Control Synthesis with Dynamic Integral Quadratic Constraints - LMI Approach

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Abstract

In this paper, we consider the control synthesis problem when the disturbance input w is within a set defined by several dynamic Integral Quadratic Constraints (IQCs). We show that the condition on the existence of a stabilizing controller can be expressed as a set of Linear Matrix Inequalities (LMIs). We also show that the stabilizing controllers, if exist, have the dimension no larger than the sum of the dimension of the open-loop system and the multipliers in IQCs.

Keywords: control synthesis, Integral Quadratic Constraint (IQC), Linear Matrix Inequality (LMI).

1 Introduction

Integral Quadratic Constraints (IQCs) has been playing an important role in the area of robust control [4]. In [1], it was shown that a class of robust control problems can be formulated in the so-called generalized l_2 framework, in which a controller $\tilde{K}$ is designed for the open-loop system $\tilde{G}$ such that

$$\sup_{e\in\mathcal{E}} \sup_{d\in\mathcal{D}} \langle e, (\tilde{G}\star\tilde{K})w\rangle < \gamma, \qquad (1)$$

where γ is the desired performance level, and the sets $\mathcal{E}$, $\mathcal{D}$ are defined by several *static* IQCs. It was also shown that such problems can be casted as a feasibility problem over a set of Linear Matrix Inequalities (LMIs). These results were further expanded in [2] to include *dynamic* IQCs on e and w, so that more important robust control problems can be treated under the generalized l_2 framework and cased as finite dimensional, convex optimization problems. While it is relatively easily to obtain the LMI expressions for static IQCs, obtaining LMIs for dynamic IQCs is more involved and appears to be a non-trivial problem. In [2], several transformations and auxiliary parameters need to be introduced in order to obtain a LMI expression.

In this paper, we consider the continuous time control synthesis problem with several IQCs applied on the system's input. The framework we adopt is different from the generalized l_2 setup. In particular, we consider the following problem: For a given system G, find a controller K such that

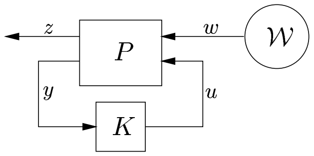

Figure 1: Control synthesis with disturbance $w \in \mathcal{W}$

$$\sup_{w\in\mathcal{W}} \int_{-\infty}^{\infty} \begin{bmatrix} Tw \\ w \end{bmatrix}^* \Pi \begin{bmatrix} Tw \\ w \end{bmatrix} d\omega < \gamma\|w\|^2,\ \Pi_{11} > 0,$$

where $T = G\star K$ is the closed-loop system, and $\mathcal{W}$ is defined by dynamic IQCs. It is possible to show that generalized l_2 synthesis (1) can be reformulated into this framework, by treating $(e,d) := w$ as an *input* to a transformed system. We will show that obtaining the LMI expression for dynamic IQCs on w can be done a bit more explicit. In particular, we do not need to work with the transformation proposed in [2], no extra parameter introduced, and the LMIs are derived directly and expressed explicitly in terms of the matrices of the open-loop system and multipliers used in the IQCs.

Notation :
The space of square summable, or finite energy, R^m-valued functions $f : (0,\infty) \to R^m$ is denoted by L_2^m. The norm on L_2^m is defined as

$$\|f\| = \left(\int_0^\infty f^T(t)f(t)dt\right)^{\frac{1}{2}}.$$

We will simply write $f \in L_2$ when the dimension of f is of no importance. The Fourier transform of a L_2 function $f(t)$ is defined as

$$\hat{f}(j\omega) = \int_{-\infty}^{\infty} e^{j\omega t} f(t)dt.$$

We will use $S_\infty^{m\times m}$ to denote the set of all proper, m by m, real rational transfer matrices $H(s)$ which have no poles on the imaginary axis and satisfy $H(s) = H(s)^*$, where the adjoint is defined as $H^*(s) = H(-s)^T$.

A matrix is called Hurwitz if all of its eigenvalues are located in the open left-half complex plane.

2 Problem Statement

In this paper, we consider the output feedback problem as shown in Figure 1. We assume that the system under consideration has a state-space representation

$$P \ : \ \begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{11} w + D_{12} u \\ y = C_2 x + D_{21} w \end{cases}, \tag{2}$$

where z is the measurement performance, y is the output feedback, u is the control input, and w represents the exogenous excitation, such as reference commands, disturbance signals, and sensor noises. We also assume that w belongs to a set $\mathcal{W}$ which is characterized by *dynamic* and/or static integral quadratic constraints. The design task is to select a controller K so that the closed-loop system satisfies certain quadratic performance criteria under the presence of w. We will make the following assumptions on the open-loop system

1. (A, B_1) is controllable and (A, C_1) is observable
2. (A, B_2) is stabilizable and (A, C_2) is detectable
3. D_{12} has the structure $D_{12}^T = \begin{bmatrix} I & \bar{D}_{12}^T \end{bmatrix}^T$.

