ON CONVEXITY IN STABILIZATION OF NONLINEAR SYSTEMS

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Abstract A stability criterion for nonlinear systems, recently derived by the first author, can be viewed as a dual to Lyapunov's second theorem. The criterion is stated in terms of a function which can be interpreted as the stationary density of a substance that is generated all over the state space and flows along the system trajectories towards the equilibrium.

The new criterion has a remarkable convexity property, which in this paper is used for controller synthesis via convex optimization. Recent numerical methods for verification of positivity of multivariate polynomials are used.

Keywords Stabilization, nonlinear systems, sums of squares, convexity

1. INTRODUCTION

Lyapunov functions have long been recognized as one of the most fundamental analytical tools for analysis and synthesis of nonlinear control systems. See for example (Artstein, 1983; Brockett, 1983; Hahn, 1963; Isidori, 1995; Krstic *et al.*, 1995; Ledyaev and Sontag, 1999).

There has also been a strong development of computational tools based on Lyapunov functions. Many such methods are based on convex optimization and solution of matrix inequalities, exploiting the fact that the set of Lyapunov functions for a given system is convex.

A serious obstacle in the problem of controller synthesis is however that the joint search for a controller $u(x)$ and a Lyapunov function $V(x)$ satisfying the condition

$$\frac{\partial V}{\partial x}[f(x) + g(x)u(x)] < 0$$

is not convex. In fact, for some systems the set of u and V satisfying the inequality is not even connected.

Given the difficulties with Lyapunov based controller synthesis, it is most striking to find that the new convergence criterion presented in (Rantzer, 2000b; Rantzer, 2000a) has much better convexity properties. Indeed, the set of $(\rho, u\rho)$ satisfying

$$\nabla \cdot [\rho(f + gu)] > 0 \tag{1}$$

is convex. In this paper, we will exploit this fact in the computation of stabilizing controllers for some example systems. For the case of polynomial (or rational) systems, the search for a candidate pair $(\rho, u\rho)$ verifying the inequality (1) can be done using the methods introduced in (Parrilo, 2000).

2. THE CONVERGENCE CRITERION

The main result of (Rantzer, 2000a) can be stated as follows:

THEOREM 1
Given the equation $\dot{x}(t) = f(x(t))$, where $f \in \mathbf{C}^1(\mathbf{R}^n, \mathbf{R}^n)$ and $f(0) = 0$, suppose there exists a non-negative $\rho \in \mathbf{C}^1(\mathbf{R}^n \setminus \{0\}, \mathbf{R})$ such that $\rho(x)f(x)/|x|$ is integrable on $\{x \in \mathbf{R}^n : |x| \geq 1\}$ and

$$[\nabla \cdot (\rho f)](x) > 0 \qquad \text{for almost all } x \tag{2}$$

Then, for almost all initial states $x(0)$ the trajectory $x(t)$ exists for $t \in [0, \infty)$ and tends to zero as $t \to \infty$. Moreover, if the equilibrium $x = 0$ is stable, then the conclusion remains valid even if ρ takes negative values. □

The proof is based on the following lemma, which can be viewed as a version of Liouville's theorem (Arnold, 1989; Mane, 1987).

LEMMA 1
Let $f \in \mathbf{C}^1(D, \mathbf{R}^n)$ where $D \subset \mathbf{R}^n$ is open and let $\rho \in \mathbf{C}^1(D, \mathbf{R})$ be integrable. For $x_0 \in \mathbf{R}^n$, let $\phi_t(x_0)$ for $t \geq 0$ be the solution $x(t)$ of $\dot{x} = f(x)$, $x(0) = x_0$. For a measurable subset Z of D, let $\phi_t(Z) = \{\phi_t(x) \mid x \in Z\}$. Then

$$\begin{aligned}
&\int_{\phi_t(Z)} \rho(x)dx - \int_Z \rho(z)dz \\
&= \int_0^t \int_{\phi_\tau(Z)} [\nabla \cdot (\rho f)](x)dxd\tau
\end{aligned}$$

□

Proof. Note that for every $\mathbf{C}^1$ matrix function $M(t)$ with $M(0) = I$

$$\frac{\partial}{\partial t} \det M(t)\Big|_{t=0} = \operatorname{trace} M'(0)$$

This follows by direct expansion of the determinant, since the first order terms in t correspond to the diagonal elements of $M(t)$.

