On Pathwise Uniqueness for the Zakai Filter Equation

Vladimir M. Lucic and Andrew J. Heunis
Department of Electrical and Computer Engineering
University of Waterloo
Waterloo, Ontario N2L 3G1, Canada
email: lucic@kingcong.uwaterloo.ca and heunis@kingcong.uwaterloo.ca

Abstract

We study a nonlinear filtering problem in which the signal to be estimated is conditioned by the observations. The main result establishes pathwise uniqueness for the unnormalized (Zakai) filter equation.

1 Introduction

An early work on uniqueness for the stochastic differential equations of nonlinear filtering is that of Szpirglas [8]. The basic viewpoint in [8] is to regard the measure-valued stochastic differential equations of nonlinear filtering as entities quite separate from the original nonlinear filtering problem, for which one can formulate the notions of solution (or weak solution), pathwise uniqueness and uniqueness in law, by analogy with the corresponding definitions proposed by Yamada and Watanabe [9] for Itô stochastic differential equations. With these notions at hand, it is then established in [8] that pathwise uniqueness and uniqueness in law hold for both the normalized (Fujisaki-Kallianpur-Kunita) and unnormalized (Duncan-Mortensen-Zakai) filter equations, in the case of a nonlinear filtering problem where the signal is a Markov process which is *independent of the Wiener process in the observation equation*, and the sensor function in the observation equation is uniformly bounded.

Our goal is to look at uniqueness for the stochastic differential equations of nonlinear filtering from a point of view similar to that of Szpirglas [8], but for a nonlinear filtering problem in which there is *dependence of the signal on the observations*. We shall in fact look at the specific nonlinear filtering problem where the signal $\{X_t\}$ is an $\mathbb{R}^d$-valued process solving an equation of the form

$$dX_t = b(X_t)\,dt + B(X_t)\,dW_t + c(X_t)\,dV_t, \tag{1.1}$$

the $\mathbb{R}^{d_1}$-valued observation process is defined by

$$Y_t = W_t + \int_0^t h(X_s)\,ds, \tag{1.2}$$

and $\{(W_t, V_t)\}$ is a standard $\mathbb{R}^{d_1+d_2}$-valued Wiener process (precise conditions on the mappings $b(\cdot)$, $B(\cdot)$, $c(\cdot)$ and $h(\cdot)$ will be stated in Section 2). The pair (1.1) and (1.2) represents a simple model of a signal and observation in which the signal $\{X_t\}$ depends on the Wiener process $\{W_t\}$ of the observation equation.

Motivated by Szpirglas [8], we shall regard the filter equations for this nonlinear filtering problem as measure-valued stochastic differential equations, defined quite independently of the filtering problem, and will formulate the notions of weak solution and pathwise uniqueness for the Zakai (or unnormalized) filter equation. Our main result (see Theorem 2.7 to follow) establishes pathwise uniqueness for the unnormalized filter equation, subject to reasonably general conditions on the mappings $b(\cdot)$, $B(\cdot)$, and $c(\cdot)$ in the signal equation (1.1), and a uniform boundedness condition on the sensor function $h(\cdot)$ in the observation equation (1.2). As will be seen from the discussion of Section 2 (see Remark 2.9) the elegant semigroup ideas used by Szpirglas [8] to establish pathwise uniqueness do not extend to the filtering problem represented by (1.1) and (1.2), where the signal $\{X_t\}$ depends on the observation Wiener process $\{W_t\}$, and we will use an approach quite different from that of [8].

2 Stochastic Differential Equations of Nonlinear Filtering

Remark 2.1. For a complete separable metric space E, let $\mathcal{B}(E)$ denote the corresponding Borel σ-algebra. $B(E)$ denotes the set of real-valued uniformly-bounded Borel measurable mappings on E, while $C(E)$ denotes the set of real-valued continuous functions on E, and $\bar{C}(E)$ denotes the members of $C(E)$ which are uniformly bounded. Also, let $\mathcal{M}^+(E)$ denote the space of all positive bounded measures on the measurable space $(E, \mathcal{B}(E))$, with the usual topology of weak convergence, and let $\mathcal{P}(E)$ denote the collection of all members of $\mathcal{M}^+(E)$ which are probability measures. Let $C^\infty(\mathbb{R}^q)$ denote the set of infinitely smooth real-valued mappings on Euclidean space $\mathbb{R}^q$, and let $C_c^\infty(\mathbb{R}^q)$ be the collection of members of $C^\infty(\mathbb{R}^q)$ with compact support. Finally, let $\hat{C}(\mathbb{R}^q)$ denote the collection of members of $\bar{C}(\mathbb{R}^q)$ which vanish at infinity.

