Parametrization of Admissible Controllers for Generalized Rosenbrock Systems[1]

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Abstract

In this work, the problem of parametrizing feedback systems with prescribed properties for general linear systems in Rosenbrock representation is considered. With this, the results of stabilizing controller parametrization for state space systems can be extended to general case where system regularity is not assumed. The stabilization theory is formulated axiomatically to permit its application to a wide variety of design problems and is extremely elementary in nature.

1. Introduction

Recently, more general linear systems than the usual state space systems have received great attention ([3], [6], [5], [1], and the references therein). In particular, some results related to coprime factorization and parametrization of all stabilizing controllers have been extended to descriptor systems ([6], [14]). As a direct extension of the result for state space systems in [9], explicit formulae for stable doubly coprime fractional representation of a transfer matrix of a descriptor system were presented in [14]. A proper stable coprime factorization and the stabilizing controller parametrization is considered in [6]. These results, however, are restricted to the class of regular systems. The large class of non-regular systems, i.e. systems without transfer matrix representation is not considered.

In this work we extend some results of controller parametrization to a system in a general system matrix representation. This representation includes systems which do not have a unique input-output representation. We consider that the desirable properties of the system can be characterized as a general notion of internal stability. Our axiomatic presentation of section 3, restricted to state-space systems, can be considered as an alternative presentation to [7], [16]. In these works, duality and special structures as Full Information and Output Estimation are used. In section 4, we consider a parametrization centred on any given stabilizing controller, extending, in this way, the result of [4] without coprime factorization arguments.

2. Systems with generalized Rosenbrock representation

Consider the generalized Rosenbrock system matrix

$$P : \mathbb{R}^k \to \mathbb{R}^{(m+r)\text{x}(n+s)}$$
$$p \mapsto P(p) = \left[\begin{array}{c|c} A(p) & B(p) \\ \hline C(p) & D(p) \end{array}\right].$$

The generalized system matrix represents a linear system of the form

$$\begin{aligned} 0 &= A(p)\xi(t) + B(p)u(t) \\ y(t) &= C(p)\xi(t) + D(p)u(t) \end{aligned}$$

where ξ is the internal variable, u is the input signal and y is the output signal. Some important systems which can be represented in this form are

i) time-varying state space systems : in this case $p = (s, t)$, $A(p) = A(t) - sI$, $B(p) = B(t)$, $C(p) = C(t)$ and $D(p) = D(t)$ where s represents the differential operator $(s\xi)(t) = \frac{d}{dt}\xi(t)$.

ii) descriptor systems : in this case $p = s$, $A(p) = A - sE$, $B(p) = B$, $C(p) = C$ and $D(p) = D$

iii) state-space systems with delay: in this case $p = (s, \theta)$ and $A(s,\theta) = A(\theta) - sI$, $B(s,\theta) = B(\theta)$, $C(s,\theta) = C$ and $D(s,\theta) = D$ where $A(\theta)$, $B(\theta) \in \mathbb{R}[\theta]$, with θ representing the delay operator $(\theta\xi)(t) = \xi(t-1)$.

iv) general n-D systems : $P(p) \in \mathbb{R}^{(m+r)\text{x}(n+s)}[p]$

v) uncertain systems : An uncertain linear system with upper LFT representation on block structure Δ [7], has a generalized Rosenbrock representation given by, e.g., $A(p) = (\Delta A - I)$, $B(p) = \Delta B$, $C(p) = C$ and $D(p) = D$.

vi) systems with fractional representation : The transfer matrix with fractional representation (not necessarily coprime) $G(s) = N_R(s)D_R^{-1}(s)$ has a generalized Rosenbrock representation with $p = t = s$ and $A(p) = D_R(s) \in RH_\infty$, $B(p) = -I$, $C(p) = N_R(s) \in RH_\infty$ and $D(p) = 0$.

The general Rosenbrock representation enables us to deal with a wide class of systems which have not input-output representation since the matrix $A(p)$ is not required to be regular ([3]). A non-regular $A(p)$ matrix is very common in system modeling and design problems, as for example in descriptor setting ([1], [5]).

A system P with input and output signals partitioned as $u = [u_1^T \;\; u_2^T]^T$, $y = [y_1^T \;\; y_2^T]^T$ is called a partitioned system. Partitioned systems enable to represent general combinations of series, parallel and feedback connections of "common systems" (without input and output signals partitioning) in one closed loop system framework, $\mathfrak{F}(P,K)$, defined below. Thus, the feedback connection $\mathfrak{F}(*,*)$ furnishes a generic representation of

