Approximate high-gain observers for uniformly observable nonlinear systems

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Abstract

A methodolgy for the design of continuous practical observers for nonlinear uniformly observable systems is presented. The system is first transformed into observability normal form, using the observability map semi-diffeomorphism. Since this normal form may have a non-Lipschitz continuous right hand side, a so called $\epsilon-$approximate high-gain observer is designed for it, constituting the dynamic part of the observer. The inverse of the transformation is used as the static part. Convergence of the observer's state trajectory to a ball around the true state trajectory is guaranteed, with the radius of the ball as small as desired.

1 Introduction

Consider the following SISO nonlinear system:

$$\left.\begin{array}{rcl} \dot{x} & = & f(x,u)\ , \quad x\,(0)=x_0 \\ y & = & h(x) \end{array}\right\} \qquad (1)$$

where the state x evolves in an open connected subset M of $\mathbb{R}^n$, the input $u \in \mathbb{R}$ and the output $y \in \mathbb{R}$. The functions $f : M \times \mathbb{R} \to \mathbb{R}^n$ and $h : M \to \mathbb{R}$ are assumed to be smooth, for simplicity. Let us denote the solution of (1) which passes through x_0 at $t = 0$, corresponding to the input function $u(t)$, as $x(t, x_0, u(t))$. In a similar way, let us denote as $y(t, x_0, u(t)) = h(x(t, x_0, u(t)))$ the corresponding output.

Many results have been reported on the design of observers for system (1) [6]. One of the most widely used is the method of the high-gain observer, which is based on a transformation into observability normal form, by way of the observability map. Two conditions must be met: the observability map must at least be a semi-diffeomorphism (uniform observability), and the normal form should have a locally Lipschitz continuous right-hand side [8]. In this paper we propose a methodolgy to design practical observers for uniformly observable systems that may not satisfy the second requirement.

In the next section, the basic methodology involved in the design of continuous high-gain observers will be briefly recalled and its basic shortcoming will be stated. Section 3 presents the Propositions that theoretically sustain the main result, which is presented in Section 4. An example is then worked out in Section 5. Finally, some conclusions are given.

2 High-gain continuous observers

Let us initially recall some notions in the design of high-gain observers. We will also concentrate on autonomous systems, later extending the results to forced systems such as (1).

Consider the autonomous system described by

$$\left.\begin{array}{rcl} \dot{x} & = & f(x)\ , \quad x\,(0)=x_0 \\ y & = & h(x) \end{array}\right\} \qquad (2)$$

where $x \in M \subset \mathbb{R}^n$, $y \in \mathbb{R}$ and f and h are sufficiently smooth[1] on M. Let us denote the solution of (2) which passes through x_0 at $t = 0$ as $x(t, x_0)$. In a similar way, denote $y(t, x_0) = h(x(t, x_0))$ as the output.

Consider another dynamical system,

$$\dot{\hat{z}} = \hat{f}(\hat{z}, y)\ , \quad \hat{z}\,(0) = \hat{z}_0 \qquad (3)$$

$$\hat{x} = \hat{h}(z, y) \qquad (4)$$

where $z \in \Omega \subset \mathbb{R}^n$, $\hat{f} : \Omega \times \mathbb{R} \to \mathbb{R}^n$, and $\hat{h} : \Omega \times \mathbb{R} \to \mathbb{R}^n$. Define $\hat{z}(t, \hat{z}_0, y)$ as the solution to (3) passing thorugh $\hat{z}_0$ at $t = 0$.

System (3-4) is called an observer on $W \subset M$ if it satisfies: [8] (i) it has unique and global solutions, (ii) acts as a simulator whenever $x_0 = \hat{x}_0 \doteq \hat{h}\,(\hat{z}_0, h(x_0))$,

[1] By "sufficiently smooth" it is meant that all required partial derivatives are defined and continuous.

and (iii) there exists $V \subset \Omega$, such that $x_0 \in W$ and $\hat{z}_0 \in V$ imply that the trajectories $\hat{x}(t, \hat{x}_0)$ and $x(t, x_0)$ remain in W for all $t \geq 0$ and converge as $t \to \infty$.