Let $M(t) = \frac{\partial \phi_t}{\partial z}(z)$ and use $|\cdot|$ to denote determinant. The differentiability of f gives that $\phi_t(z)$ is of class $\mathbf{C}^1$ in z and $\mathbf{C}^2$ in t (Lefschetz, 1977)page 40. Hence

$$\begin{aligned}
\left[\frac{\partial}{\partial t}\left|\frac{\partial \phi_t}{\partial z}(z)\right|\right]_{t=0} &= \left[\operatorname{trace} \frac{\partial^2}{\partial t \partial z}\phi_t(z)\right]_{t=0} \\
&= \operatorname{trace} \frac{\partial f}{\partial z}(z) = \left[\nabla \cdot f\right](z)
\end{aligned}$$

and with the notation $\rho_t(z) = \rho(\phi_t(z))\left|\frac{\partial \phi_t}{\partial z}(z)\right|$

$$\begin{aligned}
\frac{\partial}{\partial t}\rho_t(z)\Big|_{t=0} &= \nabla\rho \cdot f + \rho(\nabla \cdot f) = \left[\nabla \cdot (f\rho)\right](z) \\
\frac{\partial}{\partial t}\rho_t(z)\Big|_{t=\tau} &= \frac{\partial}{\partial h}\left\{\rho_h(\phi_\tau(z))\left|\frac{\partial \phi_\tau}{\partial z}(z)\right|\right\}\Big|_{h=0} \\
&= \left[\nabla \cdot (f\rho)\right](\phi_\tau(z))\left|\frac{\partial \phi_\tau}{\partial z}(z)\right|
\end{aligned}$$

Let $\chi(\cdot)$ be the characteristic function of Z. Then

$$\begin{aligned}
&\int_{\phi_t(Z)} \rho(x)dx - \int_Z \rho(z)dz \\
&= \int_{\mathbf{R}^n} \rho(x)\chi(\phi_t^{-1}(x))dx - \int_Z \rho(z)dz \\
&= \int_{\mathbf{R}^n} \rho(\phi_t(z))\chi(z)\left|\frac{\partial \phi_t(z)}{\partial z}\right| dz - \int_Z \rho(z)dz \\
&= \int_Z [\rho_t(z) - \rho(z)]\, dz \\
&= \int_Z \int_0^t [\nabla \cdot (\rho f)]\,(\phi_\tau(z))\left|\frac{\partial \phi_\tau}{\partial z}(z)\right| d\tau dz \\
&= \int_0^t \int_{\phi_\tau(Z)} [\nabla \cdot (\rho f)]\,(x)dxd\tau
\end{aligned}$$

Proof of Theorem 1, second statement. Here it is assumed that $x = 0$ is a stable equilibrium, while ρ may take negative values. The proof for the other case is omitted from this conference manuscript.

Rather than exploiting that $f \in \mathbf{C}^1(\mathbf{R}^n, \mathbf{R}^n)$, we will actually prove the result under the weaker condition that $f \in \mathbf{C}^1(\mathbf{R}^n \setminus \{0\}, \mathbf{R}^n)$ and $f(x)/|x|$ is bounded near $x = 0$. Given any $x_0 \in \mathbf{R}^n$, let $\phi_t(x_0)$ for $t \geq 0$ be the solution $x(t)$ of $\dot{x}(t) = f(x(t))$, $x(0) = x_0$. Assume first that ρ is integrable on $\{x \in \mathbf{R}^n : |x| \geq 1\}$ and $|f(x)|/|x|$ is bounded. Then ϕ_t is well defined for all t. Given $r > 0$, define

$$Z = \cap_{l=1}^{\infty} \{x_0 \ : \ |\phi_t(x_0)| > r \text{ for some } t > l\} \tag{3}$$

