

On Input-to-State Stability for Time Varying Nonlinear Systems

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Abstract. Input-to-state stability was introduced about 10 years ago. This notion is nowadays a central concept in the analysis of nonlinear systems. However, most theoretical developments dealt mainly with time invariant systems. In this work we study the Lyapunov characterizations of input-to-state stability for time varying nonlinear systems, and in particular, for periodic time varying systems. We also present a small gain theorem for time varying nonlinear systems.

Keywords: time varying nonlinear systems, periodic systems, input-to-state stability, Lyapunov methods.

1. Introduction

The notion of input-to-state stability (ISS, for short) was formulated by E.D. Sotnag in [19]. During the past decade, this notion has found wide applications in modern nonlinear feedback analysis and design, see e.g., [2, 9, 6, 8, 25, 11, 10, 16, 17, 21, 22, 1]. It is now recognized as a central concept in nonlinear system theory, and it provides a nonlinear generalization of finite gains with respect to L_∞ norms. Moreover, this notion is also mathematically natural: there are characterizations in terms of dissipation, stability margins, and classical Lyapunov-like functions, see [21, 22].

On the other hand, most theoretical developments for ISS stability dealt mainly with time invariant systems. In practice, it is very often the case that the systems under consideration are time varying. Such a situation often arises from, e.g., trajectory tracking problems. It is thus natural to understand the ISS property, in particular, the Lyapunov characterizations, for time varying systems. Our main result in this aspect is that a time varying system is ISS if and only if it admits a smooth ISS-Lyapunov function; and for periodic systems, the Lyapunov function is also periodic in t with the same period.

The Lyapunov characterization was first considered in [12], where the stability properties of a time varying system were treated as the stability properties with respect to a closed, noncompact, invariant set for a corresponding auxiliary system. The proof can be significantly simplified by applying the converse Lyapunov theorems recently developed for output stability (or partial stability) in [23, 24]. The less routine part is the periodic case, where one needs to track the peri-

odic properties for the functions involved in the proofs; and more importantly, to generalize the smooth approximation result developed in [13, Appendix] to the periodic case. The significance of the existence of periodic Lyapunov functions lies in the fact that many feedback designs, such as back-stepping, are based on Lyapunov functions. It ensures that a feedback law developed on the basis of a Lyapunov function for periodic systems is again periodic with the same period.

Another contribution of this work is the nonlinear small gain theorem for time varying systems. The small gain theorems are very powerful in treating stability and stabilization problems for inter-connected systems. The first such small gain theorem for nonlinear systems was obtained in [8], and the theorem was again derived in [2] by a much simpler proof in conjunction with the results in [22]. In [7], a small gain theorem was presented in terms of Lyapunov functions. In [5], a small gain theorem was obtained for input-output operators that can be viewed as an elaboration of both small gain theorems obtained in [14] and [8]. In this work we provide a small gain theorem for time varying nonlinear systems using the Lyapunov formulation, following the same idea as in [7]. One can also develop a small gain theorem for time varying systems by applying the results in [8] to some auxiliary time invariant systems. As Lyapunov functions play a fundamental role in system analysis and design, we adopt the Lyapunov-like approach in this work to derive a small gain theorem for the time varying case. Moreover, our proof shows how to obtain a Lyapunov function for the inter-connected system based on the Lyapunov functions for each of the subsystems.

2. Input to State Stability

Consider the time varying system

$$\dot{x}(t) = f(t, x(t), u(t)), \quad (1)$$

where, for each t , $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is locally Lipschitz. Inputs, denoted by u , are measurable, locally essentially bounded functions from \mathbb{R} to \mathbb{R}^m . We use $x(t, \xi, t_0, u)$ to denote the trajectory of the system corresponding to the initial condition $x(t_0) = \xi$ and the input function u . This solution is uniquely defined on some maximum interval $[t_0, T_{t_0, \xi, u})$ with $T_{t_0, \xi, u} \leq \infty$. If $T_{t_0, \xi, u} = \infty$ for all t_0, ξ and all u , the system is said to be *forward complete*.