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systems since for any common system K we can find systems Z and Q such that $K = \mathfrak{F}(Z,Q)$. In the same way, any partitioned system $\mathcal{T}$ can be represented as a star feedback connection $\mathfrak{F}_R(*,*)$ between two partitioned systems P and K such that $\mathcal{T} = \mathfrak{F}_R(P,K)$. In this setting, the controller synthesis problem can be stated as to solve the equation $\mathcal{T} = \mathfrak{F}(P,K)$, where P is the given partitioned plant, $\mathcal{T}$ is the closed loop system with the desired properties and the controller K is to be computed. For the generalized plant P with input $u = [u_1^T \ \ u_2^T]^T$ and output $y = [y_1^T \ \ y_2^T]^T$, and the controller K with input u_k and output y_k, given by

$$P = \left[\begin{array}{c|cc} A(p) & B_1(p) & B_2(p) \\ \hline C_1(p) & D_{11}(p) & D_{12}(p) \\ C_2(p) & D_{21}(p) & D_{22}(p) \end{array}\right] \text{and } K(p) = \left[\begin{array}{c|c} A_k(p) & B_k(p) \\ \hline C_k(p) & D_k(p) \end{array}\right],$$

the closed loop system matrix $\mathfrak{F}(P,K)$ obtained with $u_k = y_2$ and $y_k = u_2$ is well defined if $X = \begin{bmatrix} I & -D_k(p) \\ -D_{22}(p) & I \end{bmatrix}^{-1}$ is well defined and is given by

$$\mathfrak{F}(P,K) = \left[\begin{array}{c|c} \bar{A}(p) & \bar{B}(p) \\ \hline \bar{C}(p) & \bar{D}(p) \end{array}\right]$$

where

$$\bar{A}(p) = \begin{bmatrix} A & 0 \\ 0 & A_k \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_k \end{bmatrix} X \begin{bmatrix} 0 & C_k \\ C_2 & 0 \end{bmatrix} = \begin{bmatrix} A + B_2 X_{12} C_2 & B_2 X_{11} C_k \\ B_k X_{22} C_2 & A_k + B_k X_{21} C_k \end{bmatrix}$$

$$\bar{B}(p) = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_k \end{bmatrix} X \begin{bmatrix} 0 \\ D_{21} \end{bmatrix} = \begin{bmatrix} B_1 + B_2 X_{12} D_{21} \\ B_k X_{22} D_{21} \end{bmatrix}$$

$$\bar{C}(p) = [C_1 \ 0] + [D_{12} \ 0] X \begin{bmatrix} 0 & C_k \\ C_2 & 0 \end{bmatrix} = [C_1 + D_{12} X_{12} C_2 \ \ D_{12} X_{11} C_k]$$

$$\bar{D} = D_{11} + [D_{12} \ 0] X \begin{bmatrix} 0 \\ D_{21} \end{bmatrix} = D_{11} + D_{12} X_{12} D_{21}$$

The consistence condition, that is, the existence of X, is immediately fulfilled if $D_{22}(p) = 0$. If $K = 0$, that is, A_k, B_k and C_k are dropped and $D_k = 0$, we have

$$\mathfrak{F}(P,K) = \left[\begin{array}{c|c} A(p) & B_1(p) \\ \hline C_1(p) & D_{11}(p) \end{array}\right] = P_{11}.$$

For K considered as a parameter, we say that the parametrization $\mathfrak{F}(P,K)$ is centred on P_{11}. For partitioned systems P and K the star connection system matrix, $\mathfrak{F}_R(P,K)$, is defined similarly, and has the same expression as presented in [16]. For any systems of compatible dimensions G, Z and Q with $D_{22_g} = 0$ and $D_{22_z} = 0$ we have :

$$\mathfrak{F}(G,\, Z,\, Q) := \mathfrak{F}(G, \mathfrak{F}(Z,\, Q)) = \mathfrak{F}(\mathfrak{F}_R(G,\, Z),\, Q).$$

The feedback (star) connection reduces to a LFT (Redheffer star product) ([7], [16]), if the systems have transfer matrix representation. Conditions for an algebraic-integro-differential system to have a transfer matrix representation can be found in [3]. The feedback and star connections constitute the so called "state-space machinery" and motivated the LFT representation for uncertain systems in [7].

Many important internal properties in polynomial linear systems, such as absence of impulses, regularity and stability can be characterized by the system A-matrix. All these properties can be treated in an axiomatic way as a general "stability" notion if we consider that the A-matrix belongs to a class $\mathcal{S}$ satisfying some mild conditions. We will consider the notion of stability given by a non-empty class $\mathcal{S}$ of $A(p)$ matrices satisfying

(S1) for any nonsingular real matrices T and R we have $A(p) \in \mathcal{S} \Leftrightarrow TA(p)R \in \mathcal{S}$
(S2) for $A(p)$ in block triangular form

$$A(p) = \begin{bmatrix} A_{11}(p) & A_{12}(p) \\ 0 & A_{22}(p) \end{bmatrix} \text{ or } A(p) = \begin{bmatrix} A_{11}(p) & 0 \\ A_{21}(p) & A_{22}(p) \end{bmatrix}$$

we have $A(p) \in \mathcal{S} \Leftrightarrow A_{11}(p) \in \mathcal{S}$ and $A_{22}(p) \in \mathcal{S}$.
We say that $A(p)$ is $\mathcal{S}$-stable if $A(p) \in \mathcal{S}$.

