

Robustness of closed-loop stability for infinite dimensional systems under sample and hold - counterexamples*

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Abstract

We consider continuous-time, linear control systems for which a static state feedback stabilizes the system. If we construct a sampled-data controller by applying an idealized sample-and-hold process to the continuous-time stabilizing feedback, it is known that if the state and control spaces are finite dimensional, then this sampled-data controller stabilizes the system for all sufficiently small sampling times. In this paper we show that this robustness with respect to sampling times is not true in general for infinite dimensional systems.

We consider systems where the state space X and the control space U are Hilbert spaces, the system is of the form $\dot{x}(t) = Ax(t) + Bu(t)$, and A is the generator of a strongly

continuous semigroup on X . Suppose that the continuous time feedback is $u(t) = Fx(t)$, where F is compact. Then it is known that if either B is bounded, or if A generates an analytic semigroup on X (in which case B is allowed to be unbounded in a general sense), then the sampled-data controller stabilizes the system for all sufficiently small sampling times. In this paper we show that the first condition is sharp in the following sense: we give a counterexample to show that the result is not true if B is barely unbounded, that is, B is unbounded but $A^{-\delta}B$ is bounded for all $\delta > 0$. We also give an easy counterexample if F is not compact.

1 Introduction

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0 \quad (1.1)$$

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where $x(t) \in X$, a Hilbert space with norm $\|\cdot\|$, $u(t) \in U$, a Hilbert space, A is the generator of a strongly continuous semigroup $T(t)$ on X , and B is defined on U . We say that B is *bounded* if $B \in \mathcal{B}(U, X)$, and that B is *unbounded* if it has range not in X but in a larger space. The larger spaces we consider are defined as follows: For $a \in (0, 1]$ let X_{-a} denote the closure of X in the norm $\|x\|_{-a} = \|(\lambda I - A)^{-a}x\|$, where λ is any element of the resolvent set of A .

Let $\mathcal{K}(X, U)$ denote the set of compact operators from X to U . We assume that $F \in \mathcal{K}(X, U)$ is such that the control $u(t) = Fx(t)$ exponentially stabilizes (1.1), i.e. $A + BF$ is the generator of a strongly continuous exponentially stable semigroup. For a fixed sampling time $\tau > 0$ we can form the sampled-data feedback

$$u(t) = Fx(n\tau) \quad \text{for} \quad (1.2)$$

$$n\tau \leq t < (n+1)\tau, \quad n = 0, 1, \dots$$

We say that (1.1), (1.2) is exponentially stable if there exists $M, \omega > 0$ such that $\|x(t)\| \leq Me^{-\omega t}\|x(0)\|$. In the case where X is finite-dimensional, it follows from Theorem 3 in Chen and Francis [1] that (1.1), (1.2) is exponentially stable for all sufficiently small $\tau > 0$. In [3] we give two sets of sufficient conditions on (A, B) so that this conclusion is true. We will describe these below, and then give a counterexample to show that if these conditions are not satisfied, then this robustness with respect to sampling times can fail.

We first note that there is no hope that the answer to the above question is positive unless A satisfies certain necessary conditions. Since $BF \in \mathcal{K}(X)$, we can apply Theorem 1.3 and Proposition 1.4 of Rebarber and Townley [5] (with $U = X$) to see that if $x \in L^2(0, \infty; X)$, then A must satisfy the following: There exists $\beta < 0$ so that X admits a decomposition $X_1 \oplus X_2$ with

- (a) $\dim X_1 < \infty$.
- (b) $AX_1 \subseteq X_1$ and $AX_2 \subseteq X_2$.
- (c) $\sigma(A|_{X_1}) = \sigma(A) \cap \{\lambda \in \mathbb{C} \mid \operatorname{Re}(\lambda) > \beta\}$ consists of at most finitely many eigenvalues of A , each with finite algebraic multiplicity and non-negative real part.
- (d) $A|_{X_2}$ is the generator of an exponentially stable semigroup.

In [3] we give the following sufficient conditions for this robustness with respect to sampling to hold.

Theorem 1.1 *Suppose $B \in \mathcal{B}(U, X)$, $F \in \mathcal{K}(X, U)$, and $A + BF$ generates an exponentially stable strongly continuous semigroup $T_F(t)$. Then there exists $\tau^* > 0$ such that for $\tau \in (0, \tau^*)$, there exists $M \geq 1, \delta > 0$ so that all solutions of (1.1), (1.2) satisfy $\|x(t)\| \leq Me^{-\delta t}$.*

For the next result, we assume that B is unbounded, so it is not necessarily true that $A_{BF} = A + BF$ generates a semigroup (see [3], equation (4.1) for a precise definition of this operator). Therefore, we have to be careful with our hypotheses. Let $\mathbf{G}(s) := F(sI - A)^{-1}B$.

