Robust Stabilization of Nonlinear Uncertain Systems in the Presence of Input Unmodeled Dynamics†

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Abstract

In this paper, motivated by the recent results on control of nonlinear systems with input uncertainties and converse Lyapunov results with respect to an invariant set, we address the robust adaptive control issue in the presence of uncertainties including input unmodeled *nonlinear* dynamics. Nonlinear damping terms are introduced via backstepping to counteract the uncertainties and observer design is also employed to cancel the affects of the nonlinear unmodeled dynamics. To this end, nonlinear systems transformable to the *Extended Strict Feedback Form with Input Uncertainties* are considered. A robust adaptive controller is designed to achieve semiglobal stability with respect to an invariant set in the presence of uncertainties utilizing partial state feedback. The results are also extended to a class of decentralized systems.

I. Introduction

Adaptive and robust control of nonlinear systems with uncertainties have received much attention. However, the design is more challenging if there are uncertainties which are unmodeled dynamics such as parasitics or dynamics through which control input is applied (i.e., input unmodeled dynamics).

The effects of *linear* input unmodeled dynamics is treated for the first time for nonlinear systems in [1] and it is shown that even in its simplest form the input unmodeled dynamics may result in dramatic shrinking of the region of attraction and in finite escape time. Thus, a dynamic feedback control law, which may be extended to higher-order systems via integrator backstepping is introduced and global asymptotic stability for perturbed strict-feedback systems is achieved. In [2], a two-step robust design procedure is presented to handle the dynamic uncertainties at the control input. Nevertheless, the input unmodeled dynamics considered in [2] must be input strictly passive. The stabilization of nonlinear systems with nonlinear minimum-phase input unmodeled dynamics is addressed in [3,4] and semiglobal stability is obtained by applying a suitable observer design together with a saturation-type controller design. This approach is utilized in this paper as well.

The contribution of this paper is to relax two assumptions appearing in [3,4] and therefore to enlarge the class of systems considered. In the Assumption 3 in [4], it is assumed that the nominal system (the system without considering input-uncertainties) is globally asymptotically stabilizable whereas in this paper we consider inclusion of bounded unmeasurable disturbances that may not necessarily vanish at the origin or the fact that the nominal system may at best be ultimately bounded with respect to a compact invariant set. Applying Converse Lyapunov Theorem with respect to an invariant set [5], the semiglobal stability for overall states with respect to a compact set is achieved in this paper. Furthermore, the non-minimum phase assumption in [4] is weakened. We also include the adaptive case where the uncertainties with unknown magnitude appear in certain subsystem dynamics and the adaptation is introduced to counteract their effects. The control design is a two-step design procedure. A nominal control input is designed such that the system is Uniformly Globally Asymptotically Stable (UGAS) *with respect to an invariant compact set* by ignoring the terms due to the nonlinear input unmodeled dynamics. Then, an estimator along the lines of [3,4] is proposed for the terms due to input unmodeled dynamics and the controller is modified to cancel the effects due to input unmodeled dynamics. It is shown that given any initial condition for the closed-loop system, the system can be made stable as long as the gain in the dynamics of observer is chosen sufficiently large. Furthermore, the observer error and system states can even be driven arbitrarily close to an invariant compact set (i.e., practical stabilization with respect to an invariant set). Reader may refer to [5] for some definitions and concepts.

II. System Description and Problem Statement

In this section, we study a class of nonlinear dynamic systems with uncertainties which can be put into the following *Extended Strict Feedback Form with Input Uncertainties (ESFF-IU)*

$$\begin{aligned}\dot{z} &= Qz + f_0(z, y, x, \zeta, d(t))\\ \dot{y} &= Ay + B[x_1 + f_1(z, y, x, \zeta, d(t))]\\ \dot{x}_1 &= x_2 + f_2(z, y, x, \zeta, d(t))\\ &\vdots\end{aligned}$$

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$$\begin{aligned}\dot{x}_{(n-\kappa-1)} &= x_{(n-\kappa)} + f_{(n-\kappa)}(z,y,x,\zeta,d(t)) \\ \dot{x}_{(n-\kappa)} &= f_{(n-\kappa+1)}(z,y,x) + \varrho(y,x)v \\ \dot{\zeta} &= h(z,y,x,\zeta,u) \\ v &= p(z,y,x,\zeta,u) \end{aligned} \quad (1)$$

where, A and B are in the standard Brunovsky's canonical form, $z \in \mathcal{R}^{n_z}$, $y \in \mathcal{R}^{n_y}$, $x = [x_1, \ldots, x_{(n-\kappa)}]^T \in \mathcal{R}^{n-\kappa}$, and $\zeta \in \mathcal{R}^{n_\zeta}$ are the states of the dynamic system, $d(t) \in D$ where D is a compact subset of $\mathcal{R}^m$, $u \in \mathcal{R}$ is the control input. Q is a constant matrix with appropriate dimension, ϱ is a known smooth nonlinear function. f_js $(0 \le j \le n-\kappa+1)$ are uncertain continuous nonlinear functions and bounded unmeasurable disturbances $d(t)$ may enter into the system through these terms. h and p are unknown continuous nonlinear functions satisfying certain assumptions.