Some examples of properties which can be represented by the class $\mathcal{S}$ include

i) stable state space systems ($A(s) = A - sI$ with A stable)

ii) regular descriptor systems ($A(s) = A - sE$ is a square pencil with full normal rank)

iii) regular stable descriptor systems

iv) regular n-D polynomial systems ($A(p) \in \mathbb{R}^{qxq}[p]$ with $det(A(p)) \neq 0$)

v) μ stability and $\mathcal{Q}$-stability for uncertain system [7]

vi) internal stability for fractional representation: $D_R(s)$, $D_R^{-1}(s) \in RH_\infty$

With this notion of stability the problem of finding a stabilizing control in general Rosenbrock system representation can be stated as the follows. Consider two generalized Rosenbrock systems $P(p)$ and $K(p)$ with inputs u, u_k and outputs y, y_k respectively

$$P(p) = \left[\begin{array}{c|c} A(p) & B(p) \\ \hline C(p) & 0 \end{array}\right],\ K(p) = \left[\begin{array}{c|c} A_k(p) & B_k(p) \\ \hline C_k(p) & 0 \end{array}\right].$$

The closed loop system obtained with $u = y_k$ and $y = u_k$ has the A-matrix given by

$$\bar{A}(p) = \begin{bmatrix} A(p) & B(p)C_k(p) \\ B_k(p)C(p) & A_k(p) \end{bmatrix}.$$

The stabilizing problem is to find matrices $A_k(p)$, $B_k(p)$ and $C_k(p)$ such that $\bar{A}(p) \in \mathcal{S}$. A simple way to find one such controller is as follows:

1) with $T = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}$, $R = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$ we obtain

$$T\bar{A}(p)R = \begin{bmatrix} A(p) + B(p)C_k(p) & B(p)C_k(p) \\ B_k(p)C(p) - A(p) + A_k(p) - B(p)C_k(p) & A_k(p) - B(p)C_k(p) \end{bmatrix}$$

2) Choose $A_k(p)$ such that the $(2,1)$ block is zero. In this case, we have

$$A_k(p) = A(p) - B_k(p)C(p) + B(p)C_k(p)$$

and

$$T\bar{A}(p)R = \begin{bmatrix} A(p) + B(p)C_k(p) & B(p)C_k(p) \\ 0 & A(p) - B_k(p)C(p) \end{bmatrix}.$$

Using the property (S2) of the class $\mathcal{S}$, the stabilization problem reduces to the determination of $B_k(p)$ and $C_k(p)$ such that $A(p) + B(p)C_k(p) \in \mathcal{S}$ and $A(p) - B_k(p)C(p) \in \mathcal{S}$.

Definition : The pair $(A(p), B(p))$ is $\mathcal{S}$-stabilizable if there exists $F(p)$ such that $A(p) + B(p)F(p) \in \mathcal{S}$. The pair

$(C(p), A(p))$ is $\mathcal{S}$-detectable if there exists $L(p)$ such that $A(s) + L(s)C(s) \in \mathcal{S}$.

Thus, the existence of a $\mathcal{S}$-stabilizing controller is replaced by the characterization of $\mathcal{S}$-stabilizability and $\mathcal{S}$-detectability. For uncertain systems, the $\mathcal{S}$-stabilizability can be the $\mathcal{Q}$-stabilizability ([7]); for descriptor systems, the $\mathcal{S}$-stabilizability can be the existence of a constant gain such that the closed loop system is regular, stable and impulse-free ([12]). In what follows we assume the $\mathcal{S}$-stabilizability and $\mathcal{S}$-detectability of the plant and search for the characterization of all $\mathcal{S}$-stabilizing controllers. Although it is not required an external or transfer matrix representation of the system, some classes of system equivalence involving the internal and a possible input/output representation are considered. The external equivalence below is weaker than the I/O equivalence of [3] once it is not required the uniqueness of outputs.

Definition : Two generalized Rosenbrock systems $P(p) = \left[\begin{array}{c|c} A(p) & B(p) \\ \hline C(p) & D(p) \end{array}\right]$ and $P^{eq}(p) = \left[\begin{array}{c|c} A^{eq}(p) & B^{eq}(p) \\ \hline C^{eq}(p) & D^{eq}(p) \end{array}\right]$ are said to be externally *equivalent systems (EE)* and we write $P \overset{e.e}{=} P^{eq}$ if the sequence of (E1)-(E3) relations below hold

(E1) $\left[\begin{array}{c|c} A^{eq}(p) & B^{eq}(p) \\ \hline C^{eq}(p) & D^{eq}(p) \end{array}\right] = \left[\begin{array}{c|c} TA(p)R & TB(p) \\ \hline C(p)R & D(p) \end{array}\right]$

for some two nonsingular real matrices T, R, or

(E2) $\left[\begin{array}{c|c} A^{eq}(p) & B^{eq}(p) \\ \hline C^{eq}(p) & D^{eq}(p) \end{array}\right] = \left[\begin{array}{cc|c} A_1(p) & A_2(p) & B_1(p) \\ 0 & A(p) & B(p) \\ \hline 0 & C(p) & D(p) \end{array}\right]$

for some $A_1(p)$, $A_2(p)$, and $B_1(p)$, or

(E3) $\left[\begin{array}{c|c} A^{eq}(p) & B^{eq}(p) \\ \hline C^{eq}(p) & D^{eq}(p) \end{array}\right] = \left[\begin{array}{cc|c} A_1(p) & 0 & 0 \\ A_2(p) & A(p) & B(p) \\ \hline C_1(p) & C(p) & D(p) \end{array}\right]$

for some $A_1(p)$, $A_2(p)$, and $C_1(p)$.