The last assumption is not as restricted as it seems. Since all the control inputs need to be penalized to form a well-posed problem, D_{12} usually has full column rank. Therefore, we can always rearrange z so that $D_{12}^T = \begin{bmatrix} \tilde{D}_{12}^T & \hat{D}_{12}^T \end{bmatrix}$ with an invertible $\tilde{D}_{12}$. Then we can define $\tilde{u} = \tilde{D}_{12} u$ and transform the system so that the control input of the new system is $\tilde{u}$. The D_{12} matrix of the transformed system is then in the desired form. Finally, the actual control input u can be obtained by $\tilde{D}_{12}^{-1}\tilde{u}$.

3 Robust Performance Analysis for Systems with Special Disturbances

Before proceeding to the control synthesis, we will in this section briefly review the results of using IQCs to analyze the performance of a system with the presence of special disturbances. These results are essential and will be used directly in the control synthesis. We start with a few definitions.

Definition 1. A signal class is a collection of vector signals $\mathcal{W} = \{w : w \in L_2^n\}$. A class of signal $\mathcal{W}$ is said to satisfy the IQC defined by σ (denoted by $\mathcal{W} \in$ IQC(σ)) if $\sigma(w) \geq 0$, $\forall \ w \in \mathcal{W}$, where σ is of the form

$$\sigma(w) = \int_{-\infty}^{\infty} \hat{w}(j\omega)^* \Pi(\omega) \hat{w}(j\omega) \ d\omega, \tag{3}$$

$\hat{w}(j\omega)$ is the Fourier transform of w, and $\Pi(s) \in S_\infty^{n\times n}$ is referred as the multiplier of σ.

Remark 1. Although in general $\Pi(\omega)$ can be any function belongs to $S_\infty^{n\times n}$, practically we only choose those of the form $G_\pi^*(j\omega) M G_\pi(j\omega)$, where $G_\pi(s)$ is a proper, rational transfer matrix, and $M = M^T$ is constant. Therefore, by the state-space representation of $G_\pi(s)$, IQC $\sigma(w) \geq 0$ can be equivalently expressed in the time-domain as

$$\sigma(w) = \int_0^\infty \begin{bmatrix} x_\pi \\ w \end{bmatrix}^T \begin{bmatrix} G & F \\ F^T & R \end{bmatrix} \begin{bmatrix} x_\pi \\ w \end{bmatrix} \ dt \geq 0,$$
$$G = G^T, \ R = R^T, \tag{4}$$

where x_π and w satisfy

$$\dot{x}_\pi = A_\pi x_\pi + B_\pi w, \quad x_\pi(0) = 0. \tag{5}$$

Definition 2. A performance is a quadratic functional $\sigma_p(z, w)$ defined as

$$\sigma_p(z, w) = \int_0^\infty z^T \Pi_{p1} z + 2 z^T \Pi_{p2} w + w^T \Pi_{p3} w \ dt$$
$$\Pi_{p1} = \Pi_{p1}^T > 0, \ \ \Pi_{p3} = \Pi_{p3}^T.$$

A system is said to satisfy σ_p-performance criterion over a set of disturbances $\mathcal{W}$ if the system is internally stable and its performance measurement z satisfies $\sigma_p(z, w) < 0$ for all $w \in \mathcal{W}$.

For example, take $\Pi_{p1} = I$, $\Pi_{p2} = 0$, $\Pi_{p3} = -\gamma^2 I$, we have $\sigma_p(z, w) = \|z\|^2 - \gamma^2 \|w\|^2$. Then the performance criterion $\sigma_p(z, w) < 0$ simply implies that the worst case L_2-gain, or the H_∞-norm, of the system over the class of disturbances $\mathcal{W}$ is bounded by γ. The next proposition gives a sufficient condition for the system that satisfies the performance criterion $\sigma_p < 0$ over a class of signals $\mathcal{W}$ which is characterized by IQC σ_w.

Proposition 1. *[4] Let T_{zw} denote the closed-loop transfer function from the input w to z for some dynamical output feedback law $u = Ky$ applied to system (2)*

$$T_{zw} : \begin{cases} \dot{x}_{cl} = A_{cl} x_{cl} + B_{cl} w \\ z = C_{cl} x_{cl} + D_{cl} w \end{cases}. \tag{6}$$

Assume that T_{zw} is internally stable, i.e., matrix A_{cl} is Hurwitz, and the input w belongs to the signal class $\mathcal{W}$. If $\mathcal{W} \in IQC(\sigma_w)$, then the system satisfies σ_p-performance criterion if

$$\sigma_p(z, w) + \sigma_w(w) < 0, \tag{7}$$

for all $w \in L_2$.

Remark 2. In the practical situation, the signal class $\mathcal{W}$ is usually characterized by several IQCs, σ_{w1}, σ_{w2},

$\cdots$, σ_{wn}. In this case, the performance condition (7) should be stated as a convex feasibility problem over the set of IQCs.

$$\text{Find } \lambda_k > 0,\ k = 1, 2, \cdots, n, \quad \text{such that}$$
$$\sigma_p(z, w) + \sum_{k=1}^{n} \lambda_k \sigma_{wk}(w) < 0, \ \ \forall\ w \in L_2, \qquad (8)$$

This is so-called S-procedure [5], which has been broadly used in the robustness analysis techniques.