The control task is to find a robust partial state feedback control law such that the states z, y, and x may remain bounded around the origin while maintaining boundedness for all other states. Notice that the state variables $[y, x]$ are available for feedback design; however, the state variables z and ζ are not available for feedback purposes.

Assumption 1: Q is an asymptotically stable matrix.

Assumption 2: The uncertain function f_0 is bounded by a known nonlinear function, i.e.,

$$|f_0(z,y,x,\zeta,d(t))| \quad \le \quad \varphi_0(|y|) \quad (2)$$

where φ_0 is a smooth function that does *not* vanish at the origin.

Assumption 3: The uncertain functions $f_j (1 \le j \le n-\kappa)$ are bounded as:

$$|f_j(z,y,x,\zeta,d(t))| \le \rho\Big[\upsilon_j(y,x_1,\ldots,x_{(j-1)})\cdot\varphi_j(|y|) + \sigma_j(y,x_1,\ldots,x_{(j-1)})\cdot\psi_j(|y|)\cdot|z|\Big] \quad (3)$$

where υ_j is a known smooth function with $\upsilon_1 = 1$,[1], φ_j is a known smooth function that does *not* vanish at the origin, σ_j is a known smooth function with $\sigma_1 = 1$[2], ψ_j is a known smooth function that does *not* necessarily vanish at the origin, ρ is an *unknown* constant. $f_{n-\kappa+1}(z,y,x)$ is a C^1 function which is *unknown*.

Remark 1: In the Assumptions 2-3, the uncertain functions $f_j (0 \le j \le n-\kappa)$ do not necessarily vanish at the origin, i.e., the system is under the bounded unmeasurable disturbances $d(t)$ and the origin may not be the equilibrium point. In this case, asymptotic stabilization of the nominal system (the system by ignoring the input uncertainties) may not be possible; thus, the Assumption 3 in [4] can not be satisfied. However, by considering the stabilization with respect to a compact invariant set instead of origin, the results in [4] are extended in this present paper to handle such a problem.

[1] The condition $\upsilon_1 = 1$ is only used for brevity and may be relaxed if required.

[2] Similar to υ_1, the condition $\sigma_1 = 1$ is only used for brevity and may be relaxed if required.

Assumption 4:

$$\varrho(y,x) \ge \varrho_m, \quad \forall y, x \quad (4)$$

where ϱ_m is a positive constant.

Assumption 5: Relative degree of v respect to control input u is zero[3], i.e.,

$$v = u - \xi(z,y,x,\zeta,u) \quad (5)$$

where ξ is a C^1 function and there exists a positive constant $\epsilon < 1$ such that

$$|\frac{\partial \xi}{\partial u}(z,x,y,\zeta,u)| \le 1-\epsilon, \quad \forall z,x,y,\zeta,u. \quad (6)$$

Assumption 6: There exists a C^1 and radially unbounded positive definite function $U(\zeta)$ such that

$$\frac{\partial U}{\partial \zeta}(\zeta)h(z,y,x,\zeta,u) \le -\alpha_1(|\zeta|) + \alpha_2(z,y,x,v) \quad (7)$$

where α_1 is a class K_∞ function and α_2 is a continuous function. Comparing with the minimum-phase condition in [3,4], this assumption is a weakened condition since α_2 is not required to be a class K function as in [3,4].

Definition 1: Function $\mathrm{sat_a}: \mathcal{R}^n \to \mathcal{R}^n$ is a C^1 function which satisfies

$$\mathrm{sat_a}(x) \quad = \quad x, \quad |x| \le a \quad (8)$$

$$|\frac{\partial \mathrm{sat_a}}{\partial x}(x)| \quad \le \quad 1, \quad \forall x \in \mathcal{R}^n \quad (9)$$

where a is a positive constant.