The generalized Rosenbrock system matrices $P(p)$ and $P^{eq}(p)$ are externally *equivalent with internal equivalence systems (EEIE)* and we write $P \overset{i.e}{=} P^{eq}$ if they are externally equivalent with (E2) and (E3) restricted to $A_1(p) \in \mathcal{S}$. For *EEIE* systems we have $A(p) \in \mathcal{S} \Leftrightarrow A^{eq}(p) \in \mathcal{S}$. We say that $P(p)$ is *internally $\mathcal{S}$-stable* if its A-matrix is $\mathcal{S}$-stable, i.e., $A(p) \in \mathcal{S}$. A controller K is a *$\mathcal{S}$-stabilizing controller* or an *admissible controller* to the generalized plant G and we write $K \in \mathcal{A}(G)$ if the A-matrix of $\mathfrak{F}(G,K)$ is $\mathcal{S}$-stable.

3. Parametrization of $\mathcal{S}$-stabilizing controllers

Assume that the plant is characterized by the following system :

$$G(p) = \left[\begin{array}{c|cc} A(p) & B_1(p) & B_2(p) \\ \hline C_1(p) & D_{11}(p) & D_{12}(p) \\ C_2(p) & D_{21}(p) & 0 \end{array}\right]. \qquad (3.1)$$

From the definitions it can be easily verified that if $G^{eq} \overset{i.e}{=} G$ and $K^{eq} \overset{i.e}{=} K$ then $\mathfrak{F}(G^{eq},K) \overset{i.e}{=} \mathfrak{F}(G,K)$ and $\mathfrak{F}(G,K^{eq}) \overset{i.e}{=} \mathfrak{F}(G,K)$. Thus, the following lemma is immediate.

Lemma 3.1: Consider the generalized plant G with $D_{22g} = 0$ and the controller K. Let Z and Q be systems such that $K \overset{i.e}{=} \mathfrak{F}(Z,Q)$. Then for any given realization $\mathfrak{F}_R^{given}(G,Z)$ satisfying $\mathfrak{F}_R^{given}(G,Z) \overset{i.e}{=} \mathfrak{F}_R(G,Z)$ we have $K \in \mathcal{A}(G) \Leftrightarrow Q \in \mathcal{A}(\mathfrak{F}_R^{given}(G,Z))$.

Proof : By definition, $K \in \mathcal{A}(G) \Leftrightarrow$ the A-matrix of $\mathfrak{F}(G,K)$ is $\mathcal{S}$-stable $\Leftrightarrow$ the A-matrix of $\mathfrak{F}(G,\mathfrak{F}(Z,Q))$ is $\mathcal{S}$-stable $\Leftrightarrow$ the A-matrix of $\mathfrak{F}(\mathfrak{F}_R(G,\ Z),\ Q)$ is $\mathcal{S}$-stable $\Leftrightarrow$ the A-matrix of $\mathfrak{F}(\mathfrak{F}_R^{given}(G,Z),\ Q)$ is $\mathcal{S}$-stable $\Leftrightarrow$ $Q \in \mathcal{A}(\mathfrak{F}_R^{given}(G,Z))$. ☐

From Lemma 3.1, the task of determining the set of admissible controllers, $\mathcal{A}(G)$, can be replaced by the problem of stabilizing $\mathcal{T} := \mathfrak{F}_R^{given}(G,Z)$, which has trivial solution for $\mathcal{T} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ since, in this case, $\mathfrak{F}(\mathcal{T},Q) = Q$ for any Q. The problem of finding Z such that $\mathfrak{F}_R(G,Z) \overset{i.e}{=} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ corresponds to finding a 'LFT inverse' Z for the generalized system G and is considered in the next lemma, in which we also characterize $\mathcal{A}(G)$ for this case. The expression of Z and the stability conditions used in this lemma can be derived as indicated in Remark 3.2.

Lemma 3.2: Consider G given by (3.1) with D_{12} and D_{21} nonsingular and such that $A - B_1D_{21}^{-1}C_2$ and $A - B_2D_{12}^{-1}C_1$ are $\mathcal{S}$-stable. Then there exists a generalized Rosenbrock system matrix Z such that

$$\mathfrak{F}_R(G,\ Z) \overset{i.e}{=} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \text{ and } \mathfrak{F}_R(Z,\ G) \overset{i.e}{=} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

For this Z we have

(i) $Q \overset{i.e}{=} \mathfrak{F}(G,K) \ \Leftrightarrow\ K \overset{i.e}{=} \mathfrak{F}(Z,Q)$
(ii) $\mathcal{A}(G) = \{K \overset{i.e}{=} \mathfrak{F}(Z,Q) : Q$ is internally stable$\}$.

One such system is given by

$$Z := \left[\begin{array}{c|cc} A - B_1D_{21}^{-1}C_2 - B_2D_{12}^{-1}(C_1 - D_{11}D_{21}^{-1}C_2) & (B_1 - B_2D_{12}^{-1}D_{11})D_{21}^{-1} & B_2D_{12}^{-1} \\ \hline -D_{12}^{-1}(C_1 - D_{11}D_{21}^{-1}C_2) & -D_{12}^{-1}D_{11}D_{21}^{-1} & D_{12}^{-1} \\ -D_{21}^{-1}C_2 & D_{21}^{-1} & 0 \end{array}\right].$$

In the next we will consider the construction of the controller parametrization. Having in mind the general feedback characterization of the set of all $\mathcal{S}$-stabilizing controllers of the above lemma, it is reasonable to suppose that in the general case, where the plant (3.1) is considered, we can consider controllers of the form $K = \mathfrak{F}(Z,Q)$, where

$$Z = \left[\begin{array}{c|cc} A_z & B_{1z} & B_{2z} \\ \hline C_{1z} & D_{11z} & D_{12z} \\ C_{2z} & D_{21z} & 0 \end{array}\right] \text{ and } Q = \left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array}\right]. \qquad (3.2)$$

