Fault Detection System Design Based on a New Trade-off Strategy

S. X. Ding* , P. M. Frank**, E. L. Ding§ and T. Jeinsch*
* Dept. of Electrical Engineering, University of Applied Sciences Lausitz
01958 Senftenberg, Germany, E-mail: ding@e-technik.fh-lausitz.de
** Dept. of Measurement and control, University of Duisburg
47048 Duisburg, Germany, Email: p.m.frank@uni-duisburg.de
§ Dept. of Physical Engineering, University of Applied Sciences Gelsenkirchen
45877 Gelsenkirchen, Germany, Email: ed@pt.fh-gelsenkirchen.de

Abstract: In this paper, problems related to the fault detection in dynamic systems with unknown inputs are studied. Instead of designing fault detection systems from the viewpoint of increasing the system robustness against unknown inputs and the sensitivity to the faults, an approach is proposed, which allows us to design fault detection systems based on a trade-off between the false alarm rate and the missed detection rate. A further study on the relationships between the approach proposed and the existing $H_2, H_{\infty/\infty}$ and $H_{\infty/\min}$ optimization approaches demonstrates that the approach proposed in this paper provides us not only with a unified solution to the existing approaches but also with the best solution among these approaches in view of minimizing the missed detection rate under a given false alarm rate.

1 Introduction and background

In this contribution, we consider fault detection problems for linear time-invariant (LTI) dynamic systems described by

$$y(s) = G_u(s)u(s) + G_d(s)d(s) + G_f(s)f(s) \qquad (1)$$

where $G_u(s), G_f(s), G_d(s)$ are known transfer function matrices from the input vector $u(s) \in \mathbf{R}^{\mathbf{k_u}}$, the fault vector $f(s) \in \mathbf{R}^{\mathbf{k_f}}$ to be detected and from the unknown input vector $d(s) \in \mathbf{R}^{\mathbf{k_d}}$ to the output vector $y(s) \in \mathbf{R}^{\mathbf{m}}$. d is unknown but bounded by $\|d\|_2 \leq \Delta_d$. To simplify the description, we denote the minimal state-space realization of $G_u(s), G_d(s)$ and $G_f(s)$ with $(A, B, C, D), (A, E_d, C, F_d)$ and (A, E_f, C, F_f) and assume that for all ω

$$rank\left(G_d(j\omega)\right) = rank\left(G_d(s)\right) = k_d \qquad (2)$$

1.1 A brief review of fault detection methods

A typical FD system consists of a residual generator and a residual evaluation stage including an evaluation function and a threshold [2], [7], [8], [10], [11], [12], [15], [16]. For the purpose of residual generation, we use an LTI residual generator, which can be expressed in terms of a general form

$$r(s) = R(s)\left(\hat{M}_u(s)y(s) - \hat{N}_u(s)u(s)\right) \qquad (3)$$

where $(\hat{M}_u(s), \hat{N}_u(s))$ is a left coprime factorization of $G_u(s)$, i.e. $G_u(s) = \hat{M}_u^{-1}(s)\hat{N}_u(s)$ and transfer function matrix $R(s)$, also called post-filter, is a parameterization matrix being arbitrarily selectable in $\mathbf{RH}_\infty$. It has been shown by Ding and Frank [9] that the dynamics of the residual generator (3) is governed by

$$\begin{aligned} r(s) &= R(s)\left(\bar{G}_d(s)d(s) + \bar{G}_f(s)f(s)\right), \qquad (4)\\ \bar{G}_d(s) &= \hat{M}_u(s)G_d(s), \bar{G}_f(s) = \hat{M}_u(s)G_f(s) \end{aligned}$$

In this contribution, the H_2-norm of residual vector $r(s)$ is used as the residual evaluation function which evaluates the energy change in $r(s)$ caused by $f(s)$ or $d(s)$ and is presented either in the time domain

$$\|r\|_2 = \left(\int_0^\infty r^\top(t)r(t)dt\right)^{1/2} \qquad (5)$$

or in the frequency domain

$$\|r\|_2 = \left(\frac{1}{2\pi}\int_{-\infty}^\infty r^\top(-j\omega)r(j\omega)d\omega\right)^{1/2} \qquad (6)$$

The last step to a successful fault detection is the establishment of a logic decision unit. In this contribution, we consider a simple but mostly used logic:

$$\begin{aligned} &\text{If } \|r\|_2 > J_{th} \text{ (threshold)} \Longrightarrow \text{then fault} \qquad (7)\\ &\text{If } \|r\|_2 \leq J_{th} \text{ (threshold)} \Longrightarrow \text{then no fault} \qquad (8) \end{aligned}$$