Equation (3) is called the dynamical part of the observer, whereas (4) is called the static part. The system (2) is said to have a global observer if there is an observer on M, a semi-global observer if there is an observer on every compact subset of M, and a local observer at x^0 if there is an observer on a neighborhood of x^0. If the functions $\hat{f}$ and $\hat{h}$ are smooth, the observer is called smooth (C^∞), and if at least one of them is continuous, then it is said to be continuous (C^0).

For observer design purposes it is common to use a continuous, (at least once) differentiable, and invertible correspondence $\Phi : M \to \bar{M} \subset \mathbb{R}^n$, such that $z = \Phi(x)$ and system (2) is transformed into: [7]

$$\left.\begin{array}{rcl} \dot{z} &=& \bar{f}(z) \ , \quad z(0) = z_0 \\ y &=& \bar{h}(z) \end{array}\right\} \tag{5}$$

where $z \in \bar{M} \subset \mathbb{R}^n$, and

$$\left.\begin{array}{rcl} \bar{f}(z) &=& \left.\dfrac{\partial \Phi}{\partial x} f(x)\right|_{x=\Phi^{-1}(z)} \\ \bar{h}(z) &=& h\left(\Phi^{-1}(z)\right) \end{array}\right\} \tag{6}$$

The dynamical part of an observer for (2) is implemented with an identity observer for the transformed system (5)[2]. One possibility is a Luenberger observer:

$$\dot{\hat{z}} = \bar{f}(\hat{z}) + l\,(\hat{z})\left(y - \bar{h}(\hat{z})\right) \ , \quad \hat{z}(0) = \hat{z}_0 \ . \tag{7}$$

The inverse of the mapping Φ^{-1} can be used as the static part:

$$\hat{x} = \Phi^{-1}(\hat{z}) \ . \tag{8}$$

A standard assumption is that Φ is a *diffeomorphism* (smooth with a smooth inverse), which guarantees a smooth transformed system (5), with unique solutions that correspond one-to-one to those of the original system (2). Furthermore, the convergence of trajectories for the transformed system carries on to that of the original one. In this case, the observer obtained is also smooth, and by derivating the output equation (8), an identity observer for system (2) can be directly contructed.

Xia and Zeitz [8] have shown that this methodology is still applicable if Φ is assumed to be a *semi-diffeomorphism*, where just continuity (not smoothness) of the inverse transformation is required. The transformed system (5) can thus be smooth or continuous. If (5) has the property of uniqueness of solutions, a one-to-one correspondence (T-equivalence) between trajectory solutions of (2) and (5) can be established. If additionally Φ^{-1} is uniformly continuous on $\bar{M}$, asymptotic convergence of $\hat{z}(t, \Phi(\hat{x}_0), y)$ to $\Phi(x(t, x_0))$ implies that of $\hat{x}(t, \hat{x}_0)$ to $x(t, x_0)$.

[2] This only makes sense whenever this is easier than for the original system.

A particularly interesting result is obtained when the smooth observability map $q : M \to \mathbb{R}^n$ of system (2)

$$q(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ \overset{(n-1)}{y} \end{bmatrix} \tag{9}$$

is used as the transformation map Φ. If system (2) is *uniformly observable* (i.e. q is injective form M onto its range), then $q(x)$ is at least a semi-diffeomorphism (see [8, Lemma 3]).