Let us search a system $\mathfrak{F}^{eq}(G,Z,Q) \overset{i.e}{=} \mathfrak{F}(G,Z,Q)$ such that the $\mathcal{S}$-stability of the A-matrix can be easily characterized. Connecting the systems G and Z and using $T = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}$ and $R = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$, we have

$$\mathfrak{F}_R(G,Z) \overset{i.e}{=} \left[\begin{array}{cc|cc} A + B_2(D_{11z}C_2 + C_{1z}) & B_2C_{1z} & B_1 + B_2D_{11z}D_{21} & B_2D_{12z} \\ 0 & A + (B_2D_{11z} - B_{1z})C_2 & -B_1 - (B_2D_{11z} - B_{1z})D_{21} & B_{2z} - B_2D_{12z} \\ \hline C_1 + D_{12}(C_{1z} + D_{11z}C_2) & D_{12}C_{1z} & D_{11} + D_{12}D_{11z}D_{21} & D_{12}D_{12z} \\ D_{21z}C_2 + C_{2z} & C_{2z} & D_{21z}D_{21} & 0 \end{array}\right] \qquad (3.3)$$

where we set $A_z := A + B_2D_{11z}C_2 + B_2C_{1z} - B_{1z}C_2$ to make the (2,1)-block zero. Thus, for the central controller (when $Q = 0$), the dynamic of the closed loop system can be easily characterized by the (1,1) and (2,2) matrices. Denoting

$$F := C_{1z} + D_{11z}C_2, L := B_2D_{11z} - B_{1z} \qquad (3.4)$$

and considering that (A, B_2) is $\mathcal{S}$-stabilizable and (C_2, A) is $\mathcal{S}$-detectable, we can determine F and L such that $A + B_2F$ and $A + LC_2$ are $\mathcal{S}$-stable. In this case, the central controller is $\mathcal{S}$-stabilizing. Now, in order to simplify the expression of the A-matrix of $\mathfrak{F}^{eq}(G,Z,Q)$ (see the presence of convenient zero blocks in (3.6)), we impose the restrictions

$$C_{2z} := -D_{21z}C_2 \text{ , } B_{2z} := B_2D_{12z} \qquad (3.5)$$

and we have Z expressed by (3.12.2),

$$\mathfrak{F}_R(G,Z) \overset{i.e}{=} \mathfrak{F}_R^{eq}(G,Z) := (3.13.2)$$

and

$$\mathfrak{F}(G,Z,Q) \overset{i.e}{=} \mathfrak{F}^{eq}(G,Z,Q) := \mathfrak{F}(\mathfrak{F}_R^{eq}(G,Z),Q)$$
$$= \left[\begin{array}{ccc|c} A + B_2F & B_2[F - (D_{11z} + D_{12z}D_QD_{21z})C_2] & B_2D_{12z}C_Q & * \\ 0 & A + LC_2 & 0 & * \\ 0 & -B_QD_{21z}C_2 & A_Q & * \\ \hline * & * & * & * \end{array}\right] \qquad (3.6)$$

where * are unimportant terms. Let $\bar{A}$ be the A-matrix of $\mathfrak{F}(G,Z,Q)$. We have by internal equivalence and (S2) that

$$\bar{A} \in \mathcal{S} \Leftrightarrow A + B_2F, A + LC_2, A_Q \in \mathcal{S} \text{ .} \qquad (3.7)$$

In this case, with the above choice of F and L and by definition of admissible controller:

$$Q \in \mathcal{A}(\mathfrak{F}_R^{eq}(G,Z)) \Leftrightarrow \bar{A} \in \mathcal{S}$$
$$\Leftrightarrow A_Q \in \mathcal{S} \Leftrightarrow Q \text{ is internally } \mathcal{S}\text{-stable.} \qquad (3.8)$$

Hence, each controller given by (3.12) $\mathcal{S}$-stabilizes internally G. To show that (3.12) furnishes all the controllers, we impose that Z is invertible and use Lemma 3.2. The result is stated in the following theorem.

Theorem 3.1: Assume the plant (3.1) with (A, B_2, C_2) $\mathcal{S}$-stabilizable and $\mathcal{S}$-detectable. For any F, L, D_{11z}, D_{12z} and D_{21z} satisfying

(i) D_{11z} free; D_{12z} and D_{21z} are nonsingular (3.9)
(ii) $A + LC_2$ is $\mathcal{S}$-stable (3.10)
(iii) $A + B_2F$ is $\mathcal{S}$-stable (3.11)

we have

$$K \in \mathcal{A}(G) \Leftrightarrow K \overset{i.e}{=} \mathfrak{F}(Z,Q), \ Q \text{ is internally } \mathcal{S}\text{-stable} \qquad (3.12.1)$$

where

$$Z = \left[\begin{array}{c|cc} A + B_2F + LC_2 - B_2D_{11z}C_2 & B_2D_{11z} - L & B_2D_{12z} \\ \hline F - D_{11z}C_2 & D_{11z} & D_{12z} \\ -D_{21z}C_2 & D_{21z} & 0 \end{array}\right]. \qquad (3.12.2)$$