In fact, by Assumptions 4-5, the uncertain function $f_{n-\kappa+1}(z,y,x)$ in (1) can be lumped with input uncertainties. Therefore, $\dot{x}_{n-\kappa}$ can be rewritten as

$$\dot{x}_{n-\kappa} = \varrho(y,x)v' \quad (10)$$

where

$$v' \quad = \quad v + \frac{f_{n-\kappa+1}(z,y,x)}{\varrho(y,x)} = u - \xi' \quad (11)$$

and

$$\xi' \triangleq \xi - \frac{f_{n-\kappa+1}(z,y,x)}{\varrho(y,x)}. \quad (12)$$

It can be verified that the relative degree of v' with respect to the control u is zero from Assumption 5, i.e.,

$$v' = u - \xi'(z,y,x,\zeta,u) \quad (13)$$

and

$$|\frac{\partial \xi'}{\partial u}(z,x,y,\zeta,u)| \le 1-\epsilon, \quad \forall z,x,y,\zeta,u. \quad (14)$$

Rewrite the dynamics of $x_{n-\kappa}$

[3] In this paper, the relative degree is assumed to be zero just for brevity; it can be generalized to higher relative degree as in [4].

$$\dot{x}_{n-\kappa} = \varrho(y,x)(u-\xi') \tag{15}$$

and it should be noted that ξ' is the input uncertainty and an estimator $\hat{\xi}'$ is introduced in the next section to cancel its effects.

III. Observer Design for Input Uncertainties

Define

$$\gamma = \xi' + Lx_{n-\kappa} \tag{16}$$

where L is a positive design parameter to be chosen later. Differentiating γ, we obtain

$$\dot{\gamma} = \dot{\xi}' + L\varrho u - L\varrho(\gamma - Lx_{n-\kappa}). \tag{17}$$

The estimator $\hat{\gamma}$ for γ is constructed as

$$\dot{\hat{\gamma}} = L\varrho u - L\varrho(\hat{\gamma} - Lx_{n-\kappa}). \tag{18}$$

With the observer error defined as

$$e \stackrel{\triangle}{=} \hat{\gamma} - \gamma, \tag{19}$$

the dynamics of the observer error are

$$\dot{e} = -L\varrho e - \dot{\xi}'. \tag{20}$$

Define

$$\hat{\xi}' = \hat{\gamma} - Lx_{n-\kappa} \tag{21}$$

and notice that the initial value of $\hat{\xi}'$ grows with design parameter L. To eliminate this growth condition and guarantee the existence of unique explicit solution for control input as shown next, the following control scheme is proposed:

$$u = u^*(y,x,\hat{\beta}) + \mathrm{sat_a}(\hat{\xi}') \tag{22}$$

where $\hat{\beta}$ is an introduced adaptation parameter, a is a positive parameter to be designed, $u^*(y,x,\hat{\beta})$ is a nominal smooth partial state-feedback control law to be derived in the next section. The control law $u^*(y,x,\hat{\beta})$ together with the adaptation law for $\hat{\beta}$ given by

$$\dot{\hat{\beta}} = \Lambda(y,x,\hat{\beta}) \tag{23}$$

can uniformly stabilize the states $z, y, x, \hat{\beta}$ as long as the effects of ξ' is ignored, i.e., $u^*(y,x,\hat{\beta})$ is the control input which can uniformly stabilize the following system:

$$\begin{aligned}
\dot{z} &= Qz + f_0(z,y,x,\zeta,d(t))\\
\dot{y} &= Ay + B[x_1 + f_1(z,y,x,\zeta,d(t))]\\
\dot{x}_1 &= x_2 + f_2(z,y,x,\zeta,d(t))\\
&\vdots\\
\dot{x}_{(n-\kappa-1)} &= x_{(n-\kappa)} + f_{(n-\kappa)}(z,y,x,\zeta,d(t))\\
\dot{x}_{(n-\kappa)} &= \varrho(y,x)u^*(y,x,\hat{\beta})\\
\dot{\hat{\beta}} &= \Lambda(y,x,\hat{\beta}).
\end{aligned} \tag{24}$$

IV. Main Results: Robust Semiglobal Control Design

The semiglobal control design in this paper consists of two parts. In part I, a robust nominal control law $u^*(y,x,\hat{\beta})$ for stabilizing system (24) is derived where $\hat{\beta}$ is an introduced adaptation. Then, the derived nominal control law is modified as in (22) to obtain the actual control input together with the observer design given in (18) and (21).