Theorem 1.2 *Suppose A generates an analytic semigroup, $B \in \mathcal{B}(U, X_{-1})$, $F \in \mathcal{K}(X, U)$, and*

(A1) *the function $s \rightarrow (sI - A_{BF})^{-1}$ is in $H^\infty(\mathbb{C}_0, \mathcal{B}(X))$;*

(A2) *$(I - \mathbf{G})^{-1} \in H^\infty(\mathbb{C}_0, \mathcal{B}(X))$.*

Then there exists $\tau^ > 0$ such that for all $\tau \in (0, \tau^*)$, there exists $M \geq 1, \delta > 0$ so that all solutions of (1.1), (1.2) satisfy $\|x(t)\| \leq Me^{-\delta t}$.*

2 Counterexample - B unbounded

Let

$$\Delta_\tau x := T(\tau)x + \int_0^\tau T(\tau - s)BFx ds. \quad (2.1)$$

It is shown in [3] that (1.1), (1.2) is exponentially stable for $\tau > 0$ if and only if Δ_τ is power stable.

In this section we show that the conclusions of Theorem 1.1 do not necessarily hold if B is not bounded. In particular, we identify a semigroup generator A , an input operator B , and a feedback operator F such that the following properties hold:

1. A satisfies the necessary conditions (a)-(d) in Section 1;
2. $A + BF$ generates an exponentially stable semigroup;
3. F is compact (in fact, bounded and one-dimensional);
4. B is not bounded, but $A^{-\delta}B$ is bounded for all $\delta > 0$;
5. Δ_τ given by (2.1) is not power stable.

A common measure of the unboundedness of B is the smallest $\delta \in \mathbb{R}$ such that $A^{-\delta}B \in \mathcal{B}(U, X)$. Thus we see that Theorem 1.1 is sharp if we use this measure of unboundedness of B .

Furthermore, (A, B, F) satisfies conditions (A1) and (A2) in Theorem 1.2, but A does not generate an analytic semigroup, so the same example shows that analyticity is essential for Theorem 1.2 to hold.

Let $X = \ell^2$, indexed by \mathbb{N} , with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$, and let $U = \mathbb{C}$. We define the generator

$$A = \text{diag}(\lambda_k)_{k \in \mathbb{N}},$$

with

$$\lambda_k := -1 + i\pi 3^k \quad \text{for } k \in \mathbb{N},$$

and domain

$$\mathcal{D}(A) = \{(x_k)_{k \in \mathbb{N}} \mid (x_k \lambda_k)_{k \in \mathbb{N}} \in \ell^2\}.$$

Let

$$B = (b_k)_{k \in \mathbb{N}}$$

with

$$b_k := ik \quad \text{for } k \in \mathbb{N}.$$

We define the operator $F \in \mathcal{B}(X, U)$ by

$$F(x_k) = \sum_{k \in \mathbb{N}} \frac{2x_k}{k}.$$

The first steps towards proving that properties (1)-(6) hold is to prove the following Lemmas.

Lemma 2.1 $A + BF$ had no eigenvalues in $\mathbb{C}_{-.5}$.

Lemma 2.2 $A + BF$ generates an exponentially stable semigroup.

Idea for proof of Lemma 2.2: The main step in the proof of Lemma 2.2 is to show that $A + BF$ is a *discrete spectral operator*, and that all but finitely many of its eigenvalues have spectral projections with one dimensional range - see Dunford and Schwartz [2] for the definition of a discrete spectral operator. To this end we apply Theorem XIX2.7 and Corollary XIX2.86 of [2] to $(A + BF)^* = A^* + F^*B^*$. We then combine this with Lemma 2.1 to obtain exponential stability. \square

The most difficult part of the counterexample construction is to show that the closed-loop sampled-data system actually has unstable solutions.

Theorem 2.3 *There exists (τ_n) such that $\tau_n \rightarrow 0$, and Δ_{τ_n} has an eigenvalue z_n with $|z_n| > 1$.*

Idea for the proof of Theorem 2.3. It follows from this result that the sampled data feedback system (1.1), (1.2), with $\tau = \tau_n$, has a solution which grows exponentially with time. The main ideas for the proof of Theorem 2.3 are as follows. Let \mathbf{H}_τ denote the discrete time transfer function for the system

$$\begin{aligned} x_{k+1} &= e^{A\tau} x_k + \left(\int_0^\tau e^{A(\tau-\sigma)} B_\varepsilon d\sigma \right) u_k, \\ y_k &= F x_k. \end{aligned}$$

It is easy to compute that

$$\mathbf{H}_\tau(z) = - \sum_{k \in \mathbb{N}} \frac{e^{\lambda_k \tau} - 1}{e^{\lambda_k \tau} - z} \frac{2i}{\lambda_k}. \quad (2.2)$$

If z is a zero of $1 - \mathbf{H}_\tau(z)$, then z is also an eigenvalue of Δ_τ (although the converse is not guaranteed).