By the analysis condition (8), the control synthesis problem considered in this paper can be formulated as the following feasibility problem

Control Synthesis Problem : Consider dynamical system (2). Assume that the system satisfies assumption $A1$ to $A3$ and its disturbance input w is within a set $\mathcal{W}$ which satisfies IQC σ_{w1} to σ_{wn}. The problem of synthesizing a controller such that the closed-loop system satisfies σ_p-performance criterion with the presence of w is to find a set of parameters $\lambda = \begin{bmatrix}\lambda_1, \lambda_2, \cdots, \lambda_n\end{bmatrix}^T$ and a feedback control law $u = Ky$ which internally stabilize system (2) and solve feasibility problem (8).

4 LMI Formulation for Control Synthesis

In this section, we will show that finding a controller $u = Ky$ and parameter λ which solve the feasibility problem (8) is equivalent to solving a set of linear matrix inequalities. Therefore, the controller design can be done rather efficiently by well-developed numerical algorithms.

Let $\sigma_{tot}(w) = \sum_{k=1}^{n} \lambda_k \sigma_{wk}(w)$. By combining the time-domain representation of each σ_{wk}, we can express σ_{tot} as

$$\sigma_{tot}(w) = \int_0^\infty \begin{bmatrix} x_{\pi_t} \\ w \end{bmatrix}^T \begin{bmatrix} \mathbf{G}(\lambda) & \mathbf{F}(\lambda) \\ \mathbf{F}(\lambda)^T & \mathbf{R}(\lambda) \end{bmatrix} \begin{bmatrix} x_{\pi_t} \\ w \end{bmatrix} dt,$$
$$\dot{x}_{\pi_t} = A_{\pi_t} x_{\pi_t} + B_{\pi_t} w, \ \ x_{\pi_t}(0) = 0. \qquad (9)$$

Note that $\mathbf{G}$, $\mathbf{F}$, $\mathbf{R}$ are matrix-valued linear functions of the decision variable λ. To simplify the notation we will drop the λ-dependency from the notation in the rest of the paper.

We next introduce several notations to facilitate the development of the paper. Let $x_c^T = \begin{bmatrix} x^T, x_{\pi_t}^T \end{bmatrix}$. By incorporating the state x_{π_t} with system (2), we have a new open-loop system

$$G_c \ : \ \begin{cases} \dot{x}_c = A_c x_c + B_{c1} w + B_{c2} u \\ z = C_{c1} x_c + D_{11} w + D_{12} u \\ y = C_{c2} x_c + D_{21} w \end{cases}, \qquad (10)$$

where

$$A_c = \begin{bmatrix} A & 0 \\ 0 & A_{\pi_t} \end{bmatrix}, \ \ B_{c1} = \begin{bmatrix} B_1 \\ B_{\pi_t} \end{bmatrix}, \ \ B_{c2} = \begin{bmatrix} B_2 \\ 0 \end{bmatrix},$$
$$C_{c1} = \begin{bmatrix} C_1 & 0 \end{bmatrix}, \ \ C_{c2} = \begin{bmatrix} C_2 & 0 \end{bmatrix}. \qquad (11)$$

Let the dynamical output-feedback law $u = Ky$ be

$$K \ : \ \begin{cases} \dot{x}_K = A_K x_K + B_K y \\ u = C_K x_K + D_K y \end{cases}, \qquad (12)$$

and let $x_{cl}^T = \begin{bmatrix} x^T, x_{\pi_t}^T, x_K^T \end{bmatrix}$. By combining the controller (12) and the open-loop system (10), the system matrices of the closed-loop map T_{zw} have the form

$$A_{cl} = \mathcal{A}_0 + \mathcal{B}\Theta\mathcal{C}, \quad B_{cl} = \mathcal{B}_0 + \mathcal{B}\Theta\mathcal{D}_{21},$$
$$C_{cl} = \mathcal{C}_0 + \mathcal{D}_{12}\Theta\mathcal{C}, \quad D_{cl} = D_{11} + \mathcal{D}_{12}\Theta\mathcal{D}_{21}, \qquad (13)$$

where

$$\Theta = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}, \ \ \mathcal{A}_0 = \begin{bmatrix} A_c & 0 \\ 0 & 0 \end{bmatrix}, \ \ \mathcal{B}_0 = \begin{bmatrix} B_{c1} \\ 0 \end{bmatrix},$$
$$\mathcal{B} = \begin{bmatrix} 0 & B_{c2} \\ I & 0 \end{bmatrix}, \ \ \mathcal{C}_0 = \begin{bmatrix} C_{c1} & 0 \end{bmatrix}, \ \ \mathcal{C} = \begin{bmatrix} 0 & I \\ C_{c2} & 0 \end{bmatrix},$$
$$\mathcal{D}_{12} = \begin{bmatrix} 0 & D_{12} \end{bmatrix}, \ \ \mathcal{D}_{21} = \begin{bmatrix} 0 \\ D_{21} \end{bmatrix}. \qquad (14)$$

Finally, let (G_c, F_c) and (G_{cl}, F_{cl}) be the matrices such that

$$\begin{bmatrix} x_{\pi_t} \\ w \end{bmatrix}^T \begin{bmatrix} \mathbf{G} & \mathbf{F} \\ \mathbf{F}^T & \mathbf{R} \end{bmatrix} \begin{bmatrix} x_{\pi_t} \\ w \end{bmatrix} = \begin{bmatrix} x_c \\ w \end{bmatrix}^T \begin{bmatrix} G_c & F_c \\ F_c^T & \mathbf{R} \end{bmatrix} \begin{bmatrix} x_c \\ w \end{bmatrix}$$
$$= \begin{bmatrix} x_{cl} \\ w \end{bmatrix}^T \begin{bmatrix} G_{cl} & F_{cl} \\ F_{cl}^T & \mathbf{R} \end{bmatrix} \begin{bmatrix} x_{cl} \\ w \end{bmatrix}$$