Consider a nonlinear filtering problem made up of the following basic elements:

E.1 A fixed finite interval of interest $[0, T]$.

E.2 A complete probability space $(\Omega, \mathcal{F}, P)$ carrying a filtration $\{\mathcal{F}_t,\ t \in [0,T]\}$ such that $\mathcal{F}_0$ includes all null events of $(\Omega, \mathcal{F}, P)$. Defined on $(\Omega, \mathcal{F}, P)$ is an $\mathbb{R}^d$-valued continuous $\{\mathcal{F}_t\}$-adapted process $\{X_t,\ t \in [0,T]\}$ and an $\mathbb{R}^{d_1+d_2}$-valued $\{\mathcal{F}_t\}$-Wiener process $\{(W_t, V_t),\ t \in [0,T]\}$ such that (1.1) holds, where $b : \mathbb{R}^d \to \mathbb{R}^d$, $B : \mathbb{R}^d \to \mathbb{R}^{d \times d_1}$, and $c : \mathbb{R}^d \to \mathbb{R}^{d \times d_2}$ are Borel-measurable and locally bounded functions (that is, uniformly bounded over bounded subsets of $\mathbb{R}^d$).

E.3 An $\mathbb{R}^{d_1}$-valued *observation process* $\{Y_t,\ t \in [0,T]\}$ defined by (1.2), where $h : \mathbb{R}^d \to \mathbb{R}^{d_1}$ is Borel-measurable and uniformly bounded.

Define the *observation filtration* $\{\mathcal{F}_t^Y\}$ by

$$\mathcal{F}_t^Y := \sigma\{Y_u,\ u \in [0,t]\} \vee \mathcal{N}(P), \tag{2.3}$$

where $\mathcal{N}(P) := \{N \in \mathcal{F} :\ P(N) = 0\}$. From Lemma 1.1 of Kurtz and Ocone [4] there exists some $\mathcal{P}(\mathbb{R}^d)$-valued $\{\mathcal{F}_t^Y\}$-optional process $\{\pi_t,\ t \in [0,T]\}$, called the *optimal filter*, which is defined on $(\Omega, \mathcal{F}, P)$ and satisfies

$$\pi_t \phi = \mathrm{E}[\phi(X_t) | \mathcal{F}_t^Y] \quad a.s., \tag{2.4}$$

for each $t \in [0,T]$ and $\phi \in B(\mathbb{R}^d)$. The $\mathbb{R}^{d_1}$-valued *innovations process* defined by

$$I_t^k := Y_t^k - \int_0^t \pi_s h^k \, ds, \tag{2.5}$$

is a $\{\mathcal{F}_t^Y\}$-Wiener process (see Theorem VI.8.4 of Rogers and Williams [6]). Now define $m : \mathbb{R}^d \to \mathbb{R}^{d \times (d_1+d_2)}$ by

$$m(x) := \begin{bmatrix} B(x) & c(x) \end{bmatrix}, \quad \forall\, x \in \mathbb{R}^d,$$

and put

$$\begin{aligned} \mathcal{A}\phi(x) &:= \sum_{i=1}^d b^i(x) \partial_i \phi(x) \\ &+ \frac{1}{2} \sum_{i,j=1}^d [m(x) m^T(x)]^{ij} \partial_i \partial_j \phi(x), \end{aligned}$$

$$\mathcal{B}_k \phi(x) := \sum_{i=1}^d B^{ik}(x) \partial_i \phi(x), \quad k = 1, \ldots, d_1,$$

for each $x \in \mathbb{R}^d$, $\phi \in C^\infty(\mathbb{R}^d)$. Then, from Theorem VI.8.11 of Rogers and Williams [6], we see that the optimal filter $\{\pi_t,\ t \in [0,T]\}$ satisfies the relation

$$\begin{aligned} d\pi_t \phi = \pi_t(\mathcal{A}\phi)\, dt \quad &+ \sum_{k=1}^{d_1} [\pi_t(h^k \phi + \mathcal{B}_k \phi) \\ &- (\pi_t h^k)(\pi_t \phi)]\, dI_t^k, \end{aligned} \tag{2.6}$$

for each $\phi \in C_c^\infty(\mathbb{R}^d)$. This is known variously as the Fujisaki-Kallianpur-Kunita equation, the Kushner-Stratonovich equation, or the normalized filter equation.