In this case $z = q(x)$ transforms system (2) into *observability normal form*: [9]

$$\left.\begin{array}{rcl} \dot{z} &=& f^*(z) = \begin{bmatrix} z_2 \\ z_3 \\ \vdots \\ \varphi(z) \end{bmatrix} \\ y &=& z_1, \end{array}\right\} \tag{10}$$

where $z \in N = q(M)$, and $\varphi(z) = L_f^n h\left(q^{-1}(z)\right)$ can be smooth or just continuous. Now consider the following Lemma:

Lemma 1 ([8]) *Let system (2) be uniformly observable on M; denote $N = q(M)$. If $L_f^n h \circ q^{-1}(z)$ is* Lipschitz continuous *on a subset $N' \subset N$ and $q^{-1}(z)$ is uniformly continuous on N', then (2) admits a continuous observer on $M' = q^{-1}(N')$ with gain vector $l \in \mathbb{R}^n$*

$$\left.\begin{array}{rcl} \dot{\hat{z}} &=& f^*(\hat{z}) - l[\hat{z}_1 - y] \\ \hat{x} &=& q^{-1}(\hat{z}) \end{array}\right\} \tag{11}$$

One possible method to calculate the observer gain l for (11) is the method of the *high-gain observer* proposed by Gauthier et al. [2], where $l = S_\infty(\theta)^{-1} c$ for some sufficiently large θ, $c^T = [1, 0, \ldots, 0]$ (i.e. $y = c^T z$), and $S_\infty(\theta)$ is the stationary solution of $\dot{S}_t(\theta) = -\theta S_t(\theta) - A^T S_t(\theta) - S_t(\theta) A + c c^T$, with the matrix A having components $A_{i,j} = \delta_{i,j-1}$.

Although the method described here allows the design of observers for a great class of nonlinear systems, it is not applicable to many uniformly observable systems. The composition of a smooth function and a continuous one ($L_f^n h \circ q^{-1}(z)$) is in general continuous but not Lipschitz.

Example 1 *Consider the following first order system:*

$$\begin{array}{rcl} \dot{x} &=& k \\ y &=& x^p, \end{array} \tag{12}$$

where $k \in \mathbb{R} \setminus \{0\}$ and $p \geq 3$ is an odd integer. Its observability normal form is

$$\begin{aligned} \dot{z} &= kp\, z^{(p-1)/p} \\ y &= z. \end{aligned}$$

Note $\varphi(z) = kp\, z^{(p-1)/p}$ is non-Lipschitz continuous. There is also no uniqueness of solutions, since $z(t) \equiv 0$ and $z(t) = (k\,t)^p$ are two solutions for the initial condition $z(0) = 0$. Therefore, a continuous observer using Lemma 1 cannot be constructed.

The purpose of this paper is to extend the observer design method to the whole class of uniformly observable systems. Since the function φ of the observability normal form (10) is in general just continuous, it could lack unicity of solutions, as in the previous Example 1. We will introduce some concepts to relate the unique trajectories of the original system with the set of possible trajectories of the observability normal form.

Moreover, in case (10) has unique solutions it is well known that it is structurally observable, since its observability map is the identity. But what does observability mean if there is no uniqueness of solutions?

Instead of insisting on the construction of an asymptotic observer, we will take a more practical perspective by designing approximate observers for non-Lipschitz observability normal forms (10), such that the trajectories of the observer converge to a small ball around the true system trajectories, and one can make the radius of this ball as small as desired.

3 Trajectory semi-equivalence and observability

Recall that a system is called *observable* if output indistiguishability of two initial states implies the identity of these two initial states [4]. For systems with unique solutions, observability implies a one-to-one correspondence between output and state trajectories. However, for systems with no unique solutions, for the same initial condition two different state trajectories might be solutions, but if their output trajectories are the same, the system would then still be called observable. A stronger condition than observability is thus needed.

Definition 1 *System (2) is called* trajectory observable *if every output trajectory corresponds to one and only one state trajectory.*

This concept reduces to observability for systems with unique solutions.