Note that

$$G_c = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{G} \end{bmatrix} \ \ F_c = \begin{bmatrix} 0 \\ \mathbf{F} \end{bmatrix} \ \ G_{cl} = \begin{bmatrix} G_c & 0 \\ 0 & 0 \end{bmatrix} \ \ F_{cl} = \begin{bmatrix} F_{cl} \\ 0 \end{bmatrix}.$$

By the notations introduced above, the problem of controller synthesis with dynamic IQCs on disturbance input w can be formulated as

$$\text{Find } \ \Theta,\ \lambda_k > 0,\ k = 1, 2, \cdots, n \quad \text{such that}$$
$$\sigma_p(z, w) + \int_0^\infty \begin{bmatrix} x_{cl} \\ w \end{bmatrix}^T \begin{bmatrix} G_{cl} & F_{cl} \\ F_{cl}^T & \mathbf{R} \end{bmatrix} \begin{bmatrix} x_{cl} \\ w \end{bmatrix} dt < 0, \qquad (15)$$

where x_{cl}, z, and w satisfy the dynamical system (6), (11) to (14). The main result of this paper is to show that solving (15) is equivalent to solving a set of LMIs.

Proposition 2. *Assume* $\Pi_{p1} > 0$ *and let*

$$\Sigma_l := \begin{bmatrix} \mathcal{B}^T & \mathcal{D}_{12}^T \Pi_{p2} & \mathcal{D}_{12}^T \end{bmatrix}, \ \ \Sigma_r := \begin{bmatrix} \mathcal{C} & \mathcal{D}_{21} & 0 \end{bmatrix},$$

where 0 *denotes the a zero matrix with appropriate size. Let* W_l *and* W_r *be the matrices whose columns span the null spaces of* Σ_l *and* Σ_r *respectively. Then* (Θ, λ) *is feasible to (15) and* A_{cl} *is a Hurwitz matrix if and only if there exists a symmetric matrix* $P_{cl} > 0$ *such that* (P_{cl}, λ) *satisfies*

$$W_l^T \Psi_1 W_l < 0, \quad \text{and} \quad W_r^T \Psi_2 W_r < 0, \tag{16}$$

where Ψ_1 *and* Ψ_2 *are defined as*

$$\begin{aligned}
\Psi_1 &= \begin{bmatrix} \psi_1 & \psi_2 & P_{cl}^{-1}\mathcal{C}_0^T \\ \psi_2^T & \psi_3 & D_{11}^T \\ \mathcal{C}_0 P_{cl}^{-1} & D_{11} & -\Pi_{p1}^{-1} \end{bmatrix}, \\
\Psi_2 &= \begin{bmatrix} \psi_4 & \psi_5 & \mathcal{C}_0^T \\ \psi_5^T & \psi_3 & D_{11}^T \\ \mathcal{C}_0 & D_{11} & -\Pi_{p1}^{-1} \end{bmatrix}, \\
\psi_1 &= \mathcal{A}_0 P_{cl}^{-1} + P_{cl}^{-1}\mathcal{A}_0^T + P_{cl}^{-1} G_{cl} P_{cl}^{-1}, \\
\psi_2 &= \mathcal{B}_0 + P_{cl}^{-1}(F_{cl} + \mathcal{C}_0^T \Pi_{p2}), \\
\psi_3 &= \mathbf{R} + \Pi_{p3} + \Pi_{p2}^T D_{11} + D_{11}^T \Pi_{p2}, \\
\psi_4 &= P_{cl}\mathcal{A}_0 + \mathcal{A}_0^T P_{cl} + G_{cl}, \\
\psi_5 &= P_{cl}\mathcal{B}_0 + F_{cl} + \mathcal{C}_0^T \Pi_{p2},
\end{aligned}$$

Proof. Inequalities in (16) can be obtained by straightforward computation along the line in [3]. ☐

For the case $\Pi_{p1} = 0$, the necessary and sufficient conditions in Proposition 2 hold with Σ_l, Σ_r, Ψ_1 and Ψ_2 replaced by

$$\Sigma_l = \begin{bmatrix} \mathcal{B}^T & \mathcal{D}_{12}^T \Pi_{p2} \end{bmatrix}, \quad \Sigma_r = \begin{bmatrix} \mathcal{C} & \mathcal{D}_{21} \end{bmatrix},$$

$$\Psi_1 = \begin{bmatrix} \psi_1 & \psi_2 \\ \psi_2^T & \psi_3 \end{bmatrix}, \quad \Psi_2 = \begin{bmatrix} \psi_4 & \psi_5 \\ \psi_5^T & \psi_3 \end{bmatrix}.$$

By Proposition 2, we can now solve the control synthesis problem (15) by first solving the following feasibility problem

$$\begin{aligned}
&\text{Find } \; P_{cl} > 0,\ \lambda_k > 0,\ k = 1, 2, \cdots, n, \quad \text{subject to} \\
&W_l^T \Psi_1 W_l < 0, \quad \text{and} \quad W_r^T \Psi_2 W_r < 0,
\end{aligned} \tag{17}$$

and finding out the solution for S-procedure parameters λ_k. However, matrix inequalities in (17) are not linear because they involve both P_{cl} and P_{cl}^{-1}, as well as $P_{cl}^{-1} G_{cl} P_{cl}^{-1}$ and $P_{cl}^{-1} F_{cl}$. We will next show that they can be transformed to a set of *linear* matrix inequalities.