One can typically associate a simpler *unnormalized filter equation* with the normalized filter equation. For this purpose the following additional notation is useful: If $\{M_t\}$ is a continuous $\{\mathcal{F}_t\}$-semimartingale, and $\{\gamma_t\}$ is a locally bounded $\{\mathcal{F}_t\}$-progressively measurable process, then $\gamma \bullet M$ denotes the stochastic integral of γ with respect to M. Also, put

$$\mathcal{E}(M)_t := \exp\left(M_t - \frac{1}{2} \langle M \rangle_t \right),$$

$$\chi_t := \mathcal{E}\left(-\sum_{k=1}^{d_1} (\pi h^k) \bullet I^k \right)_t, \quad \forall\, t \in [0,T].$$

Define the $\mathcal{M}^+(\mathbb{R}^d)$-valued process $\{\sigma_t\}$ by

$$\sigma_t \phi := \frac{\pi_t \phi}{\chi_t}, \quad \forall\, t \in [0,T],\ \phi \in B(\mathbb{R}^d). \tag{2.7}$$

Using Itô's formula and the normalized filter equation (2.6), we easily arrive at the *Duncan-Mortensen-Zakai equation* or *unnormalized filter equation*: for each $\phi \in C_c^\infty(\mathbb{R}^d) \cup \{1\}$ we have

$$d\sigma_t\phi = \sigma_t(\mathcal{A}\phi)\, dt + \sum_{k=1}^{d_1} \sigma_t(h^k\phi + \mathcal{B}_k\phi)\, dY_t^k.$$

Furthermore, observe from the uniform boundedness of $h(\cdot)$ that $\{(\chi_t, \mathcal{F}_t),\ t \in [0,T]\}$ is a *martingale*, so that

$$Q(A) := \mathrm{E}^{\tilde{P}}[\chi_T; A], \quad \forall A \in \mathcal{F}, \tag{2.8}$$

defines a probability measure Q on $(\Omega, \mathcal{F})$ which is equivalent to the probability measure P. Now the Girsanov theorem shows that $\{(Y_t, \mathcal{F}_t),\ t \in [0,T]\}$ is a Wiener process on $(\Omega, \mathcal{F}, Q)$.

Motivated by this discussion, we next formulate the notion of a weak solution of the unnormalized filter equation on a general probability space which is generally distinct from the fixed probability space $(\Omega, \mathcal{F}, P)$ on which the nonlinear filtering problem is defined (see **E.2**).

Definition 2.2. A *weak solution* of the unnormalized filter equation is a pair $\{(\tilde\Omega, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde Q), (\tilde\sigma_t, \tilde Y_t)\}$ such that

1. $(\tilde\Omega, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde Q)$ is a complete filtered probability space;

2. $\{\tilde Y_t,\ t \in [0,T]\}$ is an $\mathbb{R}^{d_1}$-valued $\{\tilde{\mathcal{F}}_t\}$-Wiener process;

3. $\{\tilde\sigma_t,\ t \in [0,T]\}$ is a $\mathcal{M}^+(\mathbb{R}^d)$-valued continuous $\{\tilde{\mathcal{F}}_t\}$-adapted process such that the random element $\tilde\sigma_0$ takes values in $\mathcal{P}(\mathbb{R}^d)$, and, for each $\phi \in C_c^\infty(\mathbb{R}^d) \cup \{1\}$, we have

$$d\tilde\sigma_t\phi = \tilde\sigma_t(\mathcal{A}\phi)\, dt + \sum_{k=1}^{d_1} \tilde\sigma_t(h^k\phi + \mathcal{B}_k\phi)\, d\tilde Y_t^k. \tag{2.9}$$

Definition 2.3. The unnormalized filter equation has the property of *pathwise uniqueness* when the following holds: If $\{(\tilde\Omega, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde Q), (\tilde\sigma_t^1, \tilde Y_t)\}$ and $\{(\tilde\Omega, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde Q), (\tilde\sigma_t^2, \tilde Y_t)\}$ are weak solutions of the unnormalized filter equation such that $\tilde Q(\tilde\sigma_0^1 = \tilde\sigma_0^2) = 1$, then

$$\tilde Q\left(\tilde\sigma_t^1 = \tilde\sigma_t^2 \quad \forall t \in [0,T]\right) = 1.$$

Our goal is to establish pathwise uniqueness for the unnormalized filter equation. To this end we postulate the following conditions on the mappings $b(\cdot)$, $B(\cdot)$, $c(\cdot)$ in (1.1), and the mapping $h(\cdot)$ in (1.2):

Condition 2.4. *The mapping $b : \mathbb{R}^d \to \mathbb{R}^d$ is Borel-measurable, and the mappings $B : \mathbb{R}^d \to \mathbb{R}^{d\times d_1}$ and $c : \mathbb{R}^d \to \mathbb{R}^{d\times d_2}$ are continuous. There exists a constant $C \in [0,\infty)$ such that $\max_{i,j,k}\{|b^i(x)|, |B^{ij}(x)|, |c^{ik}(x)|\} \le C[1 + |x|]$, for each $x \in \mathbb{R}^d$.*