Proposition 1 *A system in observability normal form (10) is structurally trajectory observable.*

Proof: It is obvious that two different output trajectories correspond to different state trajectories, since $y = z_1$. Now we will prove that two different state trajectories $z\left(t, {z_0}^{(1)}\right) \neq z\left(t, {z_0}^{(2)}\right)$ generate different output trajectories. The *unique* solution to the scalar differential equation $\dot{z}_{k-1} = z_k$ is

$$z_{k-1}\left(t, z_0\right) = z_{0,k-1} + \int_0^t z_k\left(\tau, z_0\right) d\tau,$$

which, for different $z_k^{(i)}\left(t, {z_0}^{(i)}\right)$, $i = 1, 2$, will result in distinct scalar trajectories $z_{k-1}^{(i)}\left(t, {z_0}^{(i)}\right)$, because $z_k^{(i)}\left(t, {z_0}^{(i)}\right)$ are absolutely continuous functions of time and must be different in a set with measure not zero. Therefore their integrals are also different. Different $z_k^{(i)}\left(t, {z_0}^{(i)}\right)$ will iteratively imply different $z_1^{(i)}\left(t, {z_0}^{(i)}\right)$, which means $y^{(1)}\left(t, {z_0}^{(1)}\right) \neq y^{(2)}\left(t, {z_0}^{(2)}\right)$. In observability normal form, non-unicity of solutions can only occur for the (scalar) differential equation $\dot{z}_n = \varphi(z)$. Even if this happens for some initial condition, the previous discussion shows that their output trajectories will be different. ∎

System (5)[3] may have non unique solutions whereas the plant (2) has unique solutions. To construct an observer, we need some correspondence between their state trajectories. We will extend the concept of *trajectory equivalence* of [8] to do so.

Let $Z(t, z_0)$ represent the set of solutions (state trajectories) to system (5) passing through z_0 at $t = 0$, and $Y(t, z_0)$ the corresponding set of output trajectories.

Definition 2 *System (2) is said to be* trajectory semi-equivalent *to system (5) if there exists an injective mapping $\Phi : M \to \bar{M}$, such that if $x(t, x_0)$ and $h\left(x(t, x_0)\right)$ are the unique state and output trajectories of system (2), then $\Phi(x(t, x_0))$ and $\bar{h}\left(\Phi(x(t, x_0))\right)$ are state and output trajectories of system (5).*
Additionally, for every $z_0 \in \bar{M}$, there exists a unique state trajectory $z(t, z_0) \in Z(t, z_0)$ of (5), such that $\Phi^{-1}(z(t, z_0))$ is the unique state trajectory of system (1), and $h\left(x(t, \Phi^{-1}(z_0))\right) = \bar{h}\left(z(t, z_0)\right) \in Y(t, z_0)$.

When $z_0 = \Phi(x_0)$, this equivalence will be denoted as $x(t, x_0) \rightleftarrows z(t, z_0)$, where $z(t, z_0) \in Z\left(t, \Phi(x_0)\right)$ is unique for every $x(t, x_0)$.

Proposition 2 *If the observability map $q\left(x\right)$ (9) is a semi-diffeomorphism from M onto N, (the system*

[3] The observability normal form (10) is a special case of (5).

is uniformly observable), then system (2) is trajectory semi-equivalent to system (10).

Proof: From (6), it is obvious that if $x(t, x_0)$ is a state trajectory of (2), then $q(x(t, x_0))$ is a state trajectory of (10) passing through $q(x_0)$ at $t = 0$, thus proving the first requirement. To prove the second requirement, consider the state trajectory $x\left(t, q^{-1}(z_0)\right)$ of system (2). Using the first part of this proof, $q\left(x\left(t, q^{-1}(z_0)\right)\right)$ is a state trajectory of system (10), passing through $q\left(q^{-1}(z_0)\right) = z_0$ at $t = 0$. Notice the corresponding output trajectories in both coordinate systems are equal (see (6)). Therefore, the trajectory

$$z(t, z_0) = q\left(x\left(t, q^{-1}(z_0)\right)\right) \in Z(t, z_0) \ ,$$

is unique because of trajectory observability. Hence,

$$q^{-1}(z(t, z_0)) = x\left(t, q^{-1}(z_0)\right) \tag{13}$$

is a state trajectory of system (2). ∎

Proposition 3 *Let $\zeta(t)$ be some time function evolving on N. If q is a semi-diffeomorphism from M onto N with q^{-1} uniformly continuous on N, such that system (2) is trajectory semi-equivalent to system (10), then*

$$\|\zeta(t) - z(t, z_0)\| \to 0 \tag{14}$$

implies

$$\left\|q^{-1}(\zeta(t)) - x\left(t, q^{-1}(z_0)\right)\right\| \to 0 \tag{15}$$

whenever $x\left(t, q^{-1}(z_0)\right) \rightleftarrows z(t, z_0)$.