The main task of designing a fault detection system consists in the selection of parameter matrix $R(s) \in$

$\mathbf{RH}_\infty$ and threshold J_{th}. For the first task, a widely adopted scheme is to solve an optimization problem

$$\min_{R\in\mathbf{RH}_\infty} \frac{\left\|R(s)\hat{M}_u(s)G_d(s)\right\|}{\left\|R(s)\hat{M}_u(s)G_f(s)\right\|} \tag{9}$$

where $\|\cdot\|$ stands for some norm or a function which is used to measure the "size" of a transfer function matrix [1], [2], [7], [10]. The basic idea behind (9) is to maximize the influence of $f(s)$ on $r(s)$ and simultaneously to minimize the one of $d(s)$ on $r(s)$. The calculation of J_{th} often follows the definition

$$J_{th} = \sup_{d,f=0} \|r\|_2 = \|R(s)\bar{G}_d(s)\|_\infty \sup_d \|d\|_2 \tag{10}$$

Note that if

$$rank \begin{bmatrix} G_d(s) & G_f(s) \end{bmatrix} > rank\,(G_d(s)) = k_d$$

then there exists a $R_o(s)$ such that

$$\min_{R\in\mathbf{RH}_\infty} \frac{\left\|R(s)\hat{M}_u(s)G_d(s)\right\|}{\left\|R(s)\hat{M}_u(s)G_d(s)\right\|} = \frac{\left\|R_o(s)\hat{M}_u(s)G_d(s)\right\|}{\left\|R_o(s)\hat{M}_u(s)G_d(s)\right\|} = 0$$

and thus the threshold can be set to be zero. This is the so-called full decoupling from $d(s)$ [2], [7].

1.2 Problem formulation

Without doubt, the control engineering community has made a decisive contribution to the establishment of the model based FDI theory and technology. It is therefore a logic result that many model based FDI problems are handled in the context of control theory. So are the concepts robustness and sensitivity. What is the real idea behind these two concepts? They are indeed the "translation" of two essential requirements on a fault diagnosis system: *false alarm rate and missed detection rate*. False alarms are caused by $d(s)$, and to reduce them threshold is introduced, which leads in turn to missed detection. In fact, the real and most difficult task of designing a fault detection system is to find out a suitable trade-off between the false alarm rate and the missed detection rate.

It is evident that setting J_{th} according to (10) prevents false alarms. Thus, choosing $R(s)$ by solving optimization problem (9) is indeed an indirect optimization of the missed detection rate on the assumption that the false alarm rate is zero. This way of handling FD design problem seems elegant but has two practical problems: (i) The nice performance that the false alarm rate equals zero is achieved at the cost of missed detection rate. This problem may become more serious if the probability that $\|d\|_2$ reaches its maximum is very small; (ii) Choosing $R(s)$ by solving optimization problem (9) does not ensure a minimization of the missed detection rate. The main objective of this paper is to develop a new approach to the design of fault detection systems from the viewpoint of achieving a suitable trade-off between the false alarm rate and missed detection rate. Motivated by the above discussion, we shall try to solve the following problems: (i) Establishment of a trade-off relationship between the false alarm rate and the missed detection rate; (ii) Direct design of $R(s)$ to minimize the missed detection rate for a given false alarm rate.

2 A new design approach

2.1 Formulation of design problem

To begin with, we introduce $\delta_d \leq \sup_d \|d\|_2 = \Delta_d$ and define the threshold as follows

$$J_{th} = \sup_{\|d\|_2\leq\delta_d, f=0} \|r\|_2 \tag{11}$$

It then turns out $J_{th} = \|R(s)\bar{G}_d(s)\|_\infty \delta_d$. Since for any $d(s), \Delta_d \geq \|d\|_2 > \delta_d$ we may have $\|R(s)\bar{G}_d(s)d(s)\|_2 > \|R(s)\bar{G}_d(s)\|_\infty \delta_d$, which leads to, in fault-free case,

$$\|r\|_2 = \|R(s)\bar{G}_d(s)d(s)\|_2 > J_{th}$$

and hence results in a false alarm. Thus, the false alarm rate may be nonzero and its value depends on the probability that $\|d\|_2 > \delta_d$.