Condition 2.5. *The mapping $c : \mathbb{R}^d \to \mathbb{R}^{d\times d_2}$ is such that the matrix $c(x)c^T(x)$ is strictly positive definite for every $x \in \mathbb{R}^d$.*

Condition 2.6. *The mapping $h : \mathbb{R}^d \to \mathbb{R}^{d_1}$ is Borel-measurable and uniformly bounded.*

We can now state our main result:

Theorem 2.7. *Suppose that Conditions 2.4, 2.5, and 2.6 hold for the nonlinear filtering problem given by* **E.1**, **E.2** *and* **E.3**. *Then the unnormalized filter equation has the property of pathwise uniqueness.*

Remark 2.8. Following Yamada and Watanabe [9] one can formulate notions of *uniqueness in law* for weak solutions of the normalized and unnormalized filter equations. Then Theorem 2.7, together with a modification of the basic construction of Yamada and Watanabe [9], can be used to establish uniqueness in law for the normalized and unnormalized filter equations. See [5].

Remark 2.9. Szpirglas [8] establishes pathwise uniqueness for the unnormalized filter equations in the case where the signal $\{X_t\}$ is *independent* of the Wiener process $\{W_t\}$ in the observation equation, and adopts a somewhat stronger notion of weak solution than we do in Definition 2.2, namely it is insisted that the unnormalized filter relation hold for each and every function in the domain of the infinitesimal generator of the signal $\{X_t\}$. In the context of the nonlinear filtering problem given by **E.1**, **E.2**, and **E.3** this amounts to taking $\mathcal{B}_k \equiv 0$, so that a weak solution $\{(\tilde\Omega, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde Q), (\tilde\sigma_t, \tilde Y_t)\}$ satisfies the relation

$$d\tilde\sigma_t\phi = \tilde\sigma_t(\mathcal{L}\phi)\, ds + \sum_{k=1}^{d_1} \tilde\sigma_t(h^k\phi)\, d\tilde Y_t^k, \tag{2.10}$$

for each $\phi \in \mathcal{D}(\mathcal{L})$, the domain of the infinitesimal generator $\mathcal{L}$ of $\{X_t\}$. The nice thing about

(2.10) is that it includes reference to just one unbounded linear operator, namely the generator $\mathcal{L}$ of the signal $\{X_t\}$, and the resolvent identity can be used to eliminate $\mathcal{L}$, and re-write (2.10) in integral form

$$\tilde{\sigma}_t\phi = \tilde{\sigma}_0(P_t\phi) + \sum_{k=1}^{d_1}\int_0^t \tilde{\sigma}_s(h^k P_{t-s}\phi)d\tilde{Y}_s^k, \tag{2.11}$$

where $\{P_t\}$ is the Borel semigroup with infinitesimal generator $\mathcal{L}$ (see Théorème IV.1 of [8]). Consequently, it is enough to establish pathwise uniqueness for (2.11) in order to conclude pathwise uniqueness for the unnormalized filter equation. Now (2.11) involves only the *bounded* linear operators $\{P_t\}$, and this structure makes it possible to establish pathwise uniqueness for solutions of (2.11) by iterating a Gronwall-like integral inequality (see Section V.2 of Szpirglas [8]). Comparing (2.10) with the unnormalized filter equation (2.9) for the nonlinear filtering problem defined by (1.1) and (1.2), we see that (2.9) includes two unbounded linear operators, namely the first-order differential operator $\mathcal{B}_k$ which results from dependence of the signal $\{X_t\}$ on the Wiener process $\{W_t\}$ of the observation equation, as well as the second-order differential operator $\mathcal{A}$ corresponding to the signal process $\{X_t\}$. In this case there seems to be no clear way of adapting the elegant semigroup ideas of [8] to remove both of these unbounded operators and get an equivalent equation involving just bounded linear operators. Accordingly, the approach that we shall use to establish Theorem 2.7 is different from that of Szpirglas [8], and relies on a uniqueness theorem for measure-valued evolution equations (see Theorem 3.14 to follow).