Furthermore, the set of all $\mathcal{S}$-stable closed loop systems is parametrized by :

$$\mathfrak{F}(G,K) \overset{i.e}{=} \mathfrak{F}(\mathcal{T},Q) \text{ , } Q \text{ is internally } \mathcal{S}\text{-stable} \qquad (3.13.1)$$

where

$$\mathcal{T} = \left[\begin{array}{cc|cc} A + B_2F & B_2(F - D_{11z}C_2) & B_1 + B_2D_{11z}D_{21} & B_2D_{12z} \\ 0 & A + LC_2 & -(B_1 + LD_{21}) & 0 \\ \hline C_1 + D_{12}F & D_{12}(F - D_{11z}C_2) & D_{11} + D_{12}D_{11z}D_{21} & D_{12}D_{12z} \\ 0 & -D_{21z}C_2 & D_{21z}D_{21} & 0 \end{array}\right]. \qquad (3.13.2)$$

Proof: (*sufficiency*) To show that all controllers in (3.12) is $\mathcal{S}$-stabilizing, it suffices to perform $\mathfrak{F}(G,Z,Q) \overset{i.e}{=} \mathfrak{F}^{given}(G,Z,Q)$ with Z given by (3.12.2) and to verify the $\mathcal{S}$-stability of the A-matrix as above.

(*necessity*) We now show that all admissible controllers can be written in the form (3.12). Since D_{12z} and D_{21z} are nonsingular, and F and L are such that $A + B_2F$ and $A + LC_2$ are $\mathcal{S}$-stable, we have that Z satisfies the conditions of Lemma 3.2 and we can determine a system H such that

$$\mathfrak{F}_R(Z,H) \overset{i.e}{=} \mathfrak{F}_R(H,Z) \overset{i.e}{=} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \qquad (3.14)$$

It follows that, for this H, $\mathfrak{F}_R(G,Z,H) \overset{i.e}{=} G$. Thus, for any K we have that

$$\mathfrak{F}(G,K) \overset{i.e}{=} \mathfrak{F}(G,Z,H,K). \qquad (3.15)$$

Setting $Q := \mathfrak{F}(H,K)$, from (3.14) and Lemma 3.2(i) we have $K \overset{i.e}{=} \mathfrak{F}(Z,Q)$. By Lemma 3.1, $K \in \mathcal{A}(G)$ implies that $Q = \mathfrak{F}(H,K) \in \mathcal{A}(\ \mathfrak{F}_R^{eq}(G,Z)\ )$, where $\mathfrak{F}_R^{eq}(G,Z)$ is given by (3.13.2). By (3.8) it follows that A_Q is $\mathcal{S}$-stable. Thus, for any admissible K, we have that K is given by (3.12) for an internally $\mathcal{S}$-stable $Q(K)$. ☐

Remark 3.1:

a) Note that no result from the observer theory was used in order to derive the expression of Z as it is made in [8] for state space systems. In fact, in this paper, the notions of $\mathcal{S}$-stabilizability and $\mathcal{S}$-detectability for general linear systems were introduced motivated by the $\mathcal{S}$-stabilizing problem. In [7] and [16], for state space systems, the expression of Z is derived using duality and the special systems full information and output estimation. Also, the term D_{11z} which appears naturally in our derivation is not considered in these works. The use of $D_{11z} \neq 0$ is considered in the next subsection.

b) In the proof of necessity part of the above theorem, the expression of system H does not need to be determined. The proof of sufficiency is non-constructive and is well known in literature for state-space systems (see Theorem 1 of [8], Lemma 12.12 of [16], Theorem 11.4 of [15]). In fact, if we can determine the LFT-invertible matrix Z, the ideas of the proof can be used for any given $\mathcal{S}$-stabilizing controller. The problem of how to determine some desirable Z, when a $\mathcal{S}$-stabilizing controller is given, is considered in Section 4.

c) For descriptor systems, the above theorem extends the parametrization of stabilizing controllers of [6] to non-regular systems.

3.1. Parametrization for some particular plants

The construction of the controller parametrization was developed using only the similarity transformation (E1). Thus, some potentialities ((E2) and (E3)) of the relation $\overset{i.e}{=}$ were not explored yet. That is, we can consider now when we have stable hidden modes in, for example, the closed loop system $\mathcal{T}$ given by (3.13.2). Some simple situations where we can create hidden modes or simplify the state-space expression of $\mathcal{T}$ are given by the following conditions:

$$F - D_{11z}C_2 = 0, \ A + B_2F \text{ is } \mathcal{S}\text{-stable} \qquad (3.16.a)$$

$C_1 + D_{12}F = 0$, $A + B_2F$ is $\mathcal{S}$-stable (3.16.b)
$B_1 + LD_{21} = 0$, $A\ +\ LC_2$ is $\mathcal{S}$-stable (3.16.c)

It is clear that, if the plant has C_2, D_{12} or D_{21} with full rank, the above equations have solutions. So, we can consider the following particular plants:

(a) state feedback (SF) plant: C_2 invertible and (3.16.a) is satisfied

(b) FC and OE plants:D_{12} has full row rank and (3.16.b) is satisfied

(c) DF and FI plants: D_{21} has full column rank and (3.16.c) is satisfied .