Recall that following detection logic (7)-(8) a fault $f(s)$ can be detected if and only if $\|r\|_2 > J_{th}$. As a result, we claim that for a given $d(s)$ a fault $f(s)$ can be detected if and only if

$$\|R(s)(\bar{G}_d(s)d(s) + \bar{G}_f(s)f(s))\|_2 > \|R(s)\bar{G}_d(s)\|_\infty \delta_d \tag{12}$$

We now introduce two sets: (i) the set of detectable faults $\Omega_{R,J_{th}}(d)$

$$\Omega_{R,J_{th}}(d) = \{f \mid \|r(s)\|_2 > J_{th}\} \tag{13}$$

and (ii) the set of undetectable faults $\Omega^\perp_{R,J_{th}}(d)$

$$\Omega^\perp_{R,J_{th}}(d) = \{f \mid \|r(s)\|_2 \leq J_{th}\} \tag{14}$$

Since the missed detection rate is proportional to the number of undetectable faults, minimizing the missed detection rate under a given false alarm rate is equivalent to minimize the dimension of set $\Omega^\perp_{R,J_{th}}$. Note that

$$\Omega = \{f \mid f \neq 0\} = \Omega^\perp_{R,J_{th}}(d) \cup \Omega_{R,J_{th}}(d)$$

we further have

$$\min_{R(s)\in\mathbf{RH}_\infty} \dim \Omega^\perp_{R,J_{th}}(d) \Leftrightarrow \max_{R(s)\in\mathbf{RH}_\infty} \dim \Omega_{R,J_{th}}(d)$$

Following this, we formulate the problem of designing fault detection systems as finding $R(s) \in \mathbf{RH}_\infty$ such that for all d the dimension of the set of detectable faults reaches maximum, i.e.

$$\forall d, \max_{R(s)\in\mathbf{RH}_\infty} \dim \Omega_{R,J_{th}}(d) \tag{15}$$

2.2 A solution

We now outline the basic idea and present an approach to the solution of optimization problem (15). To this end, we first assume that $rank\,(G_d(s)) = k_d = m$, i.e. a full decoupling from the unknown inputs is impossible, and consider condition (12). We begin with the so-called co-inner-outer factorization (CIOF) of transfer function matrix $\bar{G}_d(s), \bar{G}_d(s) = G_{do}(s)G_{di}(s)$, where $G_{di}(s)$ is the co-inner of matrix of $\bar{G}_d(s)$ and $G_{do}(s)$ is the co-inner and RH_∞-left-invertible. Setting $R(s) = Q(s)G_{do}^{-1}(s)$ yields

$$\begin{gathered}\|r(s)\|_2 - J_{th} = -\|Q(s)G_{di}(s)\|_\infty \delta_d + \\ \|Q(s)G_{do}^{-1}(s)\bar{G}_f(s)f(s) + Q(s)G_{di}(s)d(s)\|_2\end{gathered}$$

where $Q(s) \in \mathbf{RH}_\infty$ stands for an arbitrarily selectable matrix of an appropriate dimension. Note that

$$\begin{gathered}\|Q(s)G_{di}(s)\|_\infty = \|\,(Q(s)G_{di}(s))^*\,\|_\infty = \|Q(s)\|_\infty \\ \|Q(s)G_{do}^{-1}(s)\bar{G}_f(s)f(s) + Q(s)G_{di}(s)d(s)\|_2 \\ \le \|Q(s)\|_\infty \|G_{do}^{-1}(s)\bar{G}_f(s)f(s) + G_{di}(s)d(s)\|_2\end{gathered}$$

It turns out: for all $Q(s) \in \mathbf{RH}_\infty$

$$\begin{aligned}&\|r(s)\|_2 - J_{th} \le \\ &\|Q(s)\|_\infty \left(\|G_{do}^{-1}(s)\bar{G}_f(s)f(s) + G_{di}(s)d(s)\|_2 - \delta_d\right)\end{aligned} \tag{16}$$

Inequality (16) shows that condition

$$\|G_{do}^{-1}(s)\bar{G}_f(s)f(s) + G_{di}(s)d(s)\|_2 - \delta_d > 0 \tag{17}$$

is a necessary condition under which fault $f(s)$ becomes detectable. Note that (16) is expressed only in terms of the model parameters $G_{do}(s), G_{di}(s), \bar{G}_f(s)$ and $f(s)$ as well as $d(s)$, moreover no assumption on $R(s)$ has been made by the derivation, thus the following theorem holds.

Theorem 1 *Given system (1) and threshold (11), a fault $f(s)$ can then be detected only if (17) holds.*

Following Theorem 1, we know that increasing δ_d reduces the false alarm rate on the one side and makes detecting $f(s)$ more difficult and thus increases the missed detection rate on the other side. From this point of view, we say that (17) expresses a relationship between the false alarm rate and the missed detection rate and further, based on it, we are able to make a suitable trade-off between them.