Remark 2.10. Uniqueness for the normalized and unnormalized filter equations has also been studied by Bhatt, Kallianpur and Karandikar [1], Rozovskii [7], and Kurtz and Ocone [4] from a somewhat different point of view than that taken by Szpirglas [8] and the present work. To see this in the context of the filtering problem given by (1.1) and (1.2), recall that the unnormalized optimal filter $\{\sigma_t\}$ defined by (2.7) solves the Duncan-Mortensen-Zakai equation, namely for each $\phi \in C_c^\infty(\mathbb{R}^d) \cup \{1\}$ we have

$$d\sigma_t\phi = \sigma_t(\mathcal{A}\phi)\,dt + \sum_{k=1}^{d_1}\sigma_t(h^k\phi + \mathcal{B}_k\phi)\,dY_t^k. \tag{2.12}$$

With this in mind, the following question is natural: Suppose that $\{\rho_t\}$ is some $\mathcal{M}^+(\mathbb{R}^d)$-valued, cadlag, and $\{\mathcal{F}_t^Y\}$-adapted process on $(\Omega, \mathcal{F}, P)$, such that for each $\phi \in C_c^\infty(\mathbb{R}^d) \cup \{1\}$ we have

$$d\rho_t\phi = \rho_t(\mathcal{A}\phi)\,dt + \sum_{k=1}^{d_1}\rho_t(h^k\phi + \mathcal{B}_k\phi)\,dY_t^k, \tag{2.13}$$

with

$$\sigma_0 = \rho_0 \quad \text{a.s.}$$

Does it necessarily follow that $\{\sigma_t\}$ and $\{\rho_t\}$ are indistinguishable? The works of Bhatt, Kallianpur and Karandikar ([1], Theorem 3.1), Rozovskii [7], and Kurtz and Ocone ([4], Theorems 4.2 and 4.7), provide conditions on the nonlinear filtering problem for which the answer is in the affirmative. It should be noted that uniqueness in this sense can be established for much more general nonlinear filtering problems than that represented by the simple model (1.1) and (1.2), but, on the other hand, the pathwise uniqueness established in [1], [7] and [4] is quite different from that given by Theorem 2.7, since the candidate solution $\{\rho_t\}$ of the filter equation (2.13) is postulated to be adapted specifically to the observation filtration $\{\mathcal{F}_t^Y\}$ (in fact, the arguments used in [1], [7], and [4], rely crucially on this restriction). In contrast, Theorem 2.7(i) establishes pathwise uniqueness in the more general sense of Definition 2.3, where the candidate solutions $\{(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde{Q}), (\tilde{\sigma}_t^1, \tilde{Y}_t)\}$ and $\{(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde{Q}), (\tilde{\sigma}_t^2, \tilde{Y}_t)\}$ are defined on an arbitrary filtered probability space, and there is no insistence that the measure-valued components $\{\tilde{\sigma}_t^1\}$ and $\{\tilde{\sigma}_t^2\}$ of the two solutions be adapted to the self-filtration of $\{\tilde{Y}_t\}$. The usefulness of this latter notion of pathwise uniqueness is that, by an adaptation to the filter equations of the construction of Yamada and Watanabe [9], it leads to uniqueness in law for the normalized and unnormalized filter equations (see Remark 2.8). Uniqueness in law turns out to be essential for studying weak limits and approximations of the nonlinear filter equations by the method of martingale problems and weak convergence.

3 Proof of Theorem 2.7

Remark 3.11. Suppose that E is a complete separable metric space, and $\mathcal{Q} : \mathcal{D}(\mathcal{Q}) \to B(E)$ is a mapping with domain $\mathcal{D}(\mathcal{Q}) \subset B(E)$. Then $\{\mu_t,\ t \in [0,T]\}$ is called an $\mathcal{M}^+(E)$-valued solution of the evolution equation for $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$, when (i) $\mu_t \in \mathcal{M}^+(E)$, $\forall\ t \in [0,T]$, and $\mu_0 \in \mathcal{P}(E)$; (ii) for each $\Gamma \in \mathcal{B}(E)$, the mapping $t \to$

$\mu_t(\Gamma) : [0,T] \to [0,\infty)$ is Borel-measurable; (iii) for each $f \in \mathcal{D}(\mathcal{Q})$ we have $\int_0^T |\mu_s(\mathcal{Q}f)| ds < \infty$, and

$$\mu_t f = \mu_0 f + \int_0^t \mu_s(\mathcal{Q}f)\, ds, \quad \forall t \in [0,T]. \tag{3.14}$$

The evolution equation for $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ is said to have uniqueness in the class of $\mathcal{M}^+(E)$-valued solutions over the interval $[0,T]$ when, for any two such solutions $\{\mu_t^i,\ t \in [0,T]\}$, $i = 1,2$, with $\mu_0^1 = \mu_0^2$, it follows that $\mu_t^1 = \mu_t^2, \quad \forall t \in [0,T]$.