In order to simplify the final expressions of the plants, one can consider $C_2 = I$; $D_{12} = I$; $D_{12} = [0\quad I]$ and $B_2 = [I\quad 0]$; $D_{21} = I$ and, $D_{21} = [0\quad I]^T$ and $C_2 = [I\quad 0]^T$ which are exactly the SF, OE, FC, DF and FI cases presented in [2], [7] and [16]. These particular plants are important in the study of H_2 and H_∞ control problems and so, the characterization of all $\mathcal{S}$-stabilizing controllers for them is interesting. As an example of how to determine all controllers for these cases, we present the following corollary.

Corollary 3.1: Assume G given by (3.1) with D_{21} full column rank. If
(i) (A, B_2) is $\mathcal{S}$-stabilizable and
(ii) $A - B_1D_{21L}^{-1}C_2 + M(\mathrm{I} - D_{21}D_{21L}^{-1})C_2$ is $\mathcal{S}$-stable for some M, where D_{21L}^{-1} is a left inverse of D_{21},
then

$$K \in \mathcal{A}(G) \ \Leftrightarrow\ K \overset{i.e}{=} \mathfrak{F}(Z,Q),\ Q \text{ internally } \mathcal{S}\text{-stable}$$

where Z is given by $(3.12.2)$ with L given by

$$L \ =\ -B_1D_{21L}^{-1} \ +\ M(\ I - D_{21}D_{21L}^{-1}\ ).$$

Moreover, all closed loop systems are parametrized by

$$\mathfrak{F}(G, K) \overset{i.e}{=} G_F + U_FQD_{21z}D_{21},\ Q \text{ internally } \mathcal{S}\text{-stable}$$

where

$$G_F = \left[\begin{array}{c|c} A\ + B_2F & B_1 + B_2D_{11z}D_{21} \\ \hline C_1 +\ D_{12}F & D_{11} + D_{12}D_{11z}D_{21} \end{array}\right], U_F = \left[\begin{array}{c|c} A\ + B_2F & B_2D_{12z} \\ \hline C_1 + D_{12}F & D_{12}D_{12z} \end{array}\right],$$

D_{11z} is free and D_{12z} and D_{21z} are nonsingular.

Proof: The assumption (ii) corresponds to the solvability of (3.16.c). In this case, every solution L is such that in (3.13.2) $A + LC_2$ becomes non-controlable, and it follows that

$$\mathcal{T} \overset{i.e}{=} \left[\begin{array}{c|cc} A\ + B_2F & B_1 + B_2D_{11z}D_{21} & B_2D_{12z} \\ \hline C_1 + D_{12}F & D_{11} + D_{12}D_{11z}D_{21} & D_{12}D_{12z} \\ 0 & D_{21z}D_{21} & 0 \end{array}\right]. \qquad \square$$

As we can see by the above corollary, for the FI case where $D_{21} = \begin{bmatrix} 0 \\ I \end{bmatrix}$, the final expression of the matrix Z, that is, the final expression of all stabilizing controllers, is not intuitive due to the freedom in matrix M. So, although it is simple to determine a single FI stabilizing control, this fact leads to difficulties in beginning the stabilization study with the FI case as in [16], [7], [2]. Working in the same way as the above corollary with the condition (3.16.a) for the SF case, the parametrization expression of [10] can be recovered as a particular case. The same do not occur if we start with the matrix Z presented in [8], [16], [7], [2], since in these works it is assumed $D_{11z} = 0$.

Remark 3.2: The assumptions of Lemma 3.2 and the expression of Z reflect the situation where we can solve (3.16.b), (3.16.c), $D_{11} + D_{12}D_{11z}D_{21} = 0$, $D_{12}D_{12z} = I$ and $D_{21z}D_{21} = I$. $\square$

4. Admissible controllers parametrization centred on an arbitrary stabilizing controller

In the previous section, we saw that, corresponding to the classical state-space formulae given by [9], [14], the obtained set of all admissible controllers is centred on an observer based controller, that is

$$\mathcal{A}(G) \ = \{K \overset{i.e}{=} \mathfrak{F}(\ Z,Q\ ) : Q \text{ internally } \mathcal{S}\text{-stable}\} \quad (4.1)$$

is such that Z_{11} is an observer based controller. For state space systems, formulae to the parametrization of all stabilizing controllers centred on any given stabilizing controller is given in [4], using coprime factorization techniques. Notwithstanding it is a simple task to verify if a set of controllers characterized as (4.1) is in fact the set of all admissible controllers (it suffices to follow the proof of Theorem 3.1), it was not considered yet how to determine a expression of Z corresponding to a known stabilizing controller. In this section, we derive the formulae to the parametrization of all stabilizing controllers centred on any given stabilizing controller for general Rosenbrock systems using only systems connections arguments.