Theorem 1. *Assume that* $\Pi_{p1} > 0$ *and* D_{12} *is of full column rank and has the structure* $D_{12}^T = \begin{bmatrix} I & \bar{D}_{12}^T \end{bmatrix}$. *Let* $\begin{bmatrix} W_{r_1}^T & W_{r_2}^T \end{bmatrix}^T$ *be the null space of* $\begin{bmatrix} C_2 & D_{21} \end{bmatrix}$. *Let* $W_{l_1}^T = \begin{bmatrix} -\bar{D}_{12} & I \end{bmatrix}$ *and* $W_{l_2}^T = \begin{bmatrix} -B_2 & 0 \end{bmatrix}$ *such that*

$$\begin{bmatrix} B_2^T & D_{12}^T \end{bmatrix} \begin{bmatrix} 0 & I \\ W_{l_1} & W_{l_2} \end{bmatrix} = 0. \tag{18}$$

Then there exist $P_{cl} = P_{cl}^T$ *and* λ *which satisfy the matrix inequalities in (17) if and only if there exist symmetric matrices* $\mathbf{S}$, $\mathbf{E}_1$, $\mathbf{E}_2$, *and matrix* $\mathbf{E}_3$ *such that*

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_1 & \mathbf{S}_3 \\ \mathbf{S}_3^T & \mathbf{S}_2 \end{bmatrix} > 0, \quad \mathbf{E}_1 > 0, \quad \mathbf{E}_2 > 0, \tag{19}$$

and

$$\begin{bmatrix} \Gamma_1 & \Gamma_2 & W_{l_1}^T C_1 \mathbf{E}_3 & W_{l_1}^T \Delta \\ \Gamma_2^T & \Gamma_3 & \Gamma_4 & \Gamma_5 \\ \mathbf{E}_3^T C_1^T W_{l_1} & \Gamma_4^T & \Gamma_6 & \Gamma_7 \\ \Delta^T W_{l_1} & \Gamma_5^T & \Gamma_7^T & \Gamma_8 \end{bmatrix} < 0, \tag{20}$$

$$\begin{bmatrix} \Gamma_9 & \Gamma_{10} & 0 \\ \Gamma_{10}^T & \Gamma_{11} & \Gamma_{12} \\ 0 & \Gamma_{12}^T & -\Pi_{p1}^{-1} \end{bmatrix} < 0, \tag{21}$$

$$\begin{bmatrix} \mathbf{S}_1 & \mathbf{S}_3 & I \\ \mathbf{S}_3^T & \mathbf{S}_2 - \mathbf{E}_2 & -\mathbf{E}_3^T \\ I & -\mathbf{E}_3 & \mathbf{E}_1 \end{bmatrix} > 0. \tag{22}$$

where

$$\begin{aligned}
\Delta &= D_{11} + \Pi_{p1}^{-1}\Pi_{p2} \\
\Gamma_1 &= -W_{l_1}^T \Pi_{p1}^{-1} W_{l_1} \\
\Gamma_2 &= W_{l_1}^T (C_1 \mathbf{E}_1 - \Pi_{p1}^{-1} W_{l_2}) \\
\Gamma_3 &= (A + W_{l_2}^T C_1)\mathbf{E}_1 + \mathbf{E}_1 (A + W_{l_2}^T C_1)^T - W_{l_2}^T \Pi_{p1}^{-1} W_{l_2} \\
\Gamma_4 &= (A + W_{l_2}^T C_1)\mathbf{E}_3 - \mathbf{E}_3 A_{\pi_t} \\
\Gamma_5 &= B_1 + W_{l_2}^T \Delta - \mathbf{E}_3 B_{\pi_t} \\
\Gamma_6 &= \mathbf{E}_2 A_{\pi_t} + A_{\pi_t}^T \mathbf{E}_2 + \mathbf{G} \\
\Gamma_7 &= \mathbf{E}_2 B_{\pi_t} + \mathbf{F} \\
\Gamma_8 &= \mathbf{R} + \Pi_{p3} - \Pi_{p2}^T \Pi_{p1}^{-1} \Pi_{p2} \\
\Gamma_9 &= \mathbf{S}_2 A_{\pi_t} + A_{\pi_t}^T \mathbf{S}_2 + \mathbf{G} \\
\Gamma_{10} &= (\mathbf{S}_3^T A + A_{\pi_t}^T \mathbf{S}_3^T) W_{r_1} + (\mathbf{S}_3^T B_1 + \mathbf{S}_2 B_{\pi_t} + \mathbf{F}) W_{r_2} \\
\Gamma_{11} &= W_{r_1}^T (\mathbf{S}_1 A + A^T \mathbf{S}_1) W_{r_1} + W_{r_1}^T (\mathbf{S}_1 B_1 + \mathbf{S}_3 B_{\pi_t}) W_{r_2} \\
&\quad + W_{r_2}^T (B_1^T \mathbf{S}_1 + B_{\pi_t}^T \mathbf{S}_3^T) W_{r_1} + W_{r_2}^T \Gamma_8 W_{r_2} \\
\Gamma_{12} &= W_{r_1}^T C_1^T + W_{r_2}^T \Delta^T
\end{aligned} \tag{23}$$