Proof of Theorem 2.7 Fix weak solutions $\{(\tilde\Omega, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde Q), (\tilde\sigma_t^i, \tilde Y_t)\}$, $i = 1,2$, of the unnormalized filter equation, such that

$$\tilde Q\left[\tilde\sigma_0^1 = \tilde\sigma_0^2\right] = 1, \tag{3.15}$$

and define product measures on $(\mathbb{R}^{2d}, \mathcal{B}(\mathbb{R}^{2d}))$ by

$$\mu_t^{12}(\cdot, \tilde\omega) := (\tilde\sigma_t^1 \times \tilde\sigma_t^2)(\cdot, \tilde\omega),$$

for all $(t, \tilde\omega) \in [0,T] \times \tilde\Omega$. A simple application of the Dynkin class theorem establishes that the mapping $(t,\tilde\omega) \to \mu_t^{12}(\Gamma, \tilde\omega) : \tilde\Omega \times [0,T] \to [0,\infty)$ is measurable with respect to the $\{\tilde{\mathcal{F}}_t\}$-progressive σ-algebra, for each $\Gamma \in \mathcal{B}(\mathbb{R}^{2d})$, Also put

$$\nu_t^{12}(\Gamma) := \mathrm{E}^{\tilde Q}[\mu_t^{12}(\Gamma)], \tag{3.16}$$

for all $\Gamma \in \mathcal{B}(\mathbb{R}^{2d})$, $t \in [0,T]$. It readily follows that ν_t^{12} defines a positive measure on $(\mathbb{R}^{2d}, \mathcal{B}(\mathbb{R}^{2d}))$ for every $t \in [0,T]$. Using uniform boundedness of $h(\cdot)$ it is easily shown that

$$\sup_{t\in[0,T]} \nu_t^{12}(\mathbb{R}^{2d}) < \infty.$$

This shows that ν_t^{12} is a positive measure on $(\mathbb{R}^{2d}, \mathcal{B}(\mathbb{R}^{2d})$, uniformly bounded with respect to $t \in [0,T]$, while Fubini's theorem shows that the mapping $t \to \nu_t^{12}(\Gamma) : [0,T] \to \mathbb{R}$ is Borel-measurable for each $\Gamma \in \mathcal{B}(\mathbb{R}^{2d})$. Next, define $\nu_t^{11}, \nu_t^{22} \in \mathcal{M}^+(\mathbb{R}^{2d})$, $t \in [0,T]$, analogously to ν_t^{12}, by

$$\nu_t^{ii}(\Gamma) := \mathrm{E}^{\tilde Q}[(\tilde\sigma_t^i \times \tilde\sigma_t^i)(\Gamma)], \tag{3.17}$$

for each $\Gamma \in \mathcal{B}(\mathbb{R}^{2d})$, $t \in [0,T]$, $i = 1,2$. In the same way as for $\nu_\cdot^{12}$, we see that $\nu_\cdot^{ii}$ are positive measures on $(\mathbb{R}^{2d}, \mathcal{B}(\mathbb{R}^{2d}))$, uniformly bounded with respect to $t \in [0,T]$, and the mappings $t \to \nu_t^{ii}(\Gamma) : [0,T] \to \mathbb{R}$ are Borel-measurable for each $\Gamma \in \mathcal{B}(\mathbb{R}^{2d})$, $i = 1,2$.

For mappings $f_1, f_2 \in B(\mathbb{R}^d)$ define the tensor product of f_1 with f_2 to be the mapping $f_1 \otimes f_2 : \mathbb{R}^{2d} \to \mathbb{R}$ given by

$$f_1 \otimes f_2(x_1, x_2) := f_1(x_1) f_2(x_2),$$

for all $x_1, x_2 \in \mathbb{R}^d$. In view of (3.16) and (3.17), for each $f_1, f_2 \in B(\mathbb{R}^d)$ we have

$$\nu_t^{12}(f_1 \otimes f_2) \quad = \quad \mathrm{E}^{\tilde Q}[(\tilde\sigma_t^1 f_1)(\tilde\sigma_t^2 f_2)], \tag{3.18}$$

$$\nu_t^{ii}(f_1 \otimes f_2) \quad = \quad \mathrm{E}^{\tilde Q}[(\tilde\sigma_t^i f_1)(\tilde\sigma_t^i f_2)], \tag{3.19}$$

for $i = 1,2$. From (3.15), (3.16), and (3.17) we see that ν_0^{11}, ν_0^{22}, and ν_0^{12} are probability measures on $\mathcal{B}(\mathbb{R}^{2d})$, and