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Throughout this work, we use $|\xi|$ to denote the Euclidean norm for $\xi \in \mathbb{R}^n$, and, for $-\infty < a < b \leq \infty$, we use $\|u\|_{(a,b)}$ to denote the L_∞^m norm of u as a function defined on the interval (a, b) . For a closed subset \mathcal{A} of \mathbb{R}^n , we use $|\xi|_{\mathcal{A}}$ to denote the point-to-set distance from ξ to \mathcal{A} , that is,

$$|\xi|_{\mathcal{A}} = \inf\{|\xi - \eta| : \eta \in \mathcal{A}\}.$$

A nonempty subset \mathcal{A} is said to be 0-invariant for system (1) if every trajectory with zero input starting from \mathcal{A} is defined for all $t \geq t_0$ and stays in \mathcal{A} for all $t \geq t_0$. That is, for the zero-input system

$$\dot{x}(t) = f(t, x(t), 0), \quad (2)$$

it holds that $x(t, \xi, t_0, 0) \in \mathcal{A}$ for all $t \geq t_0$ whenever $\xi \in \mathcal{A}$.

A function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class \mathcal{K} if it is continuous, positive definite, and strictly increasing, and is of class \mathcal{K}_∞ if it is also unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be of class \mathcal{KL} if for each fixed $t \geq 0$, $\beta(\cdot, t)$ is of class \mathcal{K} , and for each fixed $s \geq 0$, $\beta(s, t)$ decreases to 0 as $t \rightarrow \infty$.

Definition 2.1 Suppose that system (1) is forward complete. The system is *input-to-state stable* (ISS) with respect to a nonempty, closed 0-invariant set \mathcal{A} if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that, for each initial time t_0 , each initial state ξ and each input function u , it holds that

$$|x(t, \xi, t_0, u)|_{\mathcal{A}} \leq \beta(|\xi|_{\mathcal{A}}, t - t_0) + \gamma\left(\|u\|_{(t_0, \infty)}\right) \quad (3)$$

for all $t \geq t_0$. In the case when $\mathcal{A} = \{0\}$, we simply say that such a system is ISS.

Observe that in the case when \mathcal{A} is compact, the completeness assumption is redundant in the above definition.

As in the time invariant case, if system (1) is ISS, then the corresponding 0-input system (2) is *uniformly globally asymptotically stable*, that is, there exists some $\beta \in \mathcal{KL}$ such that for every trajectory $x(t, \xi, t_0)$ of system (2), it holds that

$$|x(t, \xi, t_0)|_{\mathcal{A}} \leq \beta(|\xi|_{\mathcal{A}}, t - t_0), \quad \forall t \geq 0.$$

2.1 ISS-Lyapunov Functions

A smooth function $V : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is an *ISS-Lyapunov function with respect to a set \mathcal{A}* if there exist \mathcal{K}_∞ -functions $\underline{\alpha}, \bar{\alpha}, \chi$ and a continuous positive definite function α such that

$$\underline{\alpha}(|\xi|_{\mathcal{A}}) \leq V(t, \xi) \leq \bar{\alpha}(|\xi|_{\mathcal{A}}) \quad \forall t, \forall \xi; \quad (4)$$

and

$$|\xi|_{\mathcal{A}} \geq \chi(|\mu|) \implies \frac{\partial V}{\partial t}(t, \xi) + \frac{\partial V}{\partial \xi}(t, \xi)f(t, \xi, \mu) \leq -\alpha(|\xi|_{\mathcal{A}}). \quad (5)$$

Note that it results in an equivalent definition if one requires that the estimate (5) holds for some $\alpha \in \mathcal{K}_\infty$ (in contrast to requiring α be merely positive definite) (cf. [13, 12]).

It was shown in [21] that for a time invariant system $\dot{x} = f(x, u)$, the ISS property is equivalent to the existence of an ISS-Lyapunov function V which is independent of t , and in the case when \mathcal{A} is compact, property (5) is equivalent to the existence of $\alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ such that

$$\frac{\partial V}{\partial \xi}(\xi)f(t, \xi, \mu) \leq -\alpha(|\xi|_{\mathcal{A}}) + \sigma(|\mu|). \quad (6)$$

A natural question to ask is if this equivalence relation still holds for time varying systems, that is, if (5) is equivalent to the existence of $\alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ such that

$$\frac{\partial V}{\partial t}(t, \xi) + \frac{\partial V}{\partial \xi}(t, \xi)f(t, \xi, \mu) \leq -\alpha(|\xi|_{\mathcal{A}}) + \sigma(|\mu|). \quad (7)$$

The answer to this question is in general not true for the time varying case even when $\mathcal{A} = \{0\}$. As an example, consider the one-dimensional single input system

$$\dot{x}(t) = -x(t) + (1+t)q(u(t) - |x(t)|),$$

where $q : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function satisfying $q(r) = 0$ for all $r \leq 0$ and $q(r) > 0$ for all $r > 0$. Let $V(\xi) = \xi^2$. It is not hard to see that

$$V(\xi) \geq |\mu|^2 \implies \frac{\partial V}{\partial \xi}(\xi)f(t, \xi, \mu) = -2V(\xi).$$

Yet V fails to satisfy property (7).