Proof. See appendix. ☐

If $\bar{D}_{12}$ is empty, then W_{l_1} is empty, $W_{l_2} = -B_2^T$, and inequality (20) reduces to

$$\begin{bmatrix} \Gamma_3 & \Gamma_4 & \Gamma_5 \\ \Gamma_4^T & \Gamma_6 & \Gamma_7 \\ \Gamma_5^T & \Gamma_7^T & \Gamma_8 \end{bmatrix} < 0.$$

For the case $\Pi_{p1} = 0$, we assume that[1] $D_{12}^T \Pi_{p2}$ is invertible. Let $\hat{W} = -(D_{12}^T \Pi_{p2})^{-1} B_2^T$. Then the results in Theorem 1 hold with (20) and (21) replaced by

$$\begin{bmatrix} \hat{\Gamma}_1 & \hat{\Gamma}_2 \\ \hat{\Gamma}_2^T & \hat{\Gamma}_3 \end{bmatrix} < 0, \qquad \begin{bmatrix} \mathbf{S}_2 A_{\pi_t} + A_{\pi_t}^T \mathbf{S}_2 + \mathbf{G} & \hat{\Gamma}_4 \\ \hat{\Gamma}_4^T & \hat{\Gamma}_5 \end{bmatrix} < 0$$

respectively, where

$$\begin{aligned} \hat{\Gamma}_1 = {} & A\mathbf{E}_1 + \mathbf{E}_1 A^T + (B_1 - \mathbf{E}_3 B_{\pi_t} + \mathbf{E}_1 C_1^T \Pi_{p2})\hat{W} \\ & + \hat{W}^T (B_1 - \mathbf{E}_3 B_{\pi_t} + \mathbf{E}_1 C_1^T \Pi_{p2})^T \\ & + \hat{W}^T (\mathbf{R} + \Pi_{p3} + \Pi_{p2}^T D_{11} + D_{11}^T \Pi_{p2})\hat{W} \\ \hat{\Gamma}_2 = {} & A\mathbf{E}_3 - \mathbf{E}_3 A_{\pi_t} + \hat{W}^T (\mathbf{E}_2 B_{\pi_t} + \mathbf{F} + \mathbf{E}_3 C_1^T \Pi_{p2})^T \\ \hat{\Gamma}_3 = {} & \mathbf{E}_2 A_{\pi_t} + A_{\pi_t}^T \mathbf{E}_2 + \mathbf{G} \\ \hat{\Gamma}_4 = {} & (\mathbf{S}_3^T A + A_{\pi_t}^T \mathbf{S}_3^T) W_{r_1} + (\mathbf{S}_3^T B_1 + \mathbf{S}_2 B_{\pi_t} + \mathbf{F}) W_{r_2} \\ \hat{\Gamma}_5 = {} & W_{r_1}^T (\mathbf{S}_1 A + A^T \mathbf{S}_1) W_{r_1} + W_{r_1}^T (\mathbf{S}_1 B_1 + \mathbf{S}_3 B_{\pi_t}) W_{r_2} \\ & + W_{r_2}^T (B_1^T \mathbf{S}_1 + B_{\pi_t}^T \mathbf{S}_3^T) W_{r_1} + W_{r_2}^T (\mathbf{R} + \Pi_{p3}) W_{r_2}. \\ & + W_{r_2}^T \Pi_{p2}^T (C_1 W_{r_1} + D_{11} W_{r_2}) \\ & + (W_{r_1}^T C_1^T + W_{r_2}^T D_{11}^T) \Pi_{p2} W_{r_2} \end{aligned}$$

The optimization problem (17) can now be equivalently formulated as

$$\begin{aligned} & \text{Find } \mathbf{S},\ \mathbf{E}_1,\ \mathbf{E}_2,\ \mathbf{E}_3,\ \lambda > 0, \\ & \text{subject to } \ (19) \text{ to } (23). \end{aligned} \tag{24}$$

Notice that (19) to (23) are *linearly* or *affinely* dependent on the decision variables.

Controller Reconstruction

Suppose that (24) is solvable and a solution of $(\mathbf{S}, \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \lambda)$ is obtained. We then compute

$$Q_2 = \mathbf{E}_2^{-1}, \quad Q_3 = \mathbf{E}_3 \mathbf{E}_2^{-1}, \quad Q_1 = \mathbf{E}_1 + \mathbf{E}_3 \mathbf{E}_2^{-1} \mathbf{E}_3^T.$$

It is shown in the appendix that P_{cl} and P_{cl}^{-1} are of the forms

$$P_{cl} = \begin{bmatrix} \mathbf{S} & N \\ N^T & \star \end{bmatrix} \; P_{cl}^{-1} = \begin{bmatrix} Q & M \\ M^T & \star \end{bmatrix} \; Q = \begin{bmatrix} Q_1 & Q_3 \\ Q_3^T & Q_2 \end{bmatrix}.$$

Matrix P_{cl} and a stabilizing controller K can then be reconstructed as described in [3]. Here we omit all the details.

