$\mathcal{H}_\infty$-Control and Frequency Analysis for Systems with Mixed Continuous-Time and Discrete-Time Measurements

Olof Lindgärde, Bengt Lennartson
Control and Automation Laboratory, Department of Signals and Systems
Chalmers University of Technology, SE-412 96 Göteborg, Sweden
TU.06280OL@memo.volvo.se, bl@s2.chalmers.se

Abstract

Feedback control using both continuous-time and discrete-time measurements is considered in this paper. A design method including both $\mathcal{H}_\infty$-synthesis and frequency analysis is suggested. It includes a strategy for selecting weighting filters in the control design. The presented frequency analysis method is then a powerful tool in the design process.

1 Introduction

This paper considers mixed continuous-time and discrete-time periodic systems. Such systems are linear time-periodic (LTP), which is an important class of linear systems. The objective of this work is to design $\mathcal{H}_\infty$ controllers for such periodic systems and to analyze them in the frequency domain.

Analysis of SD systems in the frequency domain is complicated by the periodic behavior of the feedback system. Therefore, frequency analysis for SD control including the inter-sample behavior has gained great interest during the past years, see e.g. [3, 8, 1]. The purpose of frequency analysis of feedback systems is often twofold. On the one hand the purpose is to study how a feedback system attenuates or amplifies inputs within certain frequency regions, on the other the purpose is to study robust stability properties. Therefore a distinction is made between the robust frequency gain (RFG), c.f. [8, 1, 7], and the performance frequency gain (PFG) [7]. In this paper the PFG case is considered. Compared to [7] the PFG analysis is extended to include mixed continuous-time and discrete-time inputs. Acctually, by the presented model structure it is not only possible to analyze systems with mixed continuous and SD measurements, but a couple of different control problems in a common framework. For instance, it is possible to analyze systems including single- and multirate sampled-data control, mixed continuous and piecewise continuous control signals, as well as general periodic continuous-time systems.

A problem in LTP design is that the closed-loop system may never reach steady state when it is subjected to step disturbances. Instead the closed-loop system reaches a periodic steady state where the output oscillates. This phenomenon is observed in [4, 6]. The problem occurs when the control signal is continuous or updated at a fast rate, while some of the measurements are updated at a slower rate. A strategy for selecting weighting filters in the control design is presented. With this strategy, the influence of step disturbances is suppressed on the controlled output and oscillations are avoided in LTP design. It is shown that the oscillation problem is further complicated if the number of measurement signals exceeds the number of control signals. The solution to this problem is to filter the continuous or the fast sampled measurement signals with high-pass filters. By introducing these high-pass filters, the continuous measurement signals will be used in the high frequency domain only, and not in the low frequency behavior.

In a concluding example, an $\mathcal{H}_\infty$-controller is applied to a lime slaker model developed in [5]. Two design cases will be studied. In the first case, only continuous-time temperature measurements are available. The resulting controller will then be linear time invariant (LTI). It will then be assumed that it is possible to take additional automatic discrete measurements and the resulting controller will be LTP. It is a difficult task to chose appropriate filters for the mixed continuous and discrete-time controller. In this example the PFG is a powerful tool for analysis of the closed-loop system. For instance, it is possible to analyze the influence of mixed continuous-time and discrete-time measurement noise on a continuous-time output.

2 Frequency analysis

In this section mixed continuous-time and discrete-time periodic systems are analyzed in the frequency domain. This is done by considering a state-space description of the closed-loop system, which includes both the plant and the controller.

Define the following signal spaces

$$\begin{aligned}
&\mathcal{S}_C(\omega) \equiv \{v(t) : v(t) = v_0 e^{j\omega t}, \|v_0\| < \infty\} \\
&\mathcal{S}_V(\omega) \equiv \{v(t) : v(t) = v_0(t) e^{j\omega t}, \\
&\qquad v_0(t) \in \mathcal{L}_2[0,T), v_0(t+T) = v_0(t)\ \forall\, t\}
\end{aligned}$$

$$\mathcal{S}_{C_d}(\omega) \equiv \{v_d(t_k) : v_d(t_k) = v_{d0}e^{j\omega t_k}, \|v_{d0}\| < \infty\}$$
$$\mathcal{S}_{V_d}(\omega) \equiv \{v_d(t_k) : v_d(t_k) = v_{d0}(t_k)e^{j\omega t_k}, v_{d0}(t_k) \in l_2[0,T], v_{d0}(t_k+T) = v_0(t_k)\ \forall\ t_k\}$$