Part I: Finding nominal control input $u^*(y,x,\hat{\beta})$:

The design objective in this part is to find a nominal control law $u^*(y,x,\hat{\beta})$, such that all the states in system (24) and the introduced adaptation parameter $\hat{\beta}$ are globally uniformly bounded. To proceed with the design, define matrices $P_1 = P_1^T > 0$ and $P_2 = P_2^T > 0$ as solutions of the following Lyapunov and Riccati equations

$$\begin{aligned}
P_1Q + Q^TP_1 &= -I_{n_z}\\
P_2A + A^TP_2 - 2\alpha P_2BB^TP_2 &= -I_{n_y}
\end{aligned} \tag{25}$$

where $\alpha > 0$ is our design parameter. Furthermore, define a smooth class K_∞ function ν such that

$$\begin{aligned}
\frac{d\nu}{dW}|y|^2 \;\geq\; & 4|P_1|^2[\varphi_0(|y|) - \varphi_0(0)]^2\\
& +(n-\kappa+1)\sum_{i=1}^{n-\kappa}[\varphi_i(|y|) - \varphi_i(0)]^2\\
& +(n-\kappa+1)\sum_{i=1}^{n-\kappa}\left[\psi_i^2(|y|) - \psi_i^2(0)\right]^2 + |y|^2
\end{aligned} \tag{26}$$

where $W = y^TP_2y$. The constructive algorithm for finding ν is given in the Appendix A and is along the lines in [6]. Then, we may start our recursive control design as follows:

Step 0: Start with the first (z,y)-subsystem of (1) and consider the composite positive definite and proper function

$$V_0 = z^TP_1z + \nu(W) + \frac{1}{2}\Gamma_\beta(\hat{\beta} - \beta^*)^2 \tag{27}$$

where Γ_β is a positive design parameter, β^* is an unknown positive constant which acts as one design freedom to counterbalance the unknown constants due to uncertainties to appear in the following derivation, $\hat{\beta}$ is an adaptation parameter, ν is a smooth K_∞ function defined above and $W = y^TP_2y$. Differentiating (27) along the trajectories of (24), one attains

$$\begin{aligned}
\dot{V}_0 \;=\; & -|z|^2 + 2z^TP_1f_0 - \frac{d\nu}{dW}|y|^2\\
& + \frac{d\nu}{dW}2y^TP_2B(x_1 + f_1 + \alpha B^TP_2y) + \Gamma_\beta(\hat{\beta} - \beta^*)\dot{\hat{\beta}}.
\end{aligned} \tag{28}$$

Using the inequality $2ab \leq a^2 + b^2 (a, b \in \mathcal{R})$ the second term in (28) is bounded as

$$2z^TP_1f_0 \;\leq\; 2|z|\cdot|P_1|\cdot|\varphi_0(|y|)|$$

$$\leq \frac{1}{4}|z|^2 + 4|P_1|^2[\varphi_0(|y|) - \varphi_0(0)]^2 + 2|z| \cdot |P_1| \cdot |\varphi_0(0)|. \quad (29)$$

Similarly, we have

$$2\frac{d\nu}{dW}y^T P_2 B f_1 \leq \rho^2|\frac{d\nu}{dW}|^2 \cdot |B^T P_2 y|^2 + [\varphi_1(|y|) - \varphi_1(0)]^2 + 2|\frac{d\nu}{dW}y^T P_2 B| \cdot |\rho\varphi_1(0)| + \frac{1}{4}|z|^2 + 4\rho^2|\frac{d\nu}{dW}|^2 \cdot |B^T P_2 y|^2 \cdot \psi_1^2(0) + 4\rho^4|\frac{d\nu}{dW}|^4 \cdot |B^T P_2 y|^4 + [\psi_1^2(|y|) - \psi_1^2(0)]^2 \quad (30)$$

Choose the virtual controller x_1^* and define a new variable s_1 as

$$\begin{aligned} x_1^* &= -\alpha B^T P_2 y - \frac{1}{2}\hat{\beta}[B^T P_2 y \frac{d\nu}{dW} + (B^T P_2 y)^3 \cdot (\frac{d\nu}{dW})^3]\Big|_{W = y^T P_2 y} \\ s_1 &= x_1 - x_1^*. \end{aligned} \quad (31)$$

Furthermore, define

$$\aleph_0 = 4|P_1|^2[\varphi_0(|y|) - \varphi_0(0)]^2 + [\varphi_1(|y|) - \varphi_1(0)]^2 + [\psi_1^2(|y|) - \psi_1^2(0)]^2. \quad (32)$$