It is clear that the dimension of the closed-loop system is the same as the dimension of matrix P_{cl}. By Lemma 1 in the appendix, P_{cl} can always be constructed with a dimension no more than twice of the matrix $\mathbf{S}$. Since the dimension of $\mathbf{S}$ is the sum of the dimension of A and A_{π_t}, we conclude that the resulting controller, if exists, always has the dimension less than or equal to the sum of the dimension of the open-loop system and the multipliers in IQCs.

5 Concluding Remarks

In this paper, we show that control synthesis with *dynamic* Integral Quadratic Constraints (IQCs) on the exogenous input can be casted as an optimization problem over a set of Linear Matrix Inequalities (LMIs). This problem has been studied in [2], where several transformations and auxiliary parameters need to be introduced in order to obtain the LMI expression. In this paper, we show that the LMIs can be obtained in a simpler way. In particular, we do not need to work with the transformation proposed in [2], no extra parameter is introduced, and the LMIs are derived directly and expressed explicitly in terms of the matrices of the open-loop system and multipliers used in the IQCs. For the future research, we would like to study how both approaches are related and the computation complexity involved with both approaches.

6 Appendix

Lemma 1. *[6] Given two matrices $X \in C^{n\times n}$, $Y \in C^{n\times n}$, and $X = X^T > 0$, $Y = Y^T > 0$. Let m be a positive integer. The there exists matrices $X_2 \in C^{n\times m}$, $X_3 = X_3^T \in C^{m\times m}$ such that*

$$\begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix} > 0, \quad \begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix}^{-1} = \begin{bmatrix} Y & \star \\ \star & \star \end{bmatrix},$$

if and only if $X - Y^{-1} \geq 0$ and $rank(X - Y^{-1}) \leq m$.

Proof of Theorem 1:

Let P_{cl} and P_{cl}^{-1} have the partition

$$\begin{aligned} P_{cl} &= \begin{bmatrix} \mathbf{S} & N \\ N^T & \star \end{bmatrix} = \left[\begin{array}{cc|c} \mathbf{S}_1 & \mathbf{S}_3 & N_1 \\ \mathbf{S}_3^T & \mathbf{S}_2 & N_2 \\ \hline N_1^T & N_2^T & \star \end{array}\right], \\ P_{cl}^{-1} &= \begin{bmatrix} Q & M \\ M^T & \star \end{bmatrix} = \left[\begin{array}{cc|c} Q_1 & Q_3 & M_1 \\ Q_3^T & Q_2 & M_2 \\ \hline M_1^T & M_2^T & \star \end{array}\right]. \end{aligned} \tag{25}$$

Since P_{cl} is positive definite, $\mathbf{S}$ and Q must be positive definite as well.

By (11) and (14), Σ_r and Σ_l can be partitioned as

$$\Sigma_r = \left[\begin{array}{cc|c|c|c} 0 & 0 & I & 0 & 0 \\ C_2 & 0 & 0 & D_{21} & 0 \end{array}\right],$$

$$\Sigma_l = \left[\begin{array}{cc|c|c|c} 0 & 0 & I & 0 & 0 \\ B_2^T & 0 & 0 & D_{12}^T \Pi_{p2} & D_{12}^T \end{array}\right], \quad D_{12} = \begin{bmatrix} I \\ \bar{D}_{12} \end{bmatrix},$$

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[1] Notice that $D_{12}^T \Pi_{p2}$ is a square matrix

and therefore their null spaces are of the forms

$$W_r = \left[\begin{array}{cc|c|c|c} 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \\ W_{r_1}^T & 0 & 0 & W_{r_2}^T & 0 \end{array}\right]^T,$$

$$W_l = \left[\begin{array}{cc|c|c|c} 0 & 0 & 0 & 0 & W_{l_1}^T \\ I & 0 & 0 & 0 & W_{l_2}^T \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & -\Pi_{p2}^T \end{array}\right]^T,$$

where $W_{l_1}^T = \begin{bmatrix} -\bar{D}_{12} & I \end{bmatrix}$, $W_{l_2}^T = \begin{bmatrix} -B_2 & 0 \end{bmatrix}$, and $\begin{bmatrix} W_{r_1}^T & W_{r_2}^T \end{bmatrix}^T$ is any basis of the kernel of $\begin{bmatrix} C_2 & D_{21} \end{bmatrix}$.

By straightforward computation, it is easy to check that $W_r^T \Psi_2 W_r$ is equal to

$$\begin{bmatrix} \Gamma_9 & 0 & \Gamma_{10} \\ 0 & -\Pi_{p1}^{-1} & C_1 W_{r_1} + D_{11} W_{r_2} \\ \Gamma_{10}^T & W_{r_1}^T C_1^T + W_{r_2}^T D_{11}^T & \Gamma_{13} \end{bmatrix},$$

where Γ_9, Γ_{10} are defined in (23), and

$$\begin{aligned} \Gamma_{13} = {} & W_{r_1}^T (\mathbf{S}_1 A + A^T \mathbf{S}_1) W_{r_1} + W_{r_1}^T (\mathbf{S}_1 B_1 + \mathbf{S}_3 B_{\pi_t}) W_{r_2} \\ & + W_{r_2}^T (B_1^T \mathbf{S}_1 + B_{\pi_t}^T \mathbf{S}_3^T) W_{r_1} + (W_{r_1}^T C_1^T + W_{r_2}^T D_{11}^T) \Pi_{p2} W_{r_2} \\ & + W_{r_2}^T \Pi_{p2}^T (C_1 W_{r_1} + D_{11} W_{r_2}) + W_{r_2}^T (\mathbf{R} + \Pi_{p3}) W_{r_2}. \end{aligned}$$