Note that setting $Q(s) = I$ and therefore $R(s) = G_{do}^{-1}(s)$ leads to $\|r(s)\|_e - J_{th} = \|G_{do}^{-1}(s)\bar{G}_f(s)f(s) + G_{di}(s)d(s)\|_2 - \delta_d$. This means that (17) is also a sufficient condition for $f(s)$ to be detectable if $R(s)$ is set to be $G_{do}^{-1}(s)$. Using this result we are able to prove the following theorem.

Theorem 2 *$R^*(s) = G_{do}^{-1}(s)$ is the optimal solution of optimization problem (15).*

We would like to emphasize that the optimal solution $R^*(s)$ is independent of d and it ensures that for all d the dimension of the set of detectable faults reaches maximum.

2.3 Interpretation and generalization

In this subsection, we discuss the optimal solution derived above and then extend it to a more general case. We begin with the dynamics of the residual signal. Substituting the optimal solution $R^*(s)$ into (4) yields

$$r(s) = G_{do}^{-1}(s)\bar{G}_f(s)f(s) + G_{di}(s)d(s)$$

Recall that $\bar{G}_d(s) = G_{do}(s)G_{di}(s)$ and so

$$\bar{G}_d(j\omega)\bar{G}_d^*(j\omega) = G_{do}(j\omega)G_{do}^*(j\omega)$$

Hence, the frequency domain profile of the "amplitude" of $\bar{G}_d(j\omega)$ is uniquely determined by $G_{do}(j\omega)$. Moreover, for $f(s) = 0$

$$\begin{aligned}\|r(j\omega)\|_2 &= \|G_{di}(j\omega)d(j\omega)\|_2 \le \|d(j\omega)\|_2 \\ \text{and } J_{th} &= \|G_{di}(j\omega)\|_\infty \delta_d = \delta_d\end{aligned}$$

These facts reveal that in the residual signal the weighting factor for the unknown inputs equals or is smaller than one and the faults are weighted by $G_{do}^{-1}(j\omega)\bar{G}_f(j\omega)$.

It is very interesting to note that in the directions as well as at the frequencies where the influence of the unknown inputs on $y - \hat{y}$ is strong, the influence of the faults on the residual signal will be, due to $G_{do}^{-1}(j\omega)$, weakly weighted. In against, the influence of the faults on the residual signal will be more strongly weighted in directions and at frequencies where the influence of the unknown inputs on $y - \hat{y}$ is weak. This observation not only provides us with an interpretation of the optimal solution but also motivates us to extend the above result to the case

$$rank\,(G_d(s)) = k_d < m \tag{18}$$

i.e. a full decoupling from d is possible. Note that in this case there exists a RH_∞-matrix $P(s)$ so that

$$P(s)\bar{G}_d(s) = \begin{bmatrix} G_{d1}(s) \\ O \end{bmatrix} \in \mathbf{R}^{m\times k_d}, rank\,(G_{d1}(s)) = k_d$$

Let $G_{d1}(s) = G_{do1}(s)G_{di1}(s)$ be a co-inner-co-outer factorization, then we have

$$\begin{bmatrix} G_{do1}^{-1}(s) & O \end{bmatrix} P(s)\bar{G}_d(s) = G_{di1}(s)$$

With the aid of this result, we are able to prove the following theorem.

Theorem 3

$$\forall d \qquad \sup_{R(s)\in\mathbf{RH}_\infty} \dim\Omega_{R,J_{th}}(d) = \lim_{\epsilon\longrightarrow 0}\dim\Omega_{R^*,J_{th}}(d)$$

$$\textit{with} \qquad R^*(s) = \begin{bmatrix} G_{do1}^{-1}(s) & O \\ O & 1/\epsilon \end{bmatrix} P(s),\, J_{th} = \delta_d$$

It is very interesting to notice that

$$\begin{aligned} \begin{bmatrix} O & 1/\epsilon \end{bmatrix} P(s)\bar{G}_d(s) &= O \\ \begin{bmatrix} O & 1/\epsilon \end{bmatrix} P(s)\bar{G}_f(s)f(s) &= 1/\epsilon\hat{f}_2(s) \end{aligned}$$

This means that a fault $f(s)$, whose influence on $r(s)$ is different from that of $d(s)$, i.e.

$$R^*(s)\bar{G}_f(s)f(s) \notin \operatorname{Im}\left(R^*(s)\bar{G}_d(s)\right)$$

can be, independent of its size, detected. This fact is indeed equivalent to a perfect decoupling of $r(s)$ from $d(s)$, although a threshold different from zero is used. Hence, the solution given above also provides us with an alternative solution to the well-known perfect decoupling FD problem. Note, however, that the idea behind our solution is different from the known ones. In fact, we weight the part of faults, whose influence satisfies

$$\begin{aligned} R^*(s)\bar{G}_f(s)f(s) &\notin \operatorname{Im}\left(R^*(s)\bar{G}_d(s)\right) \Leftrightarrow \\ G_f(s)f(s) &\notin \operatorname{Im}\left(G_d(s)\right) \end{aligned} \tag{19}$$

very strongly by factor $1/\epsilon$ so that this part of faults can always be detected in spite of the existence of a threshold larger than zero.