Therefore, $\dot{V}_0$ reduces to

$$\begin{aligned} \dot{V}_0 \leq & -\frac{1}{2}|z|^2 - \frac{d\nu}{dW}|y|^2 + \aleph_0 + (\rho^2 + 4\rho^2\psi_1^2(0) - \beta^*) \\ & \cdot|\frac{d\nu}{dW}|^2 \cdot |B^T P_2 y|^2 + (4\rho^4 - \beta^*) \cdot |\frac{d\nu}{dW}|^4 \cdot |B^T P_2 y|^4 \\ & + \Gamma_\beta(\hat{\beta} - \beta^*)\Big\{\dot{\hat{\beta}} - \Gamma_\beta^{-1}[|B^T P_2 y|^2 \cdot |\frac{d\nu}{dW}|^2 \\ & + |B^T P_2 y|^4 \cdot |\frac{d\nu}{dW}|^4] + 2\sigma_\beta(\hat{\beta} - \grave{\beta})\Big\} - 2\Gamma_\beta\sigma_\beta(\hat{\beta} \\ & - \beta^*)(\hat{\beta} - \grave{\beta}) + 2\frac{d\nu}{dW}y^T P_2 B s_1 + 2|z| \cdot |P_1| \cdot |\varphi_0(0)| \\ & + 2|\frac{d\nu}{dW}y^T P_2 B| \cdot |\rho\varphi_1(0)| \end{aligned} \quad (33)$$

where σ_β and $\grave{\beta}$ are positive design parameters.

Define

$$\begin{aligned} \omega_0 &= \Gamma_\beta^{-1}\Big[|B^T P_2 y|^2 \cdot |\frac{d\nu}{dW}|^2 + |B^T P_2 y|^4 \cdot |\frac{d\nu}{dW}|^4\Big] \\ & \quad -2\sigma_\beta(\hat{\beta} - \grave{\beta}) \quad (34) \\ \check{\beta}_0 &= \max\{\rho^2 + 4\rho^2\psi_1^2(0), 4\rho^4\}. \quad (35) \end{aligned}$$

At this step, notice that the fourth and fifth terms in the bound for $\dot{V}_0$ given by (33) may be made negative by choosing

$$\beta^* > \check{\beta}_0. \quad (36)$$

Moreover, consider the inequalities

$$(\hat{\beta} - \beta^*)(\hat{\beta} - \grave{\beta}) \leq -(\hat{\beta} - \grave{\beta})^2 + (\grave{\beta} - \beta^*)^2 \quad (37)$$

and

$$(\rho^2 + 4\rho^2\psi_1^2(0) - \beta^*) \cdot |\frac{d\nu}{dW}|^2 \cdot |B^T P_2 y|^2 + 2|\frac{d\nu}{dW}y^T P_2 B| \cdot |\rho\varphi_1(0)| \leq \frac{|\rho\varphi_1(0)|^2}{(\beta^* - \rho^2 - 4\rho^2\psi_1^2(0))} \quad (38)$$

to further bound $\dot{V}_0$ as

$$\begin{aligned} \dot{V}_0 \leq & 2|z| \cdot |P_1| \cdot |\varphi_0(0)| - \frac{1}{2}|z|^2 - \Gamma_\beta\sigma_\beta(\hat{\beta} - \grave{\beta})^2 \\ & -\frac{d\nu}{dW}|y|^2 + \aleph_0 + 2\frac{d\nu}{dW}y^T P_2 B s_1 \\ & + \Gamma_\beta(\hat{\beta} - \beta^*)(\dot{\hat{\beta}} - \omega_0) + \Pi_0 \end{aligned} \quad (39)$$

where

$$\Pi_0 = \Gamma_\beta\sigma_\beta(\grave{\beta} - \beta^*)^2 + \frac{|\rho\varphi_1(0)|^2}{(\beta^* - \rho^2 - 4\rho^2\psi_1^2(0))}. \quad (40)$$

<u>*Step m*</u> ($2 \leq m \leq n-\kappa-1$): Skipped here due to limited space. Reader may refer to [6] for details.