Finally, LMI (21) is obtained by an equivalent expression of $W_r^T \Psi_2 W_r < 0$

$$\begin{bmatrix} I & 0 & 0 \\ 0 & -\Pi_{p2} W_{r_2} & I \\ 0 & I & 0 \end{bmatrix}^T W_r^T \Psi_2 W_r \begin{bmatrix} I & 0 & 0 \\ 0 & -\Pi_{p2} W_{r_2} & I \\ 0 & I & 0 \end{bmatrix} < 0.$$

It is not difficult to see from Proposition 2 that $W_r^T \Psi_2 W_r$ depends *linearly* on $\mathcal{P}_{cl}$, $\mathbf{G}$, $\mathbf{F}$, and $\mathbf{R}$. Therefore, reduce $W_r^T \Psi_2 W_r < 0$ to LMI (21) is more or less straightforward. However, to obtain LMI (20) from $W_l^T \Psi_1 W_1 < 0$ is much more tricky since $W_l^T \Psi_1 W_1$ depends *nonlinearly* on $\mathcal{P}_{cl}^{-1}$ and other decision variables. The key to obtain LMI (20) is a set of nonlinear parameter transformation. Let

$$N_T = \begin{bmatrix} I & 0 & 0 \\ 0 & Q_p & 0 \\ 0 & 0 & I \end{bmatrix}, \text{ where } Q_p = \begin{bmatrix} I & 0 \\ -Q_2^{-1} Q_3^T & Q_2^{-1} \end{bmatrix}.$$

Since Q_p is of full rank, so is N_T. Therefore, $W_l^T \Psi_1 W_1 < 0$ is equivalent to $N_T^T W_l^T \Psi_1 W_1 N_T < 0$. By a lengthy algebraic manipulation, which can not be included here due to the limit of space, it can be verified that $N_T^T W_l^T \Psi_1 W_1 N_T < 0$ is exactly the LMI (20) with

$$\mathbf{E}_1 = Q_1 - Q_3 Q_2^{-1} Q_3^T, \quad \mathbf{E}_2 = Q_2^{-1}, \quad \mathbf{E}_3 = Q_3 Q_2^{-1}.$$

Notice that this is a one-to-one correspondence, i.e., supposed that $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$ is known, (Q_1, Q_2, Q_3) can be computed as

$$Q_2 = \mathbf{E}_2^{-1}, \quad Q_3 = \mathbf{E}_3 \mathbf{E}_2^{-1}, \quad Q_1 = \mathbf{E}_1 + \mathbf{E}_3 \mathbf{E}_2^{-1} \mathbf{E}_3^T.$$

Finally, by Lemma 1, we need to impose the constraint $\mathbf{S} - Q^{-1} > 0$ such that (25) can be realized. Notice that

$$\begin{bmatrix} Q_1 & Q_3 \\ Q_3^T & Q_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ Q_3^T & Q_2 \end{bmatrix}^T \begin{bmatrix} \mathbf{E}_1 & 0 \\ 0 & \mathbf{E}_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ Q_3^T & Q_2 \end{bmatrix},$$

and therefore we have

$$\begin{aligned} \begin{bmatrix} Q_1 & Q_3 \\ Q_3^T & Q_2 \end{bmatrix}^{-1} &= \begin{bmatrix} I & 0 \\ Q_3^T & Q_2 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{E}_1 & 0 \\ 0 & \mathbf{E}_2 \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ Q_3^T & Q_2 \end{bmatrix}^{-T} \\ &= \begin{bmatrix} I & 0 \\ -\mathbf{E}_3^T & \mathbf{E}_2 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 & 0 \\ 0 & \mathbf{E}_2 \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ -\mathbf{E}_3^T & \mathbf{E}_2 \end{bmatrix}^T \end{aligned}$$

Thus, by Schur's complement, $\mathbf{S} - Q^{-1} > 0$ can be expressed in terms of $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$

$$\left[\begin{array}{cc|cc} \mathbf{S}_1 & \mathbf{S}_3 & I & 0 \\ \mathbf{S}_3^T & \mathbf{S}_2 & -\mathbf{E}_3^T & \mathbf{E}_2 \\ \hline I & -\mathbf{E}_3^T & \mathbf{E}_1 & 0 \\ 0 & \mathbf{E}_2 & 0 & \mathbf{E}_2 \end{array}\right] > 0,$$

which is equivalent to the LMI (22)

$$\begin{bmatrix} \mathbf{S}_1 & \mathbf{S}_3 & I \\ \mathbf{S}_3^T & \mathbf{S}_2 - \mathbf{E}_2 & -\mathbf{E}_3^T \\ I & -\mathbf{E}_3 & \mathbf{E}_1 \end{bmatrix} > 0.$$

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