As a tool for analyzing a continuous-time multi-variable signal $v(t) \in \mathcal{S}_C(\omega)$ or $v(t) \in \mathcal{S}_V(\omega)$, $\|v\|_{\mathcal{P}}$ will denote the square root of the signal power averaged over time, i.e.

$$\|v\|_{\mathcal{P}} \equiv \lim_{\tau\to\infty} \sqrt{\frac{1}{2\tau}\int_{-\tau}^{\tau} \|v(t)\|^2\, dt} \qquad (1)$$

The corresponding definition for a discrete time signal $v^*(kT)$ is given by

$$\|v^*\|_p \equiv \lim_{N\to\infty} \sqrt{\frac{1}{2N}\sum_{k=-N}^{N} \|v^*(kT)\|^2} \qquad (2)$$

Let (v, v_d) denote the mixed continuous and discrete signal. Let the signal power of such mixed continuous and discrete signal be given by $\|(v,v_d)\|^2_{(\mathcal{P},p)} = \|v\|^2_{\mathcal{P}} + \|v_d\|^2_p$. Let v_0^e denote the extended vector $v_0^e = [v_0^H\ v_{d0}^H]^H$. If $(v, v_d) \in \mathcal{S}_C(\omega) \otimes \mathcal{S}_{C_d}(\omega)$ then $\|(v,v_d)\|_{\mathcal{P},p} = \sqrt{\|v_0\|^2 + \|v_{d0}\|^2} = \|v_0^e\|$. The following definition of the PFG will now be used:

Definition 1 (Performance frequency gain, PFG) *Let the input $(v, v_d) \in \mathcal{S}_C(\omega) \otimes \mathcal{S}_{C_d}(\omega)$. The performance frequency gain (PFG) of a sampled-data (SD) system Ψ_{zv} will be denoted by $\overline{\gamma}_{zv}(\omega)$ where*

$$\overline{\gamma}_{zv}(\omega) = \max_{\|v_0^e\|=1} \frac{\|z\|_{\mathcal{P}}}{\|(v,v_d)\|_{(\mathcal{P},p)}} \qquad (3)$$

Computing the PFG: Let Ψ_{zv} denote the closed-loop operator for a mixed continuous-time and discrete-time periodic system. It will be assumed that Ψ_{zv} is internally stable, piecewise continuous and T-periodic. In the analysis mixed continuous and discrete-time input signals (v, v_d) will be assumed. The following state-space representation describes the system $\Psi_{zv} : (v, v_d) \to z$ with jumps:

$$\begin{aligned} \dot{x}(t) &= A_{tot}(t)x(t) + B_{tot}(t)v(t) && (4)\\ x(t_k) &= A_{dtot}(t_k)x(t_k^-) + B_{dtot}(t_k)v_d(t_k) && (5)\\ z(t) &= C_{tot}(t)x(t) + D_{tot}(t)v(t) && (6) \end{aligned}$$

Note that the model structure in (4)–(6) includes the controller in feedback systems. In SD control the discrete controller is included in the A_{dtot} matrix.

It is furthermore assumed that A_{dtot} is invertible. However, the hold function in systems with piecewise constant control signals may cause A_{dtot} to be singular. This problem can be solved by introducing a small perturbation ϵ in the control signal update. If ϵ is sufficiently small this modification of A_{dtot} will not affect the analysis of the feedback system.