3 Interconnections

In this section, we study the relationships between the design approach presented in this contribution and the known ones [7], [8], [10], [12], [15], [16], [19].

3.1 Relationship to perfectly decoupled FD

Under the concept *perfectly decoupled fault detection* (PDFD) we understand a residual generation satisfying

$$r(s) = Q(s)\left(G_d(s)d(s) + G_f(s)f(s)\right) = Q(s)G_f(s)f(s)$$

It is well-known that the existence condition for a PDFD is

$$rank \begin{bmatrix} G_d(s) & G_f(s) \end{bmatrix} > rank(G_d(s)) = k_d \tag{20}$$

In case that a PDFD is achievable, the threshold will be set equal to zero.

In order to compare the scheme presented in this contribution with the PDFD, we denote the set of all detectable faults using the PDFD with

$$\Omega_{Q,o}(d) = \{f \mid Q(s)G_f(s)f(s) > 0\}$$

The following theorem gives a comparison of these two approaches.

Theorem 4 $\forall d, \Omega_{Q,o}(d) \subseteq \Omega_{R^*,J_{th}}(d)$ *or* $\dim\Omega_{R^*,J_{th}}(d) \geq \dim\Omega_{Q,o}(d)$.

3.2 Relationship to H_2-optimal FD

For the case when a PDFD is not achievable, Ding et al. [3] proposed to design FD in the following H_2-optimization sense:

$$\max_{q(s)} \frac{\left\|q(s)\hat{N}_f(s)\right\|_2}{\left\|q(s)\hat{N}_d(s)\right\|_2},\, q(s)\in H_2^m \tag{21}$$

where it is assumed that

$$\begin{aligned} G_d(s) &= \hat{M}_d^{-1}(s)\hat{N}_d(s) = \hat{M}_u^{-1}(s)\hat{N}_d(s) \\ G_f(s) &= \hat{M}_f^{-1}(s)\hat{N}_f(s) = \hat{M}_u^{-1}(s)\hat{N}_f(s) \end{aligned}$$

The solution of optimization problem (21) is given by

$$\begin{aligned} \max_{q(s)} \frac{\left\|q(s)\hat{N}_f(s)\right\|_2}{\left\|q(s)\hat{N}_d(s)\right\|_2} &= \frac{\left\|q_{opt}(s)\hat{N}_f(s)\right\|_2}{\left\|q_{opt}(s)\hat{N}_d(s)\right\|_2} \\ &= \sup_\omega \lambda_{\max}(\omega) = \lambda_{\max}(\omega_{opt}) \\ q_{opt}(s) &= q_b(s)v_{\max}(s) \\ v_{\max}(j\omega)(\hat{N}_f(j\omega)\hat{N}_f^*(j\omega) - \lambda_{\max}(\omega)\hat{N}_d(j\omega)\hat{N}_d^*(j\omega)) &= 0 \end{aligned}$$

Here, $\lambda_{\max}(\omega)$ is the maximal eigenvalue of the generalized eigenvalue problem and $v_{\max}(j\omega)$ the corresponding eigenvector. $q_b(s)$ represents a band pass filter at frequency ω_{opt}, which gives

$$\begin{aligned} &\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} q_{opt}(j\omega)\hat{N}_d(j\omega)\hat{N}_d^*(j\omega)q_{opt}^*(j\omega)d\omega\right)^{1/2} \\ = &\left(\frac{1}{2\pi}\int_{\omega_{opt}-\theta}^{\omega_{opt}+\theta} \left|q_b(j\omega)v_{\max}(j\omega)\hat{N}_d(j\omega)\right|^2 d\omega\right)^{1/2} \\ \approx &\left(v_{\max}(j\omega_{opt})\hat{N}_d(j\omega_{opt})\hat{N}_d^*(j\omega_{opt})v_{\max}^*(j\omega_{opt})\right)^{1/2} \end{aligned}$$