Theorem 1 *A forward complete time varying system (1) is ISS with respect to \mathcal{A} if and only if it admits a smooth ISS-Lyapunov function V with respect to \mathcal{A} . ■*

In the case when \mathcal{A} is compact, the completeness assumption is redundant.

Corollary 2.2 *Suppose 0 is an equilibrium for the 0-input system (2). Then system (1) is ISS if and only if it admits an ISS-Lyapunov function. □*

2.2 Stability Margin

Consider system (1). A locally Lipschitz \mathcal{K}_∞ -function ρ is called a \mathcal{K}_∞ -stability margin for the system with respect to \mathcal{A} if the system

$$\dot{x}(t) = f(t, x(t), d(t) \rho(|x(t)|_{\mathcal{A}})) \quad (8)$$

is UGAS for all measurable functions $d : \mathbb{R} \rightarrow B_0$, (where B_0 denotes the closed unit ball of \mathbb{R}^m), that is, for some $\beta \in \mathcal{KL}$, it holds that

$$|x(t, \xi, t_0, d)|_{\mathcal{A}} \leq \beta(|\xi|_{\mathcal{A}}, t - t_0) \quad \forall t \geq 0,$$

for all trajectories of system (8).

Proposition 2.3 A forward complete time varying system is ISS with respect to \mathcal{A} if and only if it admits a \mathcal{K}_∞ -stability margin with respect to \mathcal{A} . \square

Clearly, when the set \mathcal{A} is compact, the completeness condition is redundant.

3. Robust Uniform Asymptotic Stability

Consider system

$$\dot{x}(t) = f(t, x(t), d(t)), \quad (9)$$

where disturbances, denoted by $d(\cdot)$, are measurable functions from \mathbb{R} to Ω for some compact subset Ω of \mathbb{R}^m . We assume that $f : \mathbb{R} \times \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^n$ is continuous, and $f(\cdot, \cdot, v) : \mathbb{R} \times \mathbb{R}^n$ is locally Lipschitz uniformly on $v \in \Omega$.

Let \mathcal{A} be a nonempty subset of \mathbb{R}^n . We say that \mathcal{A} is (forward) invariant if $x(t, \xi, t_0, d) \in \mathcal{A}$ for all $t \geq t_0$ and all d whenever $\xi \in \mathcal{A}$.

We say that such a system is *uniformly globally asymptotically stable* (UGAS) with respect to a closed invariant set \mathcal{A} if there exists some $\beta \in \mathcal{KL}$ such that

$$|x(t, \xi, t_0, d)|_{\mathcal{A}} \leq \beta(|\xi|_{\mathcal{A}}, t - t_0) \quad \forall t \geq t_0, \quad (10)$$

for all t_0 , all ξ and all d .

A smooth function $V : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is called a Lyapunov function for (9) with respect to \mathcal{A} if (4) holds for some $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$, and

$$\frac{\partial V}{\partial t}(t, \xi) + \frac{\partial V}{\partial \xi}(t, \xi) f(t, \xi, \mu) \leq -\alpha(|\xi|_{\mathcal{A}}) \quad (11)$$

holds for some $\alpha \in \mathcal{K}_\infty$. In the special case when the system is disturbance free, i.e., when f is independent of d , the notations UGAS and Lyapunov function reduce to the corresponding notations in the classical literature, see e.g., [4].