Let $(v, v_d) \in \mathcal{S}_V(\omega) \otimes \mathcal{S}_{V_d}(\omega)$ and assume that the system Ψ_{zv} is in perodic steady state, i.e. the input v and v_d have affected the system Ψ_{zv} since $t = -\infty$. Then the output $z \in \mathcal{S}_V(\omega)$ and, furthermore, the state has the property

$$x(t+T) = e^{j\omega T}x(t) \qquad (7)$$

The property (7) can be shown by the lifting technique, c.f. [8], that transform the LTP system (4)–(6) into a corresponding discrete-time system at the time instant $t = \tau + kT$ where $\tau \in [0, T]$. The lifted system then satisfies the property in (7) for any $\tau \in [0, T]$. The adjoint system $\Psi_{zv}^{\sim} : z \to (\tilde{v}, \tilde{v}_d)$ has the property

$$< z, \Psi_{zv}(v, v_d) >_{\mathcal{L}_2[0,T]} = < \Psi_{zv}^{\sim} z, (v, v_d) >_{\mathcal{L}_2[0,T]\otimes l_2[0,T]}$$

and is given by

$$\begin{aligned} -\dot{p}(t) &= A'_{tot}(t)p(t) + C'_{tot}(t)z(t) \quad p(T) = p(0)e^{j\omega T}\\ p(t_k^-) &= A'_{dtot}(t_k)p(t_k)\\ \tilde{v}(t) &= B'_{tot}(t)p(t) + D'_{tot}(t)z(t)\\ \tilde{v}_d(t_k) &= B'_{dtot}(t_k)p(t_k) \end{aligned}$$

Let ξ denote the augmented state vector of the joint system $\Psi_{zv}^{\sim}\Psi_{zv}$, i.e. $\xi = [\,x^H \quad p^H\,]^H$ where x^H is the adjoint of x. The state-space description for the joint system $\Psi_{zv}^{\sim}\Psi_{zv} : (v, v_d) \to (\tilde{v}, \tilde{v}_d)$ is given by

$$\begin{aligned} \dot{\xi}(t) &= A(t)\xi(t) + B(t)v(t) && (8)\\ \xi(t_k) &= A_d(t_k)\xi(t_k^-) + B_d(t_k)v_d(t_k) && (9)\\ \tilde{v}(t) &= C(t)\xi(t) + D(t)v(t) && (10)\\ \tilde{v}_d(t_k) &= C_d(t_k)\xi(t_k^-) && (11) \end{aligned}$$

where

$$A = \begin{bmatrix} A_{tot} & 0\\ -C'_{tot}C_{tot} & -A'_{tot} \end{bmatrix} \quad B = \begin{bmatrix} B_{tot}\\ -C'_{tot}D_{tot} \end{bmatrix}$$
$$A_d = \begin{bmatrix} A_{dtot} & 0\\ 0 & (A_{dtot}^{-1})' \end{bmatrix} \quad B_d = \begin{bmatrix} B_{dtot}\\ 0 \end{bmatrix}$$
$$C = [\,D'_{tot}C_{tot} \quad B'_{tot}\,] \quad D = D'_{tot}D_{tot}$$
$$C_d = [\,0 \quad (A_{dtot}^{-1}B_{dtot})'\,]$$

Since the oscillating property of the state vector x in (7) applies both to Ψ_{zv} and $\Psi_{zv}^{\sim}$, it also applies to the open-loop joint system $\Psi_{zv}^{\sim}\Psi_{zv}$, i.e.

$$\xi(t+T) = e^{j\omega T}\xi(t) \qquad (12)$$

Finally let

$$B^e = [\,B \quad 0\,] \quad B_d^e = [\,0 \quad B_d\,]$$
$$C^e = \begin{bmatrix} C\\ 0 \end{bmatrix} \quad C_d^e = \begin{bmatrix} 0\\ C_d \end{bmatrix}$$
$$D^e = \begin{bmatrix} D & 0\\ 0 & 0 \end{bmatrix} \quad D_d^e = \begin{bmatrix} 0 & 0\\ 0 & D_d \end{bmatrix}$$

where the zero matrices are of appropriate dimensions such that $(B^e v_0^e = Bv_0)$, $(B_d^e v_{d0}^e = B_d v_{d0})$,

$((v_0^e)^H C^e = v_0^H C)$, $((v_{d0}^e)^H C_d^e = v_{d0}^H C_d)$ and that $((v_0^e)^H D^e v_0^e = v_0^H D v_0)$.

The following theorem states how the PFG can be computed.