<u>*Step n-κ*</u>: Proceeding along the same lines as before, we obtain the derivative of $s_{n-\kappa}$ as

$$\dot{s}_{(n-\kappa)} = \varrho(y,x)u^*(y,x,\hat{\beta}) + \sum_{j=1}^{n-\kappa}\phi_{(n-\kappa,j)}f_j + \chi_{(n-\kappa)} + \varpi_{(n-\kappa)}\dot{\hat{\beta}} \quad (41)$$

where $\chi_{(n-\kappa)}, \phi_{(n-\kappa,1)}, \phi_{(n-\kappa,2)}, \ldots, \phi_{(n-\kappa,n-\kappa)}$, and $\varpi_{(n-\kappa)}$ are known functions of their arguments. The final control law for u is given by

$$\begin{aligned} u^*(y,x,\hat{\beta}) = \frac{1}{\varrho(y,x)}\Bigg\{ & -s_{(n-\kappa-1)} - s_{n-\kappa} - \chi_{(n-\kappa)} \\ & -\varpi_{(n-\kappa)}\omega_{(n-\kappa-1)} - (\hat{\beta} + \vartheta_{(n-\kappa-1)} \\ & +\Gamma_\beta^{-1}s_{(n-\kappa)}\varpi_{(n-\kappa)}) \cdot (s_{(n-\kappa)}\sum_{j=1}^{n-\kappa}\upsilon_j^2\phi_{(n-\kappa,j)}^2 \\ & +s_{(n-\kappa)}\sum_{j=1}^{n-\kappa}\sigma_j^2\phi_{(n-\kappa,j)}^2 + s_{(n-\kappa)}^3\sum_{j=1}^{n-\kappa}\sigma_j^4\phi_{(n-\kappa,j)}^4)\Bigg\} \end{aligned} \quad (42)$$

We choose Λ in (24) as $\Lambda = \omega_{(n-\kappa)}$ and thus the adaptation law for $\hat{\beta}$ is chosen as

$$\dot{\hat{\beta}} = \omega_{(n-\kappa)}. \quad (43)$$

Using the augmented Lyapunov function $V_{(n-\kappa)}$

$$V_{(n-\kappa)} = V_{(n-\kappa-1)} + \frac{1}{2}s_{(n-\kappa)}^2, \quad (44)$$

finally choosing β^* such that $\beta^* \geq \check{\beta}_{(n-\kappa)}$, and differentiating $V_{(n-\kappa)}$ along the trajectory of (41) under the control law (42) with adaptation law (43), $\dot{V}_{(n-\kappa)}$ reduces to

$$\dot{V}_{(n-\kappa)} \leq 2|z| \cdot |P_1| \cdot |\varphi_0(0)| - \frac{1}{2^{n-\kappa+1}}|z|^2 - \sum_{i=1}^{n-\kappa}s_i^2$$

$$-\Gamma_\beta\sigma_\beta(\hat\beta-\check\beta)^2-\frac{d\nu}{dW}|y|^2+\aleph_{(n-\kappa)}+\Pi_{n-\kappa}. \quad (45)$$

Along the lines in (38), we have

$$2|z|\cdot|P_1|\cdot|\varphi_0(0)|-\frac{1}{2^{n-\kappa+1}}|z|^2 \le -\frac{1}{2^{n-\kappa+2}}|z|^2 +2^{n-\kappa+2}|P_1|^2\varphi_0^2(0). \quad (46)$$

Hence, $\dot V_{(n-\kappa)}$ can be rewritten as

$$\dot V_{(n-\kappa)} \le -\frac{1}{2^{n-\kappa+2}}|z|^2-\Gamma_\beta\sigma_\beta(\hat\beta-\check\beta)^2-\sum_{i=1}^{n-\kappa}s_i^2 -\frac{d\nu}{dW}|y|^2+\aleph_{(n-\kappa)}+\Pi_{n-\kappa+1} \quad (47)$$

where

$$\Pi_{n-\kappa+1}=\Pi_{n-\kappa}+2^{n-\kappa+2}|P_1|^2\varphi_0^2(0). \quad (48)$$

The properties of the above designed control law are stated in the following theorem.

Theorem 1: *For the system given by (24) satisfying Assumptions 1-4, the control law (42) along with the adaptation law (43) guarantees uniformly globally bounded regulation of z, y, and x* states.

Proof: Through the choice of Lyapunov function given by $V_{n-\kappa}$, and based on the derivation above and with the property of ν given by (26), we have

$$\dot V_{n-\kappa} \le -\frac{1}{2^{n-\kappa+2}}|z|^2-\Gamma_\beta\sigma_\beta(\hat\beta-\check\beta)^2-\sum_{i=1}^{n-\kappa}s_i^2 -\sum_{i=1}^{N}|y|^2+\Pi_{n-\kappa+1}. \quad (49)$$