It was shown in [13] under the backward completeness assumption that a time invariant system is UGAS with respect to \mathcal{A} if and only if it admits a smooth Lyapunov function V that is independent of t . This result is enhanced by Theorem 2 in [23] so that the backward completeness assumption is not needed. By applying Theorem 2 in [23] to the auxiliary system

$$\dot{x}(t) = f(\lambda(t), x(t), d(t)), \quad \dot{\lambda}(t) = 1, \quad (12)$$

one can derive the Lyapunov characterization for UGAS in the time varying case:

Proposition 3.4 A forward complete time varying system (9) is UGAS with respect to a closed invariant set \mathcal{A} if and only if it admits a Lyapunov function with respect to \mathcal{A} . \square

4. Periodic Systems

In this section, we consider the Lyapunov characterizations of UGAS and the ISS properties for periodic time varying systems.

4.1 Robust Uniform Asymptotic Stability

Consider a periodic time varying system under the effect of disturbances, that is, a system as in (9), where $f(t, x, d)$ is periodic in t with some period T .

Theorem 2 Let (9) be a forward complete periodic system. Then the system is UGAS with respect to a closed invariant set \mathcal{A} if and only if it admits a Lyapunov function $V(t, x)$ with respect to \mathcal{A} that is periodic in t with the same period T for $f(t, x, d)$.

Observe that when the set \mathcal{A} is compact, especially when $\mathcal{A} = \{0\}$, the completeness assumption is redundant.

Sketch of the Proof. The main idea of the proof is to first show that the method used in [24] can lead to a Lyapunov function with the periodic property that is locally Lipschitz on \mathcal{A}_1^c , where $\mathcal{A}_1 = \mathbb{R} \times \mathcal{A}$, and continuous everywhere; and then by a modified smooth approximation result in [13], one can obtain a smooth Lyapunov function with the periodic property. Below we provide more details.

Suppose that f is periodic in t with period T . By the uniqueness property, one sees that

$$x(t + (T + t_0), \xi, T + t_0, d) = x(t + t_0, \xi, t_0, d_T) \quad (13)$$

holds for all $t, t_0 \in \mathbb{R}$, all $\xi \in \mathbb{R}^n$ and all d , where d_T is the disturbance function defined by $d_T(t) = d(t - T)$. For the function $\beta \in \mathcal{KL}$, there exist $\rho, \phi \in \mathcal{K}_\infty$ such that $\rho(\beta(s, r)) \leq \phi(s)e^{-2r}$ (cf. [20]). Define

$$U(\lambda, \xi) = \sup_{t \geq 0, d} \rho(|x(t + \lambda, \xi, \lambda, d)|_{\mathcal{A}}) e^t. \quad (14)$$

By (13), the function $U(\lambda, \xi)$ is periodic in λ with period T . It can also be seen that

$$\rho(|\xi|_{\mathcal{A}}) \leq U(\lambda, \xi) \leq \phi(|\xi|_{\mathcal{A}}). \quad (15)$$

By Proposition 5 in [24], the function U is locally Lipschitz on \mathcal{A}_1^c , continuous everywhere, and for almost all $(t, \xi) \notin \mathcal{A}_1$,

$$\frac{\partial U}{\partial \lambda}(\lambda, \xi) + \frac{\partial U}{\partial \xi}(\lambda, \xi) f(\lambda, \xi, \mu) \leq -U(\lambda, \xi). \quad (16)$$

Thus, we get the following:

Lemma 4.5 The function U defined by (14) is locally Lipschitz on \mathcal{A}_1^c , continuous everywhere, periodic in λ with period T , and satisfies estimates (15) and (16). \square

By the smoothing argument used as in the proof of Theorem 2.8 in [13] together with the smooth approximation result Theorem 5 provided in the Appendix, one can build a smooth Lyapunov function $V(t, x)$ with respect to \mathcal{A} that is periodic in t with the period T by approximating the function U .

4.2 Input to State Stability

Consider a periodic control system as in (1) where f is periodic in t with some period T . By Proposition 2.3, one knows that such a system is ISS with respect to a closed 0-invariant set \mathcal{A} if and only if it admits a \mathcal{K}_∞ -stability margin, i.e., for some smooth function $\rho \in \mathcal{K}_\infty$, the corresponding system as in (8) is UGAS. Observe that the map $F(t, x, d) := f(t, x, d\rho(|x|_{\mathcal{A}}))$ is again periodic in t , with the same period T for f . Hence, by applying (2), one sees that there is a smooth Lyapunov function V , periodic in t with period T , that satisfies (4) and, for some $\alpha \in \mathcal{K}_\infty$,

$$\frac{\partial V}{\partial t}(t, \xi) + \frac{\partial V}{\partial \xi}(t, \xi)f(t, \xi, \mu\rho(|\xi|_{\mathcal{A}})) \leq -\alpha(|\xi|_{\mathcal{A}}) \quad (17)$$

for all t , all ξ and all $\mu \in B_0$. Observing that inequality (17) is equivalent to (5) with $\chi = \rho^{-1}$, one reaches the following conclusion:

Theorem 3 *A forward complete periodic system (1) is ISS with respect to \mathcal{A} if and only if it admits a smooth ISS-Lyapunov function, periodic in t with the same period T of f , with respect to \mathcal{A} . ■*

Again, we note that the completeness condition is not needed when the set \mathcal{A} is compact.

5. A Small Gain Theorem

In this section, we consider the ISS property for inter-connected time varying systems

$$\begin{aligned} \dot{x}_1(t) &= f_1(t, x_1(t), v_1(t), u_1(t)), \\ \dot{x}_2(t) &= f_2(t, x_2(t), v_2(t), u_2(t)), \end{aligned} \quad (18)$$

subject to the inter-connection constraints:

$$v_1(t) = x_2(t), \quad v_2(t) = x_1(t), \quad (19)$$

where, for $i = 1, 2$ and for each t , $x_i(t) \in \mathbb{R}^{n_i}$, $u_i(t) \in \mathbb{R}^{m_i}$, $v_i(t) \in \mathbb{R}^{p_i}$ with $p_1 = n_2, p_2 = n_1$, and the map $f_i : \mathbb{R} \times \mathbb{R}^{n_i} \times \mathbb{R}^{p_i} \times \mathbb{R}^{m_i} \rightarrow \mathbb{R}^{n_i}$ is locally Lipschitz, and $f_1(t, 0, 0, 0) = f_2(t, 0, 0, 0) = 0$ for all t . To make the presentation simpler, we focus on the equilibrium case.

Assume that both the x_1 - and the x_2 -subsystems are ISS with (v_1, u_1) and (v_2, u_2) as inputs. Consequently, there exist smooth ISS-Lyapunov functions V_1, V_2 such that there exist some $\underline{\alpha}_i, \bar{\alpha}_i \in \mathcal{K}_\infty$ ($i = 1, 2$) such that

$$\underline{\alpha}_i(|\xi_i|) \leq V_i(t, \xi_i) \leq \bar{\alpha}_i(|\xi_i|), \quad (20)$$

and for some \mathcal{K}_∞ -functions χ_i, γ_i and α_i ($i = 1, 2$),

$$\begin{aligned} V_i(t, \xi_i) &\geq \max\{\chi_i(|v_i|), \gamma_i(|\mu_i|)\} \implies \\ \frac{\partial V_i}{\partial t}(t, \xi_i) + \frac{\partial V_i}{\partial \xi_i}(t, \xi_i)f_i(t, \xi_i, v_i, \mu_i) &\leq -\alpha_i(V_i(t, \xi_i)). \end{aligned} \quad (21)$$

Note that when working with the inter-connected system (18)-(19), property (21) is equivalent to the existence of $\chi_i, \gamma_i \in \mathcal{K}$ and $\alpha_i \in \mathcal{K}_\infty$, ($i = 1, 2$), such that

$$V_1(t, \xi_1) \geq \max\{\chi_1(V_2(t, \xi_2)), \gamma_1(|\mu_1|)\} \implies$$

$$\begin{aligned} \frac{\partial V_1}{\partial t}(t, \xi_1) + \frac{\partial V_1}{\partial \xi_1}(t, \xi_1)f_1(t, \xi_1, \xi_2, \mu_1) &\leq -\alpha_1(V_1(t, \xi_1)), \end{aligned} \quad (22)$$

$$V_2(t, \xi_2) \geq \max\{\chi_2(V_1(t, \xi_1)), \gamma_2(|\mu_2|)\} \implies$$

$$\begin{aligned} \frac{\partial V_2}{\partial t}(t, \xi_2) + \frac{\partial V_2}{\partial \xi_2}(t, \xi_2)f_2(t, \xi_2, \xi_1, \mu_2) &\leq -\alpha_2(V_2(t, \xi_2)), \end{aligned} \quad (23)$$

for all $\xi_i \in \mathbb{R}^{n_i}$ and $\mu_i \in \mathbb{R}^{m_i}$, $i = 1, 2$. With this, we have the following:

Theorem 4 *Assume that, for $i = 1, 2$, the x_i -system admits an ISS-Lyapunov function V_i satisfying (20) and (22)-(23). If the small gain condition*

$$\chi_1 \circ \chi_2(r) < r, \quad \forall r > 0, \quad (24)$$

or equivalently, $\chi_2 \circ \chi_1(r) < r$ for all $r > 0$, then the inter-connected system (18) with (19) is ISS with (u_1, u_2) as inputs. In particular, the zero input system (18)-(19) with $u \equiv 0$ is UGAS.