Furthermore, for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$, such that

$$\lim_{t\to\infty} \|\zeta(t) - z(t, z_0)\| < \epsilon$$

implies

$$\lim_{t\to\infty} \left\|q^{-1}(\zeta(t)) - x\left(t, q^{-1}(z_0)\right)\right\| < \delta \ .$$

Proof: Trajectory semi-equivalence implies a one-to-one correspondence between valid state trajectories. Uniform continuity of q^{-1} implies from (14) that

$$\|q^{-1}(\zeta(t)) - q^{-1}(z(t, z_0))\| \to 0 \ .$$

Using the proof from Proposition 2 we know that equation (13) is fulfilled; hence (15) also. The second part also follows from the uniform continuity of q^{-1}. ∎

4 Approximate high-gain observers

Proposition 3 implies that an identity observer for the observability normal form, together with the algebraic map q^{-1}, may consitute the dynamic and static parts, respectively, of an observer for the original system. Since designing an identity observer for a non-Lipschitz normal form may not be easy, we will instead consider the design of an ϵ-*approximate observer*, based on the high-gain approach.

Definition 3 *System (3-4) is called an ϵ-*approximate observer*, $\epsilon > 0$, of system (2) on a subset $W \subset M$ if (i) it has unique and global solutions; (ii) does not necessarily behave as a simulator when $x_0 = \hat{x}_0 \doteq \hat{h}(\hat{z}_0, h(x_0))$; and (iii) there exists a subset $V \subset \Omega$ and $T \geq 0$, such that $x_0 \in W$ and $z_0 \in V$ imply that $x(t, x_0) \in W$ and $\hat{x}(t, \hat{x}_0, y) \in W$ for all $t \geq 0$, and additionally*

$$\|\hat{x}(t, \hat{x}_0, h(x(t, x_0))) - x(t, x_0)\| \leq \epsilon \quad \forall\, t \geq T. \tag{16}$$

Consider the following system:

$$\dot{\hat{z}} = \tilde{f}^*(\hat{z}) - S_\infty(\theta)^{-1} c(c^T \hat{z} - y) \ , \quad \hat{z}(0) = \hat{z}_0 \ , \tag{17}$$

where $\tilde{f}^*(z) \doteq [z_2, z_3, \cdots, \tilde{\varphi}(z)]^T$, with $\tilde{\varphi} : N \to \Re$ *Lipschitz* continuous with respect to z, and

$$\sup_{z\in\Gamma} |\tilde{\varphi}(z) - \varphi(z)| \leq \delta$$

on any compact subset $\Gamma \subset N$. Finding such a $\tilde{\varphi}$ is possible for every continuous function φ, for every δ, and for every compact set Γ because of the classical Stone-Weierstrass Theorem (see for example [5]).

Notice $\tilde{f}^*$ can also be written as

$$\tilde{f}^*(z) = f^*(z) + \Delta f(z) \tag{18}$$

where $\Delta f \doteq \tilde{f}^* - f^* = [0, \cdots, 0, \Delta\varphi]^T$, with $\Delta\varphi \doteq \tilde{\varphi} - \varphi$ non-Lipschitz continuous, and $\|\Delta f(z)\| = |\Delta\varphi(z)| \leq \delta$ for any z in a compact subset $\Gamma \subset N$.