Theorem 1 (PFG) *Consider the mixed continuous-time and discrete-time system in (4)–(6). Let Φ denote the transition matrix related to the system (8) and (9). The performance frequency gain PFG for this system is given by*

$$\overline{\gamma}_{zv}(\omega) = \sqrt{\lambda_{\max}(\bar{A}(\omega))/T}$$

where

$$\bar{A}(\omega) = \Lambda_v(T) + \Lambda_\xi(T)(Ie^{j\omega T} - \Phi(T))^{-1}\Gamma(T)$$

Here

$$\begin{aligned} \Gamma(t) &= \int_{t_{k-1}}^{t} \Phi(t,\sigma)e^{j\omega\sigma}B^e(\sigma)d\sigma && (13)\\ &+ \Phi(t,t_{k-1})\Gamma(t_{k-1}), \quad t \in]t_{k-1}, t_k[&& (14)\\ \Gamma(t_k^+) &= A_d(t_k)\Gamma(t_k^-) + B_d^e(t_k)e^{j\omega t_k} && (15)\\ \Lambda_v(T) &= \int_0^T \left[e^{-j\omega\sigma}C^e(\sigma)\Gamma(\sigma) + D^e(\sigma)\right] d\sigma && (16)\\ &+ \sum_{k=1}^{r} e^{-j\omega t_k}C_d^e(t_k)\Gamma(t_k^-) && (17)\\ \Lambda_\xi(T) &= \int_0^T e^{-j\omega\sigma}C^e(\sigma)\Phi(\sigma,t_0)d\sigma && (18)\\ &+ \sum_{k=1}^{r} e^{-j\omega t_k}C_d^e(t_k)\Phi(t_k^-,t_0) && (19) \end{aligned}$$

where $\Gamma(0) = 0$.

Proof: See [5] ∎

3 Design strategy

A standard objective in control design is to include integral action in the controller so that the error of the controlled output, $z_\epsilon(t) = (z(t) - z_{ref})$, tends to zero after a step disturbance, i.e. $\lim_{t\to\infty} z_\epsilon(t) = 0$. However, when the control signal is updated with a faster rate than the sampling rate of the measurement signal, it is non-trivial to fulfill this objective. As observed in e.g. [4, 6, 5], the closed-loop system may oscillate and never reach steady state when it is subjected to step disturbances. This section will show how weighting filters can be selected so that

1. steady state is achieved in LTP design, and
2. the error $\lim_{t\to\infty} z_\epsilon(t) = 0$.

One way of achieving integral action on the measurement signal, is to filter the measurement signal related to the regulated output with an integral filter. However, if the measurement signal is sampled this design strategy is not possible. Instead the weighting filters will be added at the input of the system.

A simple example will justify the design strategy. Consider the following continuous process with discrete-time measurements:

$$\begin{aligned} \dot{x}_c &= -0.2x_c + u + v && (20)\\ z &= 0.2x_c && (21)\\ y_d(t_k) &= 0.2x_c(t_k) && (22) \end{aligned}$$

The sampling interval is here $T = 10$ sec. The weighting filters used in the synthesis are shown in Fig. 1. After the synthesis, the two filters W_u and W_{yd} are implemented directly at the controller input and output. This is illustrated in Fig. 2. The regulated output is $\tilde{z}' = [\, z' \quad \tilde{u}' \,]'$. Following the $\mathcal{H}_\infty$-synthesis strategy in [2], the controller will have the following structure:

$$\begin{aligned} \dot{x}_r &= A_K x_r && (23)\\ x_r(t_k^+) &= A_{dK}x_r(t_k) + B_{dK}y_d(t_k) && (24)\\ u &= C_K x_r && (25) \end{aligned}$$

Note that the system matrices in (23)–(25) are time invariant, in contrast to controllers with both continuous and discrete measurements which have system matrices that vary periodically. From (20)–(22) and (23)–(25) it follows that the state-space description of the closed-loop system is given by

$$\dot{x} = \begin{bmatrix} -0.2 & C_K \\ 0 & A_K \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v \equiv A_{tot}x + B_{tot}v \quad (26)$$

$$x(t_k^+) = \begin{bmatrix} 1 & 0 \\ 0.2B_{dK} & A_{dK} \end{bmatrix} x(t_k) \equiv A_{dtot}x(t_k) \quad (27)$$

$$z = [\, 0.2 \quad 0 \,]\, x \equiv C_{tot}x \quad (28)$$

where $x' = [\, x_c' \quad x_r' \,]$.