Since

$$\begin{aligned} &q_{opt}(j\omega)\hat{N}_d(j\omega)\hat{N}_d^*(j\omega)q_{opt}^*(j\omega) \\ = &\frac{v_{\max}(j\omega_{opt})\hat{N}_f(j\omega_{opt})\hat{N}_f^*(j\omega_{opt})v_{\max}^*(j\omega_{opt})}{\lambda_{\max}(\omega_{opt})} \end{aligned}$$

the threshold $J_{th,2}$ is set as

$$\sqrt[2]{\frac{v_{\max}(j\omega_{opt})\hat{N}_f(j\omega_{opt})\hat{N}_f^*(j\omega_{opt})v_{\max}^*(j\omega_{opt})}{\lambda_{\max}(\omega_{opt})\Delta_\theta}}\delta_d$$

with $\Delta_\theta = 2\theta$. Denote the set of faults which are detectable using H_2-optimal FD system with

$$\Omega_{q_{opt},J_{th,2}}(d) = \\ \left\{f \mid \|q_{opt}(s)(\hat{N}_d(s)d(s) + \hat{N}_f(s)f(s))\|_2 > J_{th,2}\right\}$$

The following theorem gives a comparison between the H_2-optimal FD system and the approach proposed here.

Theorem 5 $\forall d, \Omega_{q_{opt},J_{th,2}}(d) \subseteq \Omega_{R^*,J_{th}}(d)$ *or* $\dim\Omega_{R^*,J_{th}}(d) \geq \dim\Omega_{q_{opt},J_{th,2}}(d)$

3.3 Relationship to $H_{\infty/\infty}$-optimal FD

The concept of $H_{\infty/\infty}$-optimal FD systems stands for the FD systems which are designed under the performance index [9], [17], [19]

$$\min_{R(s)\in\mathbf{RH}_\infty} \frac{\left\|R(s)\bar{G}_d(s)\right\|_\infty}{\left\|R(s)\bar{G}_f(s)\right\|_\infty} \tag{22}$$

and the threshold is set to be

$$J_{th,\infty/\infty} = \left\|R(s)\bar{G}_d(s)\right\|_\infty \delta_d$$

We now analyze the influence of the optimization on the behavior of the residual signal and the performance of the fault detection system. To this end, we rewrite $\|r(s)\|_2$ into

$$\|r(s)\|_2 = \left\|\begin{bmatrix} R(s)\bar{G}_d(s) & R(s)\bar{G}_f(s) \end{bmatrix}\begin{bmatrix} d(s) \\ f(s) \end{bmatrix}\right\|_2$$

It can be shown that for some f if

$$\left(1 + \frac{\left\|R(s)\bar{G}_f(s)\right\|_\infty}{\left\|R(s)\bar{G}_d(s)\right\|_\infty}\right)\left\|\begin{bmatrix} d(s) \\ f(s) \end{bmatrix}\right\|_2 \leq \delta_d \Leftrightarrow$$

$$\left\|\begin{bmatrix} d(s) \\ f(s) \end{bmatrix}\right\|_2 \leq \delta_d\left(1 + \frac{\left\|R(s)\bar{G}_f(s)\right\|_\infty}{\left\|R(s)\bar{G}_d(s)\right\|_\infty}\right)^{-1} \tag{23}$$

holds, then f becomes undetectable. We introduce value $T_{undetec}$,

$$T_{undetec} = \delta_d\left(1 + \frac{\left\|R(s)\bar{G}_f(s)\right\|_\infty}{\left\|R(s)\bar{G}_d(s)\right\|_\infty}\right)^{-1}$$

and define

$$\Omega^{\perp 1}_{R_{\infty/\infty},J_{th,\infty/\infty}}(d) = \left\{f \mid \left\|\begin{bmatrix} d(s) \\ f(s) \end{bmatrix}\right\|_2 \leq T_{undetec}\right\} \tag{24}$$

It is evident that $\Omega^{\perp 1}_{R_{\infty/\infty},J_{th,\infty/\infty}}(d) \subseteq \Omega^{\perp}_{R_{\infty/\infty},J_{th,\infty/\infty}}(d)$, where $R_{\infty/\infty}(s)$ denotes the optimal solution of (22), and thus optimization problem (22) is equivalent to

$$\min_{R(s)\in\mathbf{RH}_\infty} T_{undetec} \Leftrightarrow \min_{R(s)\in\mathbf{RH}_\infty} \dim\Omega^{\perp 1}_{R_{\infty/\infty},J_{th,\infty/\infty}}(d)$$