Theorem 1 *For every $\epsilon > 0$, a non-Lipschitz continuous observability normal form (10) admits an ϵ-approximate high-gain observer of the form (17).*

Proof: Define $e(t) \doteq \hat{z}(t, \hat{z}_0, y(t)) - z(t, z_0)$ and $e(0) = e_0$, where $z(t, z_0)$ and $\hat{z}(t, \hat{z}_0, y(t))$ are, respectively, the solutions of (10), and (17) driven by the output of (10).

Following the proof of the high-gain observer (see [2]) and using the same Lyapunov arguments for the non-Lipschitz dynamics of $e(t)$ (this is allowed since Lyapunov theory is also valid for continuous systems [1]),

$$\frac{d}{dt}\|e\|_{S_\infty(\theta)} \leq -\gamma\|e\|_{S_\infty(\theta)} + \|\Delta f\|_{S_\infty(\theta)}$$

where $\|x\|_S^2 \doteq x^T S x$ and $\gamma = \frac{\theta}{2} - C_1 k n \sqrt{S}$, with n the order of the system, k the Lipschitz constant of $\tilde{\varphi}$, $S \doteq \sup_{i,j} |S_\infty(1)_{i,j}| = S_\infty(1)_{n,n}$, and $C_1 = (\lambda_{\min}[S_\infty(1)])^{-\frac{1}{2}}$, i.e. $\|x\| \le C_1 \|x\|_{S_\infty(1)}$.

The previous (scalar) differential inequality yields

$$\begin{aligned}\|e\|_{S_\infty(\theta)} \;\le\; & \exp(-\gamma t)\left(\|e_0\|_{S_\infty(\theta)} - \|\Delta f\|_{S_\infty(\theta)}\right) \\ & + \|\Delta f\|_{S_\infty(\theta)}.\end{aligned}$$

Hence, exponentially, for some $T \ge 0$ and $\beta > 1$

$$\|e\|_{S_\infty(\theta)} \le \beta \|\Delta f\|_{S_\infty(\theta)} \quad \forall\, t \ge T \;. \tag{19}$$

As $\|\Delta f\| \to 0$, the Lipschitz constant k of $\tilde{\varphi}$ increases, forcing the observer to need a higher gain θ in order to achieve $\gamma > 0$. We need to show that despite this, the ultimate bound on $\|e\|$ does not become too big.

Since

$$S_\infty(\theta)_{i,j} = \frac{1}{\theta^{i+j-1}} S_\infty(1)_{i,j} \;, \tag{20}$$

then

$$\|\Delta f\|_{S_\infty(\theta)}^2 = S_\infty(\theta)_{n,n} |\Delta\varphi|^2 = \frac{S}{\theta^{2n-1}} |\Delta\varphi|^2. \tag{21}$$

Consider $e = [e_1, e_2, \cdots, e_n]^T$ and $\zeta_i \doteq e_i/\theta^i$. Using (20), it can be shown that $\|e\|_{S_\infty(\theta)}^2 = \theta \|\zeta\|_{S_\infty(1)}^2$, which together with (21) and (19) yields

$$\|\zeta\|_{S_\infty(1)}^2 = \frac{1}{\theta}\|e\|_{S_\infty(\theta)}^2 \le \frac{\beta^2}{\theta}\|\Delta f\|_{S_\infty(\theta)}^2 = \frac{\beta^2 S}{\theta^{2n}} |\Delta\varphi|^2 \;.$$

Because $\|\zeta\| \le C_1 \|\zeta\|_{S_\infty(1)}$, then $\|\zeta\| \le \frac{\beta C_1 \sqrt{S}}{\theta^n} |\Delta\varphi|$. Since $|\zeta_i| = |e_i|/\theta^i \le \|\zeta\|$ and $|\Delta\varphi| \le \delta$,

$$|e_i| \le \frac{\beta C_1 \sqrt{S}}{\theta^{n-i}} \delta \;.$$

As $\delta \to 0$, even if θ must grow, the final bounds on each component e_i do not grow accordingly, since β is a fixed constant and S and C_1 depend only on the system dimension. In conclusion

$$\|e\| \le \beta C_1 \sqrt{S} \|\Theta\|\, \delta \le K_n \delta = \epsilon$$

where $\Theta \doteq \left[\frac{1}{\theta^{n-1}}, \cdots, \frac{1}{\theta}, 1\right]^T$. Since $1 < \|\Theta\| \le \sqrt{n}$ for $\theta \ge 1$, then $K_n = \beta C_1 \sqrt{nS}$. Given ϵ, choose $\delta \le \epsilon/K_n$ and for a big enough θ, requirement (16) of the definition is satisfied. ∎