Conditions for steady state: An advantage of this simple example is that it is easy to state the conditions under which the system is to be in steady state. Assume a unit step input. A steady state solution $\bar{x}$ exist for the system (26)–(28) iff

$$\begin{aligned} 0 &= A_{tot}\bar{x} + B_{tot} && (29)\\ 0 &= (A_{dtot} - I)\bar{x} && (30) \end{aligned}$$

Following the design strategy in Fig. 1 and Fig. 2 three controllers of increasing complexity will now be designed. Simulations of the closed-loop behavior are shown in Fig. 3.

1. First the most simple case will be considered, i.e. let $W_v = W_u = 1$. Also let $W_{yd} = 10$. When implementing an $\mathcal{H}_\infty$-controller, the conditions (29) and (30) are not fulfilled and hence no steady state solution $\bar{x}$ exists for this closed-loop system.

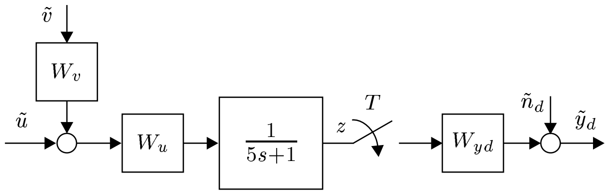

Figure 1: The plant and the filters used in the synthesis

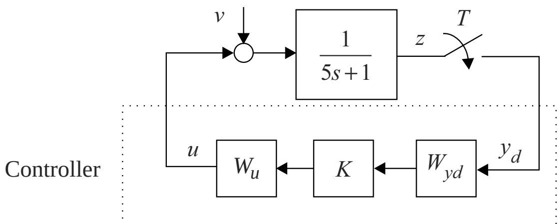

Figure 2: The plant and the filters used in the analysis

2. Comparing (29) with (26) it follows that A_K must have at least a zero eigenvalue in order for a steady state solution to exist. One way to obtain a zero pole in the controller is to include a zero pole in the weighting filter W_v at the process disturbance input v. Let $W_v = (1 + \frac{1}{10s+\epsilon})$ where $\epsilon = 10^{-3}$. Also let $W_u = 1$ and $W_{yd} = 10$. The "perturbation" ϵ is needed for the system to be stabilizable. Implementing an $\mathcal{H}_\infty$-controller a "nearly" steady state solution is obtained. The reason this solution is not completely in steady state is due to the ϵ perturbance in the filter W_v. It is interesting to observe that $z = C_{tot}\bar{x} = 0.5 \neq 0$, i.e. $\lim_{t\to\infty} z_\epsilon(t) \neq 0$, even if the filter W_v contains a zero pole.

3. One way of combining integral action in the controller with the modeling of the process disturbance, is to include a zero pole in the filter W_u instead of in W_v. Furthermore, this approach eliminates the problem of stabilizabilty, i.e. the ϵ perturbance is not needed. Let $W_v = 1$, $W_u = (1 + \frac{1}{10s})$ and let $W_{yd} = 10$. The conditions (29) and (30) are then fulfilled and $\lim_{t\to\infty} z_\epsilon(t) = 0$.

The oscillation problem occurs when the control signal is continuous or updated at a fast rate, while some of the measurements are updated at a slow rate. The $\mathcal{H}_\infty$-filter in the controller, cf. [2], then loses information about the controlled output during the period, which leads to an open-loop behavior of the control signal. This can be seen in the simulations in Fig. 3 where the control signal tends to zero between the discrete-time updates.