Remark 5.6 If V_1 and V_2 are ISS-Lyapunov functions for the subsystems satisfying (20), and, for some $\alpha_i \in \mathcal{K}_\infty$, $\theta_i^x, \theta_i^u \in \mathcal{K}_\infty$ ($i = 1, 2$), it holds that

$$\begin{aligned} \frac{\partial V_1}{\partial t}(t, \xi_1) + \frac{\partial V_1}{\partial \xi_1}(t, \xi_1)f_1(t, \xi_1, \xi_2, \mu_1) &\leq -\alpha_1(V_1(t, \xi_1)) + \theta_1^x(V_2(t, \xi_2)) + \theta_1^u(|\mu_1|), \\ \frac{\partial V_2}{\partial t}(t, \xi_2) + \frac{\partial V_2}{\partial \xi_2}(t, \xi_2)f_2(t, \xi_2, \xi_1, \mu_2) &\leq -\alpha_2(V_2(t, \xi_2)) + \theta_2^x(V_1(t, \xi_1)) + \theta_2^u(|\mu_2|), \end{aligned}$$

then χ_1 and χ_2 can be chosen as

$$\begin{aligned} \chi_1(r) &= \alpha_1^{-1} \circ (\text{id} + \varepsilon) \circ \theta_1^x(r), \\ \chi_2(r) &= \alpha_2^{-1} \circ (\text{id} + \varepsilon) \circ \theta_2^x(r), \end{aligned}$$

for any given $\varepsilon > 0$, where id stands for the identity function: $\text{id}(r) = r$ for all r . Thus, the small gain condition (24) becomes that, for some $\varepsilon > 0$,

$$\alpha_1^{-1} \circ (\text{id} + \varepsilon) \circ \theta_1^x \circ \alpha_2^{-1} \circ (\text{id} + \varepsilon) \circ \theta_2^x(r) < r,$$

for all $r > 0$. □

Applying the small gain theorem together with Theorems 4 and 1, we have the following result for cascades of time varying systems:

Corollary 5.7 For the cascade system

$$\begin{aligned} \dot{x}_1(t) &= f_1(t, x_1(t), x_2(t)), \\ \dot{x}_2(t) &= f_2(t, x_2(t)), \end{aligned} \quad (25)$$

if the x_1 -subsystem is ISS with x_2 as inputs, and the x_2 -subsystem is UGAS, then the system is UGAS. □

5.1 Proof of Theorem 4

To prove Theorem 4, note that, by letting $\zeta_i = (t, \xi_i)$, $\mathcal{A}_i = \{(t, \xi_i) : \xi_i = 0\}$ for $i = 1, 2$, properties (20) and (22)-(23) can be re-written as

$$\underline{\alpha}_i(|\zeta_i|_{\mathcal{A}_i}) \leq V_i(\zeta_i) \leq \bar{\alpha}_i(|\zeta_i|_{\mathcal{A}_i}),$$

and

$$\begin{aligned} V_1(\zeta_1) &\geq \max\{\chi_1(V_2(\zeta_2)), \gamma_1(|\mu_1|)\} \implies \\ &\frac{\partial V_1}{\partial \zeta_1}(\zeta_1) \hat{f}_1(\zeta_1, \zeta_2, \mu_1) \leq -\alpha_1(V_1(\zeta_1)), \\ V_2(\zeta_2) &\geq \max\{\chi_2(V_1(\zeta_1)), \gamma_2(|\mu_2|)\} \implies \\ &\frac{\partial V_2}{\partial \zeta_2}(\zeta_2) \hat{f}_2(\zeta_2, \zeta_1, \mu_2) \leq -\alpha_2(V_2(\zeta_2)), \end{aligned}$$