Note, on the other side, that $\Omega^{\perp 1}_{R_{\infty/\infty},J_{th,\infty/\infty}}(d)$ is not the complement of the set of detectable faults $\Omega_{R_{\infty/\infty},J_{th,\infty/\infty}}(d)$, since there does exist $f(s)$ so that

$$\left\|\begin{bmatrix} d(s) \\ f(s) \end{bmatrix}\right\|_2 > T_{undetec} \text{ but} \tag{25}$$

$$\|r(s)\|_2 \leq J_{th,\infty/\infty} \tag{26}$$

Denote the set of such faults by $\Omega^{\perp 2}_{R_\infty,J_{th,\infty/\infty}}(d)$,

$$\Omega^{\perp 2}_{R_\infty,J_{th,\infty/\infty}}(d) = \{f \mid \text{satisfies (25) and (26)}\}$$

then we have: $\Omega_f = \{f \mid f \neq 0\}$ equals to

$$\Omega^{\perp 1}_{R_{\infty/\infty},J_{th,\infty/\infty}}(d) \cup \Omega^{\perp 2}_{R_{\infty/\infty},J_{th,\infty/\infty}}(d) \cup \Omega_{R_{\infty/\infty},J_{th,\infty/\infty}} \tag{27}$$

It follows from (27) that minimizing $\dim\Omega^{\perp 1}_{R_{\infty/\infty},J_{th,\infty/\infty}}(d)$ is not equivalent to the maximization of $\dim\Omega_{R_{\infty/\infty},J_{th,\infty/\infty}}(d)$. The following theorem can then be proven.

Theorem 6 $\forall d, \Omega_{R_{\infty/\infty},J_{th,\infty/\infty}}(d) \subseteq \Omega_{R^*,J_{th}}(d)$ *or* $\dim\Omega_{R^*,J_{th}}(d) \geq \dim\Omega_{R_{\infty/\infty},J_{th,\infty/\infty}}(d)$.

3.4 Relationship to $H_{\infty/\min}$-optimal FD

$H_{\infty/\min}$-optimal FD systems are systems which are optimized under the performance index

$$\min_{R(s)\in\mathbf{RH}_\infty} \frac{\left\|R(s)\bar{G}_d(s)\right\|_\infty}{\left\|R(s)\bar{G}_f(s)\right\|_{\min}} \tag{28}$$

where $\left\|R(s)\bar{G}_f(s)\right\|_{\min}$ is a kind of measurement of the minimum influence of the faults on the residual signal, usually defined by the nonzero minimal singular value of $R(s)\bar{G}_f(s)$ [3], [13], [18]. Again, the threshold is given by $J_{th,\infty/\min} = \left\|R(s)\bar{G}_d(s)\right\|_\infty \delta_d$. Let's introduce

$$T_{detec} = \frac{\left\|R(s)\bar{G}_d(s)\right\|_\infty (\|d(s)\|_2 + \delta_d)}{\left\|R(s)\bar{G}_f(s)\right\|_{\min}} \tag{29}$$

It can be shown that all $f(s)$ satisfying

$$\|f(s)\|_2 > T_{detec} \tag{30}$$

can be detected. Denoting the optimal solution by $R_{\infty/\min}(s)$ and the set of faults satisfying (30) by

$$\Omega^1_{R_{\infty/\min},J_{th,\infty/\min}}(d) = \{f \mid \|f(s)\|_2 > T_{detec}\}$$

it becomes clear that performance index (28) is equivalent to

$$\max_{R(s)\in \mathbf{RH}_\infty} \dim \Omega^1_{R,J_{th,\infty/\min}}(d) = \dim \Omega^1_{R_{\infty/\min},J_{th,\infty/\min}}$$

On the other side, we know that $\Omega^1_{R_{\infty/\min},J_{th,\infty/\min}}(d)$ is only a sub-set of $\Omega_{R_{\infty/\min},J_{th,\infty/\min}}(d)$ defined by

$$\left\{ f \mid \left\| R_{\infty/\min}(s)(\bar{G}_d(s)d + \bar{G}_f(s)f) \right\|_2 > J_{th,\infty/\min} \right\}$$

and moreover there does exist $f(s)$ satisfying $f(s) \in \Omega_{R_{\infty/\min},J_{th,\infty/\min}}(d)$ but $f(s) \notin \Omega^1_{R_{\infty/\min},J_{th,\infty/\min}}(d)$. We thus have the following theorem.