Additional measurement signals: It is clear that the design strategy for the third controller meets the two requirements listed at the beginning of this section. However, it turns out that in the special MIMO case when the number

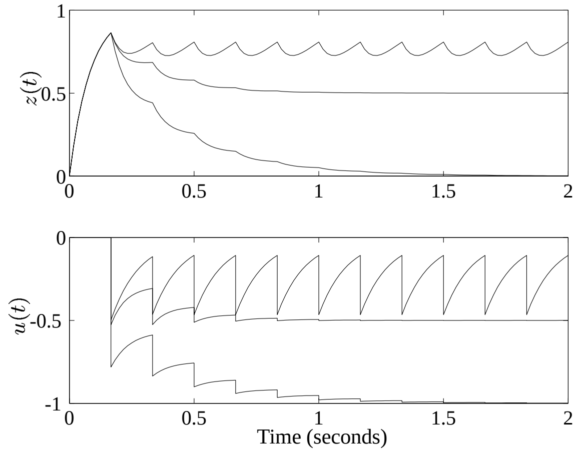

Figure 3: Step responses of the feedback system in Fig. 2. The three controllers (no. 1, 2 and no. 3) are used.

of measurement signals m_y exceeds the number of control signals m_u, integral action will not be obtained and oscillations may appear even if the W_u filter contains a zero pole. The solution to this problem is illustrated in Fig. 4. As have been mentioned previously, a standard technique is to filter the measurement signal related to the regulated output z with an integral filter. Since the measurement is sampled, the integral filter is instead moved to the control signal at the input. In order to compensate for the filter at the input, a high-pass filter has to be added to the continuous-time measurement signal y. This means that the measurement signal y is only used in the mid-band and for high input frequencies so that "$m_y = m_u$" when $\omega = 0$.

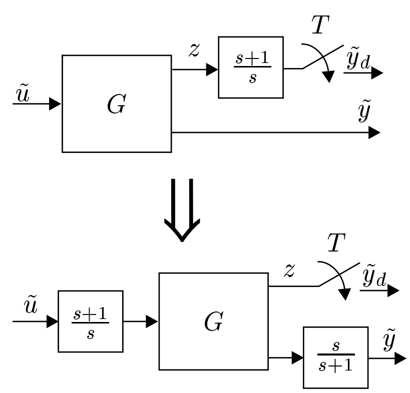

Figure 4: Example of plant and filters when $m_y > m_u$

To summarize, the following design strategy will be used

1. Include a zero pole in the W_u filter by e.g. a PI-filter. Since the disturbance $\tilde{v}$ is entered before the W_u filter, this filter models the process disturbance so that oscillations are avoided. This W_u filter also achieves $\lim_{t\to\infty} z_\epsilon = 0$ when it is included in the controller after the synthesis.

2. Filter the continuous and fast sampled measurement

signals with "high-pass filters" so that $m_y = m_u$ for low frequencies. This step is necessary if the number of measurement signals exceed the number of control signals, i.e. if $m_u < m_y$.

In the following section, a controller is designed for a plant with mixed continuous-time and discrete-time measurements and where $m_y > m_u$.

4 Example

An $\mathcal{H}_\infty$-controller is applied to an advanced lime slaker model developed in [5]. In this example the slaked model is linearized and reduced to a sixth order model. The lime slaker is modeled as two ideal stirred tank in series. The model G gives a dynamical relation between the lime and water inflow, Q^i_{AW} and Q^i_B, to the moisture content z_{B2} and to the temperature T_2 in the second tank, see Fig 5. The bandwidth from Q^i_B to z_{B2} and Ts_2 is about $2 \cdot 10^{-3}$ rad/s. The sampling interval is assumed to be one minute, i.e. $T = 60\ s$.

Two design cases will be studied. In the first case, only continuous-time temperature measurements ($y = T_2$) are available. The resulting controller then becomes LTI. In the second case it will be assumed that it is possible to take additional automatic discrete measurements of the moisture content ($y_d = z_{B2}$). The resulting controller then becomes LTP, cf. [2].

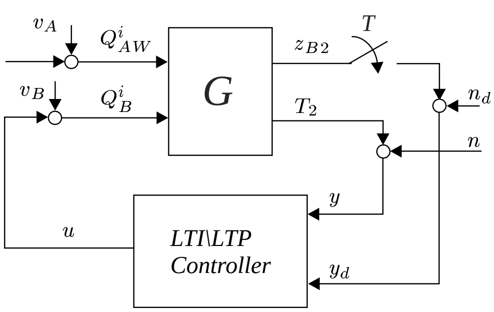

Figure 5: Plant and the controller used in the analysis

The design strategy has been to make both the controllers fulfill the following two requirements; the maximum gain from the measurement noise to the control signal ($\max(KS)$) should be equal for the two controllers, and the closed-loop systems should be equally robust $\max(T)$.