where $\hat{f}_1(\zeta_1, \zeta_2, \mu) = (1, f_1(t, \xi_1, \xi_2)^T)^T$, $\hat{f}_2(\zeta_2, \zeta_1, \mu) = (1, f_2(t, \xi_2, \xi_1)^T)^T$. Even though the proof of the small gain theorem in [7] was for the equilibrium case, it is also valid for the set case without much modification. The main idea is to first find $\hat{\chi}_1 \in \mathcal{K}_\infty$ such that $\hat{\chi}_1(r) \geq \chi_1(r)$ for all $r \geq 0$; and $\hat{\chi}_1 \circ \chi_2(r) < r$ for all $r > 0$. (If $\chi_1 \in \mathcal{K}_\infty$, one may simply take $\hat{\chi}_1 = \chi_1$.) For the detailed construction of such $\hat{\chi}_1$, see [7]. By replacing χ_1 by $\hat{\chi}_1$ if necessary, one may always assume that $\chi_1 \in \mathcal{K}_\infty$. According to Lemma A.1 in [7], there exists a \mathcal{K}_∞ -function σ that is continuously differentiable in $(0, \infty)$ with $\sigma'(r) > 0$ for all $r > 0$ such that

$$\chi_2(r) < \sigma(r) < \chi_1^{-1}(r), \quad \forall r > 0.$$

Without loss of generality, one may assume that σ is locally Lipschitz on $[0, \infty)$. Define

$$V(t, \xi_1, \xi_2) = \max\{\sigma(V_1(\zeta_1)), V_2(\zeta_2)\}. \quad (26)$$

By following the same steps as in the proof of Theorem 3.1 of [7], one can show that V is an ISS Lyapunov function for the inter-connected system (18)-(19) that is locally Lipschitz on $\{(t, \xi_1, \xi_2) : (\xi_1, \xi_2) \neq 0\}$. With such a Lyapunov function, it can be shown by standard methods that the inter-connected system is ISS.

On the other hand, by applying the smoothing argument as in the proof of Theorem 2.8 in [13], one can obtain a smooth ISS-Lyapunov function W . See also the proof of Theorem 3.1 in [7].

We again remark that in the case when the smooth condition is not a concern, the function V defined in (26) can be taken as an ISS-Lyapunov function. Indeed in many situations, it is much easier to find Lyapunov functions that are merely locally Lipschitz than to find smooth ones. Yet any suitably defined Lyapunov function can be used to guarantee the stability property. For more detailed discussions, see [18].

Appendix. Smooth Approximations

In this work, we need to generalize the smooth approximation result, Theorem B.1 in [13], so that the resulted smooth function still has the same periodic property if the function to be approximated is periodic in one of the variables.

Theorem 5 Let \mathcal{O} be an open subset of \mathbb{R}^n , and Ω be a compact subset of \mathbb{R}^m , and assume given:

- a locally Lipschitz function $\Phi : \mathbb{R} \times \mathcal{O} \rightarrow \mathbb{R}$, $(\lambda, \xi) \mapsto \Phi(\lambda, \xi)$ that is periodic in λ with period T ;
- a continuous map $f : \mathbb{R} \times \mathcal{O} \times \Omega, (\lambda, \xi, v) \mapsto f(\lambda, \xi, v)$, periodic in λ with period T , that is locally Lipschitz in (λ, ξ) on $\mathbb{R}^n \times \mathcal{O}$, uniformly in $v \in \Omega$;
- a continuous function α such that for almost all $(\lambda, \xi) \in \mathbb{R}^{n+1}$,

$$\frac{\partial \Phi}{\partial \lambda}(\lambda, \xi) + \frac{\partial \Phi}{\partial \xi}(\lambda, \xi) f(\lambda, \xi, v) \leq \alpha(\xi) \quad (27)$$

for each $v \in \Omega$;

- two continuous functions $\mu, \nu : \mathcal{O} \rightarrow \mathbb{R}_{>0}$.