We are interested in the determination of new systems Z_{new} and Q_{new} which parametrize $\mathcal{A}(G)$ and such that the central controller is a given controller K_0, that is

$$\mathcal{A}(G) \ = \{K \overset{i.e}{=} \mathfrak{F}(\ Z_{\text{new}},Q_{\text{new}}\ ) : Q_{\text{new}} \text{ internally } \mathcal{S}\text{-stable}\}$$

with $(Z_{\text{new}})_{11} \overset{i.e}{=} K_0$. In order to generate a new parametrization from (4.1), consider any systems W and M satisfying $\mathfrak{F}_R(W,M) \overset{i.e}{=} \mathfrak{F}_R(M,W) \overset{i.e}{=} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. We have that

$$\begin{aligned} K \overset{i.e}{=} \mathfrak{F}(Z,Q) &\overset{i.e}{=} \mathfrak{F}(Z,\mathfrak{F}_R(W,M),Q) \\ &\overset{i.e}{=} \mathfrak{F}(\mathfrak{F}_R(Z,W),\mathfrak{F}(M,Q)) \overset{i.e}{=} \mathfrak{F}(Z',Q') \end{aligned}$$

where we defined $Z' := \mathfrak{F}_R(Z,W)$ and $Q' := \mathfrak{F}(M,Q)$. By Lemma 3.2 (ii) it follows that

$$\{\mathfrak{F}(M,Q) : Q \text{ internally } \mathcal{S}\text{-stable}\} = \mathcal{A}(W).$$

And so, the parametrization (4.1) becomes

$$\mathcal{A}(G) \ = \{K \ \overset{i.e}{=}\ \mathfrak{F}(\ Z',Q'\ ),\ Q' \in \mathcal{A}(W)\}.$$

But by Theorem 3.1,

$$\mathcal{A}(W) = \{Q' \ \overset{i.e}{=}\ \mathfrak{F}(\ Z_W,\ Q_{\text{new}}\ ),\ Q_{\text{new}} \text{ internally } \mathcal{S}\text{-stable}\}$$

where we can choose Z_W as

$$Z_W = \left[\begin{array}{c|cc} A_W + B_{2W}F_W + L_WC_{2W} & -L_W & B_{2W} \\ \hline F_W & 0 & I \\ -C_{2W} & I & 0 \end{array}\right]$$

with F_W and L_W matrices such that $A_W + B_{2W}F_W$ and $A_W + L_W C_{2W}$ are $\mathcal{S}$-stable. Hence a new parametrization to all $\mathcal{S}$-stabilizing controllers is given by

$$K \in \mathcal{A}(G) \Leftrightarrow K \overset{i.e}{=} \mathfrak{F}(\, Z_{\text{new}}, Q_{\text{new}}),\ Q_{\text{new}} \text{ internally } \mathcal{S}\text{-stable}$$

where $Z_{\text{new}} = \mathfrak{F}_R(Z',Z_W)$. It follows that it is only necessary to find a convenient Z', F_W and L_W such that $W = \mathfrak{F}_R(H, Z')$ is "LFT invertible" and $(Z_{\text{new}})_{11} \overset{i.e}{=} K_0$ (H is the "LFT inverse" of Z and K_0 is the given controller). With some suitable choices of these variables we have the following theorem.

Theorem 4.1: Assume the plant (3.1) with (A, B_2, C_2) $\mathcal{S}$-stabilizable and $\mathcal{S}$-detectable. Consider an internally $\mathcal{S}$-stabilizing controller $K_0 = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ and matrices F_K and L_K such that $A_K + B_K F_K$ and $A_K + L_K C_K$ are $\mathcal{S}$-stable. Then, parametrizations of $\mathcal{A}(G)$ centred on K_0 can be given by

$$K \in \mathcal{A}(G) \ \Leftrightarrow\ K \overset{i.e}{=} \mathfrak{F}(\, Z,Q\,),\ Q \text{ internally } \mathcal{S}\text{-stable}$$

where the system Z can be chosen as one of the following :

$$\text{(i)}Z = \left[\begin{array}{ccc|cc} A+B_2D_KC_2 & B_2C_K & 0 & 0 & B_2 \\ B_KC_2 & A_K & 0 & 0 & -L_K \\ 0 & 0 & A_K & B_K & -L_K \\ \hline 0 & 0 & C_K & D_K & I \\ -C_2 & F_K & -F_K & I & 0 \end{array}\right]$$

$$\text{(ii) } Z = \left[\begin{array}{cc|cc} A+LC_2 & B_2C_K & B_2D_K - L & B_2 \\ 0 & A_K & B_K & -L_K \\ \hline 0 & C_K & D_K & I \\ -C_2 & 0 & I & 0 \end{array}\right]$$

$$\text{(iii) } Z = \left[\begin{array}{cc|cc} A+B_2F & 0 & 0 & B_2 \\ -B_KC_2 & A_K & B_K & 0 \\ \hline F-D_KC_2 & C_K & D_K & I \\ -C_2 & -F_K & I & 0 \end{array}\right]$$

with F and L such that $A+B_2F$ and $A+LC_2$ are $\mathcal{S}$-stable.□

From the above considerations, it is clear that there are many other possible realizations centred on K_0.

5. Conclusions

This work was focused on the controller parametrization for generalized Rosenbrock systems. It was intended to put some state-space formulae, obtained by many ways in the literature, unified in a single "LFT context". We first presented an alternative construction of the parametrization to that presented by [16], [15], [8], [11]. The obtained state space expression for the coefficient matrix Z has extra freedom in the D_{11z}, D_{12z} and D_{21z} matrices which is useful in the characterization of all controllers for special cases full information, state feedback, and their duals. We solved also the problem of construction of the parametrization of all stabilizing controllers centred on any given stabilizing controller using only system connections arguments. For state space systems, this is a state-space result corresponding to the existence of coprime factors satisfying the Bezout identity once given the plant and any stabilizing controller ([13],[4]).

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