Consider now the system G with the weighting filters W_u, W_y and W_{y_d} in Fig. 6 and Fig. 7. The system and the weighting filters used in the analysis are shown in Fig. 5. The same W_u filter, a PI filter, has been chosen for the two controllers.

$$W_u = 0.15\left(1 + \frac{1}{100s}\right) \qquad \text{(LTP) and (LTI)}$$

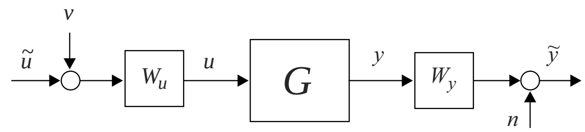

Figure 6: The plant and filters used in the synthesis of an LTI controller

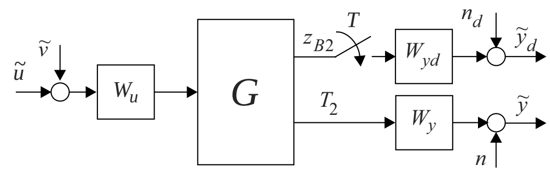

Figure 7: Plant and filters used for synthesis of an LTP controller

As observed in the previous section, the output filter W_y has a great impact on the behavior of the oscillations. A high gain of W_y means that the performance can be improved with preserved stability margins. On the other hand, a high gain of W_y in the low frequency band again means that the oscillations are amplified and that the time before they fade away after a step disturbance is prolonged. The W_y filter has therefore been modeled as a second order high-pass filter with a high gain and with a high corner frequency. The following W_y filter has been chosen for the LTP controller:

$$W_y = \frac{213.3s^2}{(s + 1/45)^2} \qquad \text{(LTP)}$$

The remaining filters have been chosen as $W_y = 80$ (LTI) and $W_{yd} = 40$ (LTP). Also, the controlled outputs of the two controllers are $z = [\, z'_{B2} \quad \tilde{u}' \,]'$ (LTI) and $z = [\, 80z'_{B2} \quad \tilde{u}' \,]$ (LTP). The analysis of the closed-loop system in the frequency domain is shown in Fig. 8–9. A step response of the closed-loop system is shown in Fig. 10.

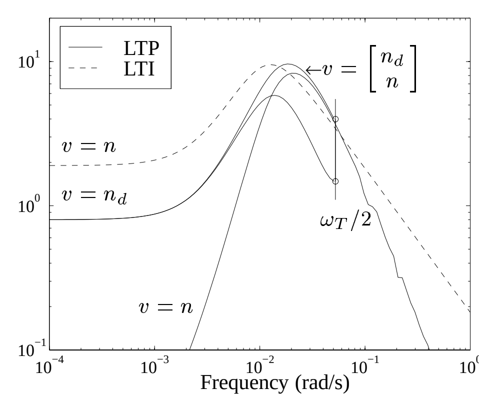

Figure 8: Influence of measurement noise on the control signal.

Because of the choice of W_u filter and W_y filter, the LTP controller only uses the continuous measurements in the high frequency band whereas the discrete measurements are used in the low frequency band. The influence of these two filters on the control behavior becomes apparent when studying Fig 8 where the PFG from the measurement noises n_d, n_y to the control signal u are shown. Observe that in the LTP case $\max(KS)$ is computed for the mixed continuous and discrete input signal $v = [\,n_d' \quad n'\,]'$ i.e. the plot shows the "worst case" gain (in the PFG sense) from the measurements noises n_d, n_y to the control signal u.