$$\nu_0^{11} = \nu_0^{22} = \nu_0^{12}. \tag{3.20}$$

Using this fact, we shall establish

$$\nu_t^{11} = \nu_t^{22} = \nu_t^{12}, \quad t \in [0,T], \tag{3.21}$$

from which pathwise uniqueness follows. Indeed, if (3.21) holds, then for each $f \in B(\mathbb{R}^d)$ we have

$$\nu_t^{11}(f \otimes f) = \nu_t^{22}(f \otimes f) = \nu_t^{12}(f \otimes f),$$

and therefore from (3.18) and (3.19),

$$\begin{aligned} &\mathrm{E}^{\tilde Q}[(\tilde\sigma_t^1 f - \tilde\sigma_t^2 f)^2] \\ &= \nu_t^{11}(f \otimes f) - 2\nu_t^{12}(f \otimes f) + \nu_t^{22}(f \otimes f) \\ &= 0. \end{aligned}$$

Thus, for each $t \in [0,T]$ and $f \in B(\mathbb{R}^d)$, we have

$$\tilde Q\left[\tilde\sigma_t^1 f = \tilde\sigma_t^2 f\right] = 1. \tag{3.22}$$

Since $\hat C(\mathbb{R}^d)$, equipped with the supremum norm, is separable and separates bounded positive measures on $\mathcal{B}(\mathbb{R}^d)$ (see Problem 5.4.25 of Karatzas and Shreve [3]), it follows from (3.22) that $\tilde Q[\tilde\sigma_t^1 = \tilde\sigma_t^2] = 1$ for each $t \in [0,T]$. Now Theorem 2.7 follows from the fact that $\{\tilde\sigma_t^i,\ t \in [0,T]\}$ are continuous (see Definition 2.2).

It remains to establish (3.21) in order to prove Theorem 2.7. To this end, for each $x_1, x_2 \in \mathbb{R}^d$ define the $2d \times 2d$ matrix $\bar a(x_1, x_2)$, the $2d$ vector $\bar b(x_1, x_2)$, and the real number $\bar h(x_1, x_2)$ by

$$\bar a(x_1, x_2) := \begin{bmatrix} cc^T(x_1) & 0 \\ 0 & cc^T(x_2) \end{bmatrix} \tag{3.23a}$$

$$+ \begin{bmatrix} B(x_1) \\ B(x_2) \end{bmatrix} \begin{bmatrix} B^T(x_1) & B^T(x_2) \end{bmatrix} \tag{3.23b}$$

$$\bar b(x_1, x_2) := \begin{bmatrix} b(x_1) + B(x_1)h(x_2) \\ b(x_2) + B(x_2)h(x_1) \end{bmatrix} \tag{3.23c}$$

$$\bar h(x_1, x_2) := \sum_{k=1}^{d_1} h^k(x_1) h^k(x_2). \tag{3.23d}$$

Remark 3.12. Observe that the matrix $\bar{a}(x_1, x_2)$ is symmetric and strictly positive-definite (see Condition 2.5), that $\bar{a}(\cdot)$ is continuous on $\mathbb{R}^{2d}$, $\bar{b}(\cdot)$ is Borel-measurable on $\mathbb{R}^{2d}$, and $\bar{h} \in B(\mathbb{R}^{2d})$. Moreover, from Conditions 2.4 and 2.6, there is a constant $K \in [0, \infty)$ such that $|\bar{b}^i(x)| \leq K[1 + |x|]$ and $|\bar{a}^{ij}(x)| \leq K[1 + |x|^2]$ for all $x \in \mathbb{R}^{2d}$. Let $\bar{\mathcal{A}}$ be the second order linear differential operator corresponding to the matrices $\bar{a}$ and $\bar{b}$, namely

$$\bar{\mathcal{A}}\phi(x) := \sum_{i=1}^{2d} \bar{b}^i(x)\partial_i\phi(x) \tag{3.24}$$

$$+ \frac{1}{2}\sum_{i,j=1}^{2d} \bar{a}^{ij}(x)\partial_i\partial_j\phi(x), \tag{3.25}$$

for all $\phi \in C^\infty(\mathbb{R}^{2d})$, $x \in \mathbb{R}^{2d}$. Then $\bar{\mathcal{A}}$ has the following property which is established in [5]:

Lemma 3.13. *Suppose that Conditions 2.4–2.6 hold. Then* $\{\nu_t^{11},\ t \in [0, T]\}$, $\{\nu_t^{12},\ t \in [0, T]\}$, *and* $\{\nu_t^{22},\ t \in [0, T]\}$, *given by* (3.16) *and* (3.17), *are* $\mathcal{M}^+(\mathbb{R}^{2d})$*-valued solutions of the evolution equation for* $(\bar{\mathcal{A}} + \bar{h}, \mathrm{span}\{1, C_c^\infty(\mathbb{R}^{2d})\})$.