Theorem 7 *$\forall d, \Omega_{R_{\infty/\min},J_{th,\infty/\min}}(d) \subseteq \Omega_{R^*,J_{th}}(d)$ or $\dim \Omega_{R^*,J_{th}}(d) \geq \dim \Omega_{R_{\infty/\min},J_{th,\infty/\min}}(d)$.*

4 Concluding remarks

From the viewpoint of a trade-off between the false alarm rate and missed detection rate, an approach to the design of fault detection system has been developed. Core of our study is the formulation of the design problem, minimizing the missed detection rate under a given false alarm rate, as an optimization problem and the derivation of an optimal solution. A further study on the relationships between the approach proposed and the existing $H_2, H_{\infty/\infty}, H_{\infty/\min}$ optimization approaches as well as the full decoupling from the unknown inputs demonstrates that the approach proposed in this paper provides us not only with a unified solution to the existing approaches but also with the best solution among these approaches in view of minimizing the missed detection rate under a given false alarm rate.

References

[1] Basseville, ”Information criteria for residual generation and fault detection and isolation”, *Automatica*, Vol.33, No.5, pp.783-803, 1997.

[2] J. Chen and P. R. Patton, *Robust model-based fault diagnosis for dynamic systems,* Kluwer Academic Publishers, 1999.

[3] X. Ding, L. Guo and P.M. Frank, ”A frequency Domain Approach to Fault Detection of Uncertain Dynamic Systems”, *Proc. the 32nd IEEE CDC*, pp.1722-1727, 1993.

[4] S. X. Ding, T. Jeinsch, P. M. Frank and E. L. Ding, ”A unified approach to the optimization of fault detection systems”, *Int. J. of Adaptive Contr. and Signal Processing*, 2000, to be published.

[5] R. K. Douglas and J. L. Speyer, ”An H_∞ bounded fault detection filter”, *Proc. ACC*, pp.86-90, 1995.

[6] A. Edelmayer, J. Bocor, F. Szigeti and L. Keviczky, ”Robust detection filter design in the presence of time-varying system perturbations”, *Automatica*, Vol.33, No.3, pp.471-475, 1997.

[7] P.M. Frank, ”Fault diagnosis in dynamic systems using analytical and knowledge-based redundancy - A survey,” *Automatica*, vol-26, pp.459-474, 1990.

[8] P.M. Frank, ”Enhancement of robustness in observer-based fault detection,” *Int. J. Contr.*, vol.59, pp.955-981, 1994.

[9] P.M. Frank and X. Ding, ”Frequency domain approach to optimally robust residual generation and evaluation for model-based fault diagnosis”, *Automatica*, pp.789-904, 1994.

[10] P.M. Frank and X. Ding, ”Survey of robust residual generation and evaluation methods in observer-based fault detection systems”, *J of Process Control*, pp.403-424, 1997.

[11] J.J. Gertler, ”Survey of model-based failure detection and isolation in complex plants,” *IEEE Contr. Syst. Magazine*, vol-3, pp.3-11, 1988.

[12] J.J. Gertler, ”Analytical redundancy methods in fault detection and isolation,” In: *Proc. IFAC/IMACS Symp. SAFEPROCESS '91*, Baden-Baden, 1991.

[13] M. Hou and R.J. Patton, ”An LMI approach to H_-/H_∞ observers”, *Proc. IEE Conf. Contr. 96*, 1996.

[14] H.H. Niemann and J. Stoustrup, ”Filter design for failure detection and isolation in the presence of modeling errors and disturbances”, *Proc. the 35th IEEE CDC*, p.1155-1160, 1996.

[15] R.J. Patton and J. Chen, ”Parity space approach to model-based fault diagnosis - A tutorial survey and some new results,” In: *Proc. IFAC/IMACS Symp. SAFEPROCESS '91*, Baden-Baden, 1991.

[16] R.J. Patton, P.M. Frank and R.N. Clark, (Eds.) *Fault Diagnosis in Dynamic Systems, Theory and Application*, Prentice Hall, 1989.

[17] Z. Qui and J. Gertler, ”Robust FDI systems and H_∞ optimization”, *Proc. the 32nd IEEE CDC*, p.1710-1715, 1993.

[18] M.A. Sadrnia, R.J. Patton and J. Chen, ”Robust H_∞/μ observer-based residual generation for fault diagnosis”, *Proc. SAFEPROCESS '97*, p.147-153, 1997.

[19] D. Sauter, F. Rambeaux and F. Hamelin, ”Robust fault diagnosis in an H_∞ setting”, *Proc. SAFEPROCESS '97*, p.879-884, 1997.

[20] M. Tadjine, H. Berkhedda and R.J. Patton, ”Frequency domain trade-off robust fault detection: towards an integrated approach”, *Proc. SAFEPROCESS '97*, p.166-170, 1997