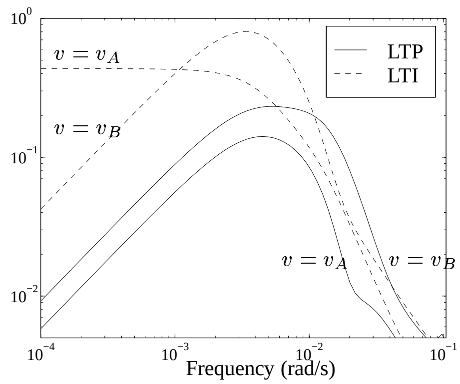

Figure 9: Reduction of process disturbances. PFG from v_A, v_B to z_{B2}

The advantage of the LTP controller over the LTI controller is apparent when studying Fig. 9 and the simulations in Fig. 10. The closed-loop LTP system has a higher bandwidth (≈ 0.02 rad/s) than the closed-loop LTI system (≈ 0.01 rad/s). The bandwidth of the closed-loop LTP system is about ten times faster compared to the open-loop system G. It should also be observed that the LTP controller suppresses disturbances on the lime inflow (v_A), i.e. $\lim_{t\to 0} z_\epsilon(t) = 0$, something the LTI controller naturally is not able to do.

The simulation in Fig. 10 shows the transient oscillating behavior of the control signal. Compared to the LTI controller, the LTP control signal is notched and not very smooth. Observe that the PFG only evaluates the stationary behavior. The jumps in the control signal are therefore not revealed in the frequency analysis. Simulations in the time domain must therefore be used as a complement to the frequency analysis.

5 Concluding remarks

This work includes design and analysis in the frequency domain for mixed continuous-time and discrete-time systems. In the frequency analysis the controller and the plant are presented in one joint state space description. This approach gives a very "clean" solution. In addition, by considering a general closed-loop system with discrete jumps, a wide class of mixed continuos-time and discrete-time controllers can be studied.

In the derivations of PFG, a key observation is the oscillating property of the state vector. The solution of the PFG is obtained by integrating differential matrix equations over the

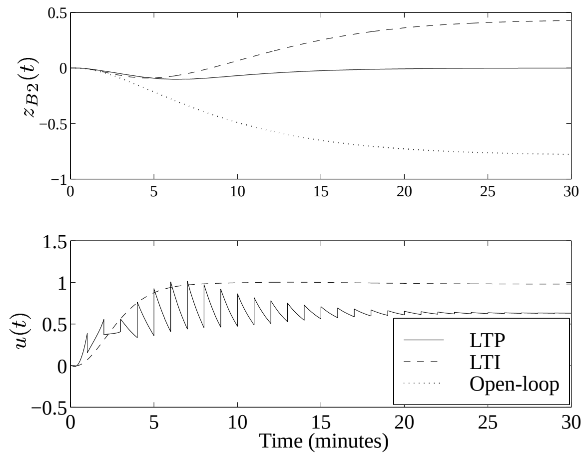

Figure 10: The response from a step disturbance $v = v_A$

system period. Observe that no gamma iteration is needed in order to find the maximum gain, cf. [8]. This is not needed for the PFG since the maximum gain is given for a single sinusoidal input vector.

References

[1] M. Araki, Y. Ito, and T. Hagiwara. Frequency-response of sampled-data systems. *Automatica*, 32(4):483–497, 1996.

[2] A. Christiansson, B. Lennartson, and H. Toivonen. Mixed continuous-time and discrete-time output feedback $\mathcal{H}_\infty$ control – a unified approach. In *Proceeding of the 5th European Control Conference*, 1999. Paper No F1040-4.

[3] G.C. Goodwin and M. Salgado. Frequency domain sensitivity functions for continuous time systems under sampled data control. *Automatica*, 30(8):1263–1270, 1994.

[4] B. Lennartson. Multirate sampled-data control of two-time-scale systems. *IEEE Trans. on Automatic Control*, 34(6):642–644, 1989.

[5] O. Lindgärde. *Frequency Analysis of Sampled-Data Systems Applied to a Lime Slaking Process*. PhD thesis, Department of Signals and Systems, Chalmers University of Technology, 1999.

[6] O. Lindgärde and B. Lennartson. Mixed continuous-time and multirate sampled-data $\mathcal{H}_\infty$-control applied to a packed bed distillation column. In *Proc. of the 3rd European Control Conference*, pages 601–606, 1995.

[7] O. Lindgärde and B. Lennartson. Frequency analysis for mixed continuous-time and discrete-time periodic systems. In *Proceeding of the 5th European Control Conference*, 1999. Paper No F1040-3.

[8] Y. Yamamoto and P.P. Khargonekar. Frequency response of sampled-data systems. *IEEE Trans. on Automatic Control*, 41(2):166–176, 1996.