Let

$$\tau_n = 3^{-n} \quad \text{for } n \text{ even.}$$

Then we write

$$-\mathbf{H}_{\tau_n}(z) = h_{1,n}(z) + h_{2,n}(z) + h_{3,n}(z) + h_{4,n}(z),$$

where

$$\begin{aligned} h_{1,n} &:= \sum_{k=1}^{n/2} \frac{e^{i\pi 3^{k-n}} e^{-3^{-n}} - 1}{e^{i\pi 3^{k-n}} e^{-3^{-n}} - z} \left(\frac{2i}{-1 + i\pi 3^k} \right), \\ h_{2,n} &:= \sum_{k=\frac{n}{2}+1}^{n-1} \frac{e^{i\pi 3^{k-n}} e^{-3^{-n}} - 1}{e^{i\pi 3^{k-n}} e^{-3^{-n}} - z} \left(\frac{2i}{-1 + i\pi 3^k} \right), \\ h_{3,n} &:= \sum_{k=n}^{\infty} \frac{e^{i\pi 3^{k-n}} e^{-3^{-n}} - 1}{e^{i\pi 3^{k-n}} e^{-3^{-n}} - z} \left(\frac{2i}{i\pi 3^k} \right), \\ h_{4,n} &:= \sum_{k=n}^{\infty} \frac{e^{i\pi 3^{k-n}} e^{-3^{-n}} - 1}{e^{i\pi 3^{k-n}} e^{-3^{-n}} - z} \left(\frac{2i}{i\pi 3^k (-1 + i\pi 3^k)} \right). \end{aligned}$$

The main part of the proof is to show that there exists a root of $1 - \mathbf{H}_{\tau_n}(z)$ outside of the unit circle. This is done by showing that $h_{3,n}(z) + 1$ has a root outside of the unit circle, and then using Rouché's Theorem. \square

This counterexample also shows that Theorem 1.2 requires the analyticity of the semi-group generated by A . This follows from the following result.

Lemma 2.4 *(A, B, F) satisfies conditions (A1) and (A2) in Theorem 1.2.*

Idea for the proof of Lemma 2.4: (A1) follows immediately from the fact that A_{BF} is the generator of a strongly continuous semi-group, see [4]. (A2) is essentially a statement of input-output stability of the closed-loop system, but because of the unboundedness of B this does not directly follow exponential stability of $A + BF$. However, we can show that \mathbf{G} is also the transfer function function of $(A, \tilde{B}, \tilde{F})$, where \tilde{B} and \tilde{F} are input and output operators such that: (1) $A + \tilde{B}\tilde{F}$ is maximal dissipative, hence by the Lumer-Phillips theorem is the generator of an exponentially stable semigroup; and (2) $(A, \tilde{B}, \tilde{F})$ is a *regular* system (see [6]), I is an admissible feedback operator for $(A, \tilde{B}, \tilde{F})$ (again, see [6]). In the setup from [6], exponential stability of $A + \tilde{B}\tilde{F}$ *does* imply input-output stability of the closed-loop system, i.e. \mathbf{G} satisfies (A2). \square

3 Counterexample - F non-compact

It might seem that we have made the problem easier by restricting to compact feedbacks and systems which can be stabilized by them. However, if we relax the compactness of F assumption, then it is easy to find counter-examples. Indeed consider

the case where $X = l^2$, indexed by \mathbb{Z} , $A = \text{diag}(1+ki)_{k=-\infty}^{\infty}$ and $B = I$. It is easy to see that $F = -2I$ is a continuous-time exponentially stabilizing feedback. Indeed $A + BF$ generates the exponentially stable semigroup

$$T_F(t) = \text{diag}(e^{(-1+ki)t})_{k=-\infty}^{\infty}$$

However, applying the sampled data feedback given by (1.2) results in the discrete-time system

$$x_{n+1} = \text{diag}(e^{(1+ki)\tau}) - 2\text{diag}\left(\frac{e^{(1+ki)\tau} - 1}{1 + ki}\right)$$

which is not exponentially stable for any $\tau > 0$.

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