Define $s=[s_1,\ldots,s_{n-\kappa}]^T$. Applying Lyapunov theorem, global bounded regulation of states z, y, $\hat\beta$, and s has been established. Since the virtual control x_i^* in each step is a function of $y,\hat\beta,s_1,s_2,\ldots,s_{i-1}$, the boundedness of x_i^* can be shown, which implies that

$$x_i=s_i+x_i^*(\hat\beta,s_1,\ldots,s_{i-1}) \quad (50)$$

is also bounded. Furthermore, define

$$V_f=\frac{1}{2^{n-\kappa+2}}|z|^2+\Gamma_\beta\sigma_\beta(\hat\beta-\check\beta)^2+\sum_{i=1}^{n-\kappa}s_i^2+\sum_{i=1}^{N}|y|^2. \quad (51)$$

Thus, it follows that the $V_{n-\kappa}$ decreases monotonically until the solution reaches a compact set

$$\Omega_f=\{[z,y,s,\hat\beta]:V_f(z,y,s,\hat\beta)\le\Pi_{n-\kappa+1}\}. \quad (52)$$

Since the Lyapunov function $V_{n-\kappa}$ is radially unbounded, we may find a positive constant c and define the compact set $\Omega_c=\{[z,y,s,\hat\beta]\in\mathcal{R}^{n_z}\times\mathcal{R}^{n_y}\times\mathcal{R}^{n_{n-\kappa}}\times\mathcal{R}: V_{n-\kappa}(z,y,s,\hat\beta)\le c\}$ such that $\Omega_f\in\Omega_c$. Therefore, Ω_c is an invariant set and is globally attractive. So solutions $[z,y,s,\hat\beta]$ are UGAS with respect to Ω_c.[4] In addition, set Ω_c can be made arbitrarily small if parameter σ or Γ is arbitrarily small and β^* is chosen to be arbitrarily large. ◊

Part II: Semiglobal controller design:

Once we find the nominal control input $u^*(y,x,\hat\beta)$ in Part I, the actual control input is given as in (22) and adaptation law for $\hat\gamma$ is given as in (18). The design parameter a is chosen as (55) in the following proof.

Our main results are stated in the following Lemma and Theorem.

Lemma 1: *For any function* $\mathrm{sat_a}:\mathcal{R}^n\to\mathcal{R}^n$ *that satisfies Definition 1, there exists a unique C^1 function $\Omega_{sat_a}(\pi,e,z,y,s,\hat\beta,\zeta)$ solving the equation*

$$u=\pi+\mathrm{sat_a}[\xi'(z,y,x,\zeta,u)+e]. \quad (53)$$

Proof: Lemma 1 can be established along the same lines in [4] by applying Contraction Mapping Theorem. Therefore, the control law for u can be rewritten in the form of explicit solution

$$u=\Omega_{sat_a}[u^*(y,s,\hat\beta),e,z,y,s,\hat\beta,\zeta]. \quad (54)$$

Theorem 2: *Under Assumptions 1-6, there exists a compact set $\Omega\subset\mathcal{R}^{n_z}\times\mathcal{R}^{n_y}\times\mathcal{R}^{n-\kappa}\times\mathcal{R}^{n_\zeta}\times\mathcal{R}$ such that for any compact set S which contains Ω, if design parameter L is chosen sufficiently large, the control law (22) together with the observer dynamics (18) guarantees the boundedness of all the closed-loop states with respect to S, i.e., the closed-loop system is practically stable with respect to the compact set Ω with a domain of attraction which contains S.*

Proof: For including the input unmodeled dynamics, the following proof is along the lines in [3,4]. However, since the nominal system is not asymptotically stable, the converse Lyapunov theorem with respect to compact set in [5] has to be utilized to replace the converse Lyapunov theorem in [3,4] to complete the proof. Due the limited space, the complete proof is skipped here and it is available upon request.

Similar to [3,4], the design parameter a is chosen as

$$a=\sup_{(z,y,s,\hat\beta,\zeta,e)\in\mathrm{T}_2}|\xi'_{\mathrm{Id}}(z,y,s,\hat\beta,\zeta,e)+e| \quad (55)$$

where ξ'_{Id} is obtained from $\xi'_{\mathrm{sat_a}}$ by choosing the identity as $\mathrm{sat_a}$ function. ◊

V. Extension to Decentralized Systems

In [7], the above design methodology has been generalized to the decentralized control for a class of large-scale systems, namely, the class of large-scale nonlinear dynamic systems which can be put into the following *Form of Large-scale Nonlinear Systems with Input Uncertainties (FLNS-IU)*

$$\dot x_i = f_i(x_i)+g_i(x_i)v_i$$

[4] Refer to [5] for the definition of UGAS with respect to an invariant set.