It remains to show that the evolution equation for $(\bar{\mathcal{A}} + \bar{h}, \mathrm{span}\{1, C_c^\infty(\mathbb{R}^{2d})\})$ has uniqueness in the class of $\mathcal{M}^+(\mathbb{R}^{2d})$-valued solutions over the interval $[0, T]$, since this fact, along with (3.20) and Lemma 3.13, gives (3.21), as required to establish Theorem 3.14(i). To this end we need the following result from [2] on uniqueness of measure-valued solutions of the evolution equation corresponding to a multiplicatively perturbed linear second-order differential operator on Euclidean space:

Theorem 3.14. *Let* $\mathcal{C}$ *be the linear second-order differential operator on the finite-dimensional Euclidean space* $\mathbb{R}^q$ *defined by* $\mathcal{D}(\mathcal{C}) := \mathrm{span}\{1, C_c^\infty(\mathbb{R}^q)\}$ *and*

$$\mathcal{C}f(x) := \sum_i \beta^i(x)\partial_i f(x) \tag{3.26}$$

$$+ \frac{1}{2}\sum_{i,j} \alpha^{ij}(x)\partial_i\partial_j f(x), \tag{3.27}$$

for all $x \in \mathbb{R}^q$, $f \in \mathcal{D}(\mathcal{C})$, *where* $\beta : \mathbb{R}^q \to \mathbb{R}^q$ *is Borel measurable,* $\alpha : \mathbb{R}^q \to \mathbb{S}_{++}^{q\times q}$ *(the space of* q *by* q *real symmetric strictly positive definite matrices) is continuous, and there exists a constant* $K \in [0, \infty)$ *such that* $|\beta^i(x)| \leq K(1 + |x|)$ *and* $|\alpha^{ij}(x)| \leq K(1 + |x|^2)$ *for all* $x \in \mathbb{R}^q$. *If* $\lambda \in B(\mathbb{R}^q)$ *then the evolution equation for* $(\mathcal{C} - \lambda, \mathcal{D}(\mathcal{C}))$ *has uniqueness in the class of* $\mathcal{M}^+(\mathbb{R}^q)$*-valued solutions over the interval* $[0, T]$.

That the evolution equation for $(\bar{\mathcal{A}} + \bar{h}, \mathrm{span}\{1, C_c^\infty(\mathbb{R}^{2d})\})$ has uniqueness in the class of $\mathcal{M}^+(\mathbb{R}^{2d})$-valued solutions over the interval $[0, T]$ now follows from Remark 3.12 and Theorem 3.14 with $q := 2d$, $\beta(\cdot) := \bar{b}(\cdot)$, $\alpha(\cdot) := \bar{a}(\cdot)$, and $\lambda(\cdot) := -\bar{h}(\cdot)$. □

References

[1] A. G. Bhatt, G. Kallianpur and R. L. Karandikar. Uniqueness and robustness of solution of measure-valued equations of nonlinear filtering. *Ann. Prob.*, 23:1895–1938, 1995.

[2] A. G. Bhatt and R. L. Karandikar. Evolution equations for Markov processes: application to the white-noise theory of filtering. *Appl. Math. Optim.*, 31:327–348, 1995.

[3] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus.* Springer–Verlag, 1991.

[4] T. G. Kurtz and D. L. Ocone. Unique characterization of conditional distributions in nonlinear filtering. *Ann. Prob.*, 16:80–107, 1988.

[5] V.M. Lucic and A.J. Heunis. On uniqueness of solutions of the stochastic differential equations of nonlinear filtering. *Ann. Appl. Probab. (to appear).*

[6] L. C. G. Rogers and D. Williams. *Diffusions, Markov Processes and Martingales Vol. 2: Itô Calculus.* J. Wiley & Sons, 1987.

[7] B. L. Rozovskii. A simple proof of uniqueness for Kushner and Zakai equations. In E. Mayer-Wolf, E. Merzbach and A. Shwartz, editor, *Stochastic Analysis*, pages 449–458. Academic Press, 1991.

[8] J. Szpirglas. Sur l'équivalence d'équations différentielles stochastiques á valeurs mesures intervenant dans le filtrage markovien non linéaire. *Ann. Inst. Henri Poincaré*, XIV(1):33–59, 1978.

[9] T. Yamada and S. Watanabe. On the uniqueness of solutions of stochastic differential equations. *J. Math. Kyoto Univ.*, 11:155–167, 1971.