Then there exists a smooth function $\Psi : \mathbb{R} \times \mathcal{O} \rightarrow \mathbb{R}$, $(\lambda, \xi) \mapsto \Psi(\lambda, \xi)$ that is periodic in λ with period T , such that

$$|\Phi(\lambda, \xi) - \Psi(\lambda, \xi)| < \mu(\xi), \quad \forall \xi \in \mathcal{O}, \forall \lambda \in \mathbb{R},$$

and for each $(\lambda, \xi) \in \mathbb{R} \times \mathcal{O}$ and $v \in \Omega$, it holds that

$$\frac{\partial \Psi}{\partial \lambda}(\lambda, \xi) + \frac{\partial \Psi}{\partial \xi}(\lambda, \xi) f(\lambda, \xi, v) \leq \alpha(\xi) + \nu(\xi).$$

The proof of the theorem basically follows the same idea used in [13]. But cares should be taken to ensure the periodic property for the resulted smooth function.

Let $\psi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ be a smooth nonnegative function which vanishes outside of the unit disk and satisfies

$$\int_{\mathbb{R}^{n+1}} \psi(s) ds = 1.$$

Write $s = (s_0, s_1)$ where $s_0 \in \mathbb{R}$, and $s_1 \in \mathbb{R}^n$. For any measurable, locally essentially bounded function Φ and any $\sigma \in (0, 1]$, define the function Φ_σ by convolution with $\frac{1}{\sigma^{n+1}} \psi(s/\sigma)$, that is

$$\Phi_\sigma(\lambda, \xi) = \int_{\mathbb{R}^{n+1}} \Phi(\lambda + \sigma s_0, \xi + \sigma s_1) \psi(s_0, s_1) ds. \quad (28)$$

Observe that if Φ is periodic in λ with period T , then Φ_σ has the same periodic property. Combining this with Lemmas B.5 of [13], we have:

Lemma 5.8 For each compact subset K of $\mathbb{R} \times \mathcal{O}$, and for any $\varepsilon > 0$, there exists a smooth function Ψ_K defined on K , periodic in λ with period T , such that

$$\sup_{(\lambda, \xi) \in K, v \in \Omega} L_{\hat{f}_v}(\Psi_K)(\lambda, \xi) \leq \alpha(\xi) + \varepsilon,$$

where for any locally Lipschitz function W and any $v \in \Omega$,

$$L_{\hat{f}_v} W(\lambda, \xi) = \frac{\partial W}{\partial \lambda}(\lambda, \xi) + \frac{\partial W}{\partial \xi}(\lambda, \xi) f(\lambda, \xi, v),$$

when the right-hand side is defined. \square

To complete the proof of Theorem 5, we follow the same idea of “patching by the argument of partition of unity” as in [13]. But we need to make sure that the “patching” functions again have the periodic property. For the sake of simplicity, we assume that $T = 1$.

First, we identify the interval $(-1/2, 1/2]$ with S_1 , the unit circle in \mathbb{R}^2 . This can be done by considering the map $\tau : \mathbb{R} \rightarrow S_1$ defined by $\tau(r) = r$ if $r \in (-1/2, 1/2]$, and $\tau(r_1) = \tau(r)$ if $r - r_1$ is an integer. Any mapping $W : \mathbb{R} \times \mathcal{O} \rightarrow \mathbb{R}$, $(\lambda, \xi) \mapsto W(\lambda, \xi)$ can be treated as a function W_1 defined on $S_1 \times \mathcal{O}$. In the case when W is periodic in λ with period 1, then W_1 has the same regularity property as W does, that is, $W_1 \in C^k$ ($0 \leq k \leq \infty$) if $W \in C^k$.

Since the theorem of partition of unit (cf. [26, Theorem 1.11]) also applies to $S_1 \times \mathcal{O}$, the same proof toward the end of Theorem B.1 in [13] can be used to show that there exists a smooth function Ψ_0 defined on $S_1 \times \mathcal{O}$, such that

$$|\Phi(\lambda, \xi) - \Psi_0(\lambda, \xi)| < \mu(\xi), \quad \forall \lambda \in S_1, \forall \xi \in \mathcal{O},$$

and for any $\nu \in \Omega$,

$$L_{f_v} \Psi_0(\lambda, \xi) \leq \alpha(\xi) + \nu(\xi).$$

To get a desired function Ψ defined on $\mathbb{R} \times \mathcal{O}$, we let $\Psi(\lambda, \xi) = \Psi_0(\tau(\lambda), \xi)$. Then Ψ is smooth on $\mathbb{R} \times \mathcal{O}$, and is periodic in λ with period 1.

Finally, we remark that the statement of Theorem 5 is still true if the function α in (27) depends on all the three variables: v, λ and ξ .

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