$$\begin{aligned} \dot{\zeta}_i &= h_i(x, \zeta, u, d(t)) \\ v_i &= p_i(x, \zeta, u), \qquad 1 \le i \le N \end{aligned} \tag{56}$$

where, $x_i \in \mathcal{R}^{n_{x_i}}$ and $\zeta_i \in \mathcal{R}^{n_{\zeta_i}}$ are the states of the ith subsystem, disturbances $d(t) \in D$ where D is a compact subset of $\mathcal{R}^m$, $u_i \in \mathcal{R}^{n_{u_i}}$ is the control input vector for ith subsystem, $v_i \in \mathcal{R}^{n_{u_i}}$; $x \stackrel{\triangle}{=} [x_1^T, x_2^T, \ldots, x_N^T]^T$, $\zeta \stackrel{\triangle}{=} [\zeta_1, \zeta_2, \ldots, \zeta_N]$, and $u \stackrel{\triangle}{=} [u_1^T, u_2^T, \ldots, u_N^T]$ are the vectors for the overall large-scale nonlinear system, f_i is a known vector field and g_i is a known function with appropriate dimensions. h_i and p_i are unknown continuous nonlinear functions satisfying certain assumptions.

In (56), x_i dynamics denote the certain part of the ith subsystem and they are the only states for feedback purposes and ζ_i dynamics denote the uncertain part. As seen in (56), the dynamics of ζ_i are affected by the control input and thus they are input unmodeled dynamics. In addition, bounded persistent disturbances $d(t)$ are included in ζ_i dynamics. To utilize the similar control design in this paper, the following Assumptions 7-10 holds [7].

Assumption 7: For each ith subsystem $(1 \le i \le N)$, there exists a known continuous nominal control law $u_i^*(x_i)$ such that for the dynamical system given by

$$\dot{x}_i = f_i(x_i) + g_i(x_i)u_i^*(x_i), \tag{57}$$

state vector x_i is uniformly globally asymptotically stable (UGAS) with respect to a compact set, namely, Ω_{i1} around the origin.

Assumption 8: For each ith subsystem $(1 \le i \le N)$, there exists a C^1 function $l_i(x_i) : \mathcal{R}^{n_{x_i}} \to \mathcal{R}^{n_{u_i}}$ such that

$$\frac{\partial l_i}{\partial x_i}(x_i)g_i(x_i) = c_i(x_i)I \tag{58}$$

where $c_i(x_i) \ge \delta_c > 0$.

Assumption 9: For each ith subsystem $(1 \le i \le N)$, relative degree of v_i (output of the input unmodeled dynamics) respect to control input u_i is zero, i.e., for each subsystem $(1 \le i \le N)$,

$$v_i = u_i - \xi_i(x, \zeta, u) \tag{59}$$

where ξ_i is a C^1 function and there exists a positive constant $\epsilon < 1$ such that

$$\left|\frac{\partial \xi_i}{\partial u}(x, \zeta, u)\right| \le 1 - \epsilon, \quad \forall x, \zeta, u. \tag{60}$$

Assumption 10: For each ith subsystem $(1 \le i \le N)$, there exists a C^1 and radially unbounded positive definite function $U_i(\zeta_i)$ such that

$$\frac{\partial U_i}{\partial \zeta_i}(\zeta_i)h_i(x, \zeta, u, d(t)) \le -\alpha_{i1}(|\zeta_i|) + \alpha_{i2}(x, v) \tag{61}$$

where α_{i1} is a class K_∞ function and α_{i2} is a continuous function.

VI. An Illustrative Example

To illustrate the design procedure detailed above, the following example is considered:

$$\begin{aligned} \dot{z} &= -z + y^2 \cos(zx_1)\cos(t^2\zeta) \\ \dot{y} &= \rho_0 y^2 \sin t + \rho_1 \sin^2(t^2\zeta)\sin x_1(1+y)z + \cos t + x_1 \\ \dot{x}_1 &= v \\ \dot{\zeta} &= -(3 + \sin y)\zeta + \zeta^2(\cos y + z^4 + y^2 + x_1 + u) \\ v &= u + \zeta \end{aligned} \tag{62}$$

where ρ_0 and ρ_1 are unknown constants.

Notice none of the existing design methodologies for input unmodeled dynamics may handle this problem. Since there is a persistent disturbance $\cos t$ in the dynamics of y, it is impossible to find a nominal control law to asymptotically stabilize the system without considering input uncertainty as in [3,4] and the Assumption 3 in [4] can not be satisfied. However, utilizing the methodology developed in this paper, one more adaptation parameter $\hat{\beta}$ is introduced to counteract the effects due to ρ_0 and ρ_2 and UGAS of the overall system with respect to a compact set is guaranteed.

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