Notice that Z contains all trajectories with $\limsup_{t\to\infty} |x(t)| > r$. The set Z, being the intersection of a countable number of open sets, is measurable. Moreover, $\phi_t(Z) = \{\phi_t(x) \mid x \in Z\}$ is equal to Z for every t. By stability of the equilibrium $x = 0$, there is a positive lower bound ε on the norm of the elements in Z, so Lemma 1 with $D = \{x \ : \ |x| > \varepsilon\}$ gives

$$0 = \int_{\phi_t(Z)} \rho(x)dx - \int_Z \rho(z)dz \tag{4}$$

$$= \int_0^t \int_{\phi_\tau(Z)} [\nabla \cdot (\rho f)]\,(x)dxd\tau \tag{5}$$

By the assumption (2), this implies that Z has measure zero. Consequently, $\limsup_{t\to\infty} |x(t)| \leq r$ for almost all trajectories. As r was chosen arbitrarily, this proves that $\lim_{t\to\infty} |x(t)| = 0$ for almost all trajectories.

When $|f(x)|/|x|$ is unbounded, there may not exist any nonzero t such that $\phi_t(z)$ is well defined for all z. We then introduce

$$\rho_0(x) = \left[\frac{e^{-|x|}}{1+|\rho(x)|^2} + \frac{|f(x)|^2}{|x|^2}\right]^{1/2} \rho(x)$$
$$f_0(x) = \frac{f(x)\rho(x)}{\rho_0(x)}$$

Then $|f_0(x)|/|x|$ is bounded and ρ_0 is integrable on $\{x \in \mathbf{R}^n : |x| \geq 1\}$, so the argument above can be applied to f_0 together with ρ_0 to prove that $\lim_{\tau\to\infty} |y(\tau)| = 0$ for almost all trajectories of the system $dy/d\tau = f_0(y(\tau))$. However, modulo a transformation of the time axis

$$t = \int_0^\tau \frac{\rho(y(s))}{\rho_0(y(s))} ds$$

the trajectories are identical: $x(t) = y(\tau)$. This, together with the boundedness of $f(x)/|x|$ near $x = 0$, also shows that $x(t)$ exists for $t \in [0, \infty)$ and tends to zero as $t \to \infty$ provided that $\lim_{\tau\to\infty} |y(\tau)| = 0$. Hence the proof of the second statement in Theorem 1 is complete.

3. A COMPUTATIONAL APPROACH

In order to understand the possibilities and limitations of computational approaches to nonlinear stability, an issue that has to be addressed is how to deal numerically with functional inequalities such as the standard Lyapunov one, or the divergence inequality (1).

Even in the restricted case of polynomial functions, it is well-known that the problem of checking global nonnegativity of a polynomial of quartic (or higher) degree is computationally hard. For this reason, we need tractable sufficient conditions that guarantee nonnegativity, and that are not significantly conservative. A particularly interesting sufficient condition is given by the existence of a sum of squares decomposition: can the polynomial $P(x)$ be written as $P(x) = \sum_i p_i^2(x)$, for some polynomials $p_i(x)$? Obviously, if this is the case, then $P(x)$ takes only nonnegative values. Notice that in the case of quadratic forms, for instance, the two conditions (positivity and sum of squares) are equivalent.

In this respect, it is interesting to notice that many methods used in control theory for constructing Lyapunov functions (for example, backstepping) use either implicitly of explicitly a sum of squares approach.

As shown in (Parrilo, 2000), the problem of checking if a given polynomial can be written as a sum of squares can be solved using semidefinite programming. We refer the reader to that work for a discussion of the specific algorithms. For our purposes, however, it will enough to know that while the standard LMI machinery can be interpreted as searching for a positive definite element over an affine family of quadratic forms, the new tools provide a way of *finding a sum of squares, over an affine family of polynomials*. The former problem is clearly a special case of the latter (in fact, they are equivalent).

