(t) = V(0) - V(t)$. Due to (A.6), $V(t)$ is positive. Function $V(t)$ is non-increasing due to $\dot{V} < 0$. Hence, the increment $\Delta V(t) > 0$ is limited. Hence

$$\Delta V(t) = (\mathbf{w}(t_o) - \mathbf{w}^*)^T\,\Gamma^{-1}\,(\mathbf{w}(t_o) - \mathbf{w}^*) - \\ -(\mathbf{w}(t) - \mathbf{w}^*)^T(\mathbf{w}(t) - \mathbf{w}^*) \geq \int_{t_0}^{t} \gamma(t)\,g\,\partial\tilde{Q}/\partial g\,dt.$$

It is evident that the integral

$$\int_{t_0}^{\infty} \gamma(t)\,g\,\partial\tilde{Q}/\partial g\,dt < \infty \quad \text{(A.8)}$$

exists because $\Delta V(t)$ is limited. From the theorem formulation, $\gamma(t)\,g\,\partial\tilde{Q}/\partial g > 0$, and only $\partial\tilde{Q}/\partial g = 0$ if $g = 0$. Thus, $g \to 0$ for $t \to \infty$ follows from (A.8), i.e., the next equation holds for $t \to \infty$: $T\dot{\psi} + \phi(\psi) = 0$.

Let us define function $V_1(\psi) = 0.5\psi^2$. It can be shown that $\dot{V}_1$. Indeed, $\dot{V}_1 = -T^{-1}\psi\phi(\psi) < 0$. Hence, $\psi \to 0$ for $t \to \infty$. Moreover, $\dot{\psi} \to 0$ at $\psi \to 0$. Thus, the control goal is achieved in the system (10). But this is equivalent to the goal condition (3). *The theorem is proved.*

Theorem 4 proof. In order to prove this theorem, we use the next lemma.

Lemma 1 . *Assume that there exists a function $\mathbf{f}(\cdot) : [0, \infty] \to \mathbf{R}^n$, which is a strong limiting non-degenerate function, and piece-wise smooth function $\mathbf{r}(\cdot) : [0, \infty] \to \mathbf{R}^n$, such that $\dot{\mathbf{r}} \to 0$, $\mathbf{f}^T\mathbf{r} \to 0$ for $t \to \infty$. Then $\mathbf{r} \to 0$ for $t \to \infty$.*

This lemma is proved in [4]. Theorems 2 and 3 guarantee that $\psi \to 0$ for $t \to \infty$ for any $\Gamma > 0$. Then $f_0(0, \mathbf{r}(\mathbf{t})) \to 0$ for $\psi \to 0$, and the adaptive control system (12) can be written in form: $\dot{\psi} = (\mathbf{w} - \mathbf{w}^*)^T\mathbf{f}_1(\psi(\mathbf{x}), \mathbf{r}(t))$. As shown in proofs for Theorems 2 and 3, the derivative $\dot{\psi} \to 0$ for $t \to \infty$, and $\psi \to 0$. Hence, from the lemma above one can see that $\mathbf{w} \to \mathbf{w}^*$ for $t \to \infty$ if the function $\mathbf{f}_1(0, \mathbf{r}(t))$ is a strong limiting non-degenerate. *The theorem is proved.*