Remark 1 *The approximate observer (17) converges to a neighborhood of* any *trajectory of the non-Lipschitz normal form (10), although it is based on a system with unique solutions!*

Theorem 2 *For every uniformly observable nonlinear system (2) an ϵ-approximate high-gain observer can be designed for any $\epsilon > 0$.*

Proof: Theorem 1 can be used to design an identity ϵ-approximate observer for its observability normal form. Because the observer is driven by the unique output of system (2), the solution to which $\hat{z}(t, \hat{z}_0, y(t))$ will approximately converge is the one corresponding to $x(t, q^{-1}(z_0))$ via trajectory semi-equivalence. Proposition 3 completes the proof. ∎

Remark 2 *These results can easily be extended to include forced systems such as (1), whenever the input function is sufficiently smooth, by using the input dependent observability map $q\left(x, u, \dot{u}, \ldots, \overset{(n-2)}{u}\right)$ [9, 10].*

5 Example

Instead of returning to Example 1, for which a trivial continuous observer can be easily designed (not high-gain, though) (see [3]), consider the following second order uniformly observable system:

$$\left.\begin{aligned} \dot{x}_1 &= a x_1 &&, \quad x_1(0) = x_{1,0} \\ \dot{x}_2 &= -a x_1 - x_2^3 &&, \quad x_2(0) = x_{2,0} \\ y &= x_1 + x_2 && \end{aligned}\right\} \tag{22}$$

The observability map $z = q(x) = \binom{x_1 + x_2}{-x_2^3}$ yields

$$\left.\begin{aligned} \dot{z}_1 &= z_2, && z_1(0) = z_{1,0} \\ \dot{z}_2 &= 3\left(a z_2 + (a z_1 - z_2)\sqrt[3]{z_2^2}\right), && z_2(0) = z_{2,0} \\ y &= z_1 \;. && \end{aligned}\right\} \tag{23}$$

The function $\varphi(z) = 3\left(a z_2 + (a z_1 - z_2)\sqrt[3]{z_2^2}\right)$ is non-Lipschitz continuous, and the system has multiple solutions for some initial conditions. The unique equilibrium point of (22) is $x = 0$, while (23) has multiple equilibria at $z_2 = 0$. Consider the initial condition $z_0 = [\bar{z}_{1,0}, 0]^T$; then $z(t) \equiv z_0$ is a solution to (23). Consider now $x(t, x_0)$, where $x_0 = q^{-1}(z_0)$. It is easy to see that $q(x(t, x_0))$ does not remain constant for all $t > 0$, but it is another solution to (23) for the same intial condition z_0.

To construct the ϵ-approximate observer, φ must be approximated by a Lipschitz function $\tilde{\varphi}$ in the vicinity of $z_2 = 0$. A possibility is:

$$\tilde{\varphi}(z) = \begin{cases} 3\left(a z_2 - \sqrt[3]{z_2^5} + a z_1 \sqrt[3]{z_2^2}\right) & ; |z_2| \ge \mu \\ 3\left(a z_2 - \sqrt[3]{z_2^5} + a z_1 \sqrt[3]{\mu^2}\, \psi\left(\frac{z_2}{\mu}\right)\right) & ; |z_2| < \mu \end{cases}$$

where $\psi(m) = m^2\left(\frac{5}{3} - \frac{2}{3}m^2\right)$ is a fourth order polynomial that approximates $g(m) = \sqrt[3]{m^2}$. In this case

$$\delta = 3 a Z_1 \sqrt[3]{\mu^2} \sup_{|m|<1} \left|\psi(m) - \sqrt[3]{m^2}\right| \approx 0.9125 a Z_1 \sqrt[3]{\mu^2} \;,$$

where $Z_1 = \sup_\Gamma |z_1|$ for some subset $\Gamma \subset N$; δ can be made as small as desired within Γ, by letting $\mu \to 0$.