To apply these tools to the stabilization problem analyzed in the paper, consider the parameterized representation for ρ and $u\rho$:

$$\rho(x) = \frac{a(x)}{b(x)^\alpha}, \qquad u(x)\rho(x) = \frac{c(x)}{b(x)^\alpha},$$

where a, b, c are polynomials, $b(x)$ is positive, and α is chosen to satisfy the integrability constraint. In this case, the condition (1) can be written as:

$$\nabla \cdot [\rho(f + gu)] = \nabla \cdot [\frac{1}{b^\alpha}(fa + gc)]$$
$$= \frac{1}{b^{\alpha+1}}[b\nabla \cdot (fa + gc) - \alpha\nabla b \cdot (af + gc)].$$

Since b is positive, we only need to satisfy the inequality:

$$b\nabla \cdot (fa + gc) - \alpha\nabla b \cdot (af + gc) > 0. \qquad (6)$$

For fixed b, α, the inequality is linear in a, c. If instead of checking positivity, we check that the left-hand side is a *sum of squares*, for the case of polynomial (or rational) vector fields, the problem can be solved using LMI methods.

4. AN EXAMPLE

A simple numerical example is the following:

$$\dot{x} = y - x^3 + x^2$$
$$\dot{y} = u$$

The function $b(x)$ is chosen based on the linearization of the system. We picked $b(x) := 3x^2 + 2xy + 2y^2$, which is a control Lyapunov function for the linearized system, and therefore, $b(x)^{-\beta}$ (for some β) will be a good choice for a ρ-function near the origin. Since we will be using cubic polynomials in x, y for c (a is taken

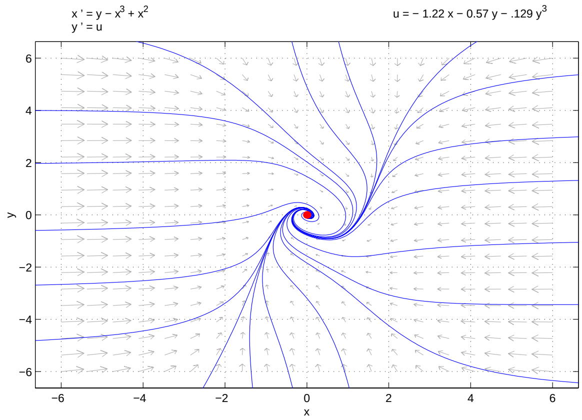

Fig. 1 Phase plot of the closed-loop system for the Example.

to be a constant), we choose $\alpha = 4$ to satisfy the integrability condition.

In this case, after solving the LMIs corresponding to the condition that the left-hand side of (6) is a sum of squares, we obtain an explicit expression for the controller, as a third order polynomial in x and y. The optimization criterion chosen is the ℓ_1 norm of the coefficients. This way, we approximately try to minimize the number of nonzero terms. The expression for the final controller is:

$$u(x, y) = -1.22x - 0.57y - 0.129y^3$$

A phase plot of the closed-loop system in presented in Figure 4.

This example has been chosen for its relative simplicity: in this particular case, it is possible to solve it directly using other methodologies. For instance, it can be noted that in this particular case $b(x)$ is actually a control Lyapunov function for the system, and from that obtain a controller (e.g., using Sontag's formula). There is no requirement in the present framework that forces $b(x)$ to be a clf. The main difference would be in terms of the computational difficulty of approximating the controller when the choice of the denominator $b(x)$ is not optimal. Further research is needed in order to fully understand the practical implications.

5. CONCLUDING REMARKS

A new computational approach to nonlinear control synthesis has been introduced. The basis is a recent convergence criterion introduced by the first author. The new criterion makes it possible to state the synthesis problem in terms of convex optimization and has earlier been exploited for optimal control problems in (Young, 1969; Vinter, 1993). Polynomials are used for parameterization and positivity is verified using the ideas in (Parrilo, 2000).

The numerical example should be viewed as a first attempt to demonstrate the power of the approach. However, many modifications are possible and much research in the area remains to be done.

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