In Figure 1 some simulations are shown, using an ϵ-approximate high-gain observer as (17) as the dynamic part and

$$\hat{x} = q^{-1}(\hat{z}) = \begin{bmatrix} \hat{z}_1 + \sqrt[3]{\hat{z}_2} \\ -\sqrt[3]{\hat{z}_2} \end{bmatrix}$$

as the static part. Simulations were performed for two different approximations (generated by varying μ), with system (22) unstable ($a > 0$).

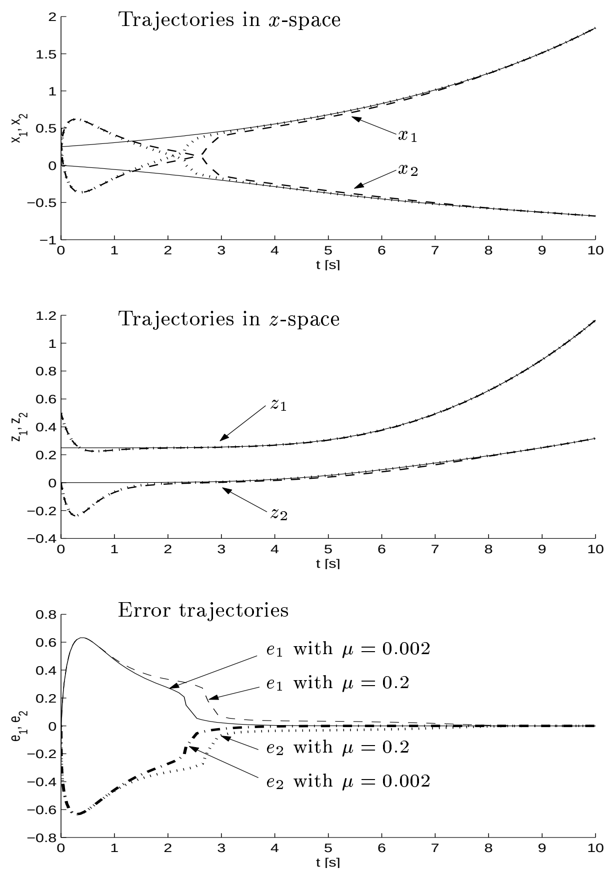

Figure 1: Trajectories of the system (—) and the observer when $x_0 = [0.25, 0]^T$, $\hat{x}_0 = [0.5, 0]^T$, $a = 0.2$, and $\theta = 3$, with two approximations: $\mu = 0.2$ (- -) and $\mu = 0.002$ ($\cdots$).

Notice from the simulations that as soon as $\hat{z}_2(t)$ leaves the "approximation zone" $|z_2| < \mu$, convergence improves considerably (see the error trajectories $e_i(t) = x_i(t) - \hat{x}_i(t)$ for $\mu = 0.2$ around $t = 7.5$). Still, a small error remains (not perceptible in the figure). Of course, in this unstable case $z_1(t) \to \infty$ as $t \to \infty$, and the observer cannot guarantee (approximate) convergence when $z(t)$ leaves some subset Γ'.

6 Conclusions

The usual methodolgy for the design of continuous high-gain observers has been extended to be applicable to the whole class of uniformly observable systems. Whenever the observability normal form results non-Lipschitz continuous, an ϵ-approximate observer may be designed. Ultimate boundedness of the observation error on a ball with radius as small as desired is thus acheived. The methodolgy can also be easily extended to forced systems with sufficiently smooth inputs.

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