

Uniform Asymptotic Controllability to a Set implies Locally Lipschitz Control-Lyapunov Function ¹

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Abstract

We show that uniform global asymptotic controllability to a closed (not necessarily compact) set for a locally Lipschitz nonlinear control system implies the existence of a locally Lipschitz control-Lyapunov function (clf), and from this clf we construct a feedback that is robust to measurement noise.

1 Introduction

We will consider the nonlinear control system

$$\dot{x} = f(x, u) \quad (1)$$

where $x \in \mathbb{R}^n$ and u belongs to some admissible control set. The regularity properties of f will be specified later. We show that if there exist admissible (open-loop) controls to steer the state to a desired closed (not necessarily compact) set \mathcal{A} then there exists a locally Lipschitz control-Lyapunov function (clf) for the system (1) with respect to \mathcal{A} , i.e., a locally Lipschitz function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, positive definite, decrescent and proper with respect to \mathcal{A} , such that, $\forall x \in \mathbb{R}^n$, the minimum over all u 's in the admissible control set of the Dini subderivate of $V(\cdot)$ is negative definite w.r.t. \mathcal{A} . (See Definition 4 for further clarification.)

The significance of this result stems from the important role that clfs have played in the development of stabilizing state feedbacks over the years. As examples, we refer the reader to [2], [14], [8], and [10] for the case of continuously differentiable clfs, and to [5], [16] and [4] for locally Lipschitz clfs. Similar to the latter articles, in section 4 we present the design of a (discontinuous) stabilizing state feedback that is robust to small additive disturbances and measurement noise using our derived clf.

An early result related to the existence of clfs appeared in [12] where, essentially, a lower semicontinuous clf is generated given asymptotic controllability to the origin. Without limiting the controls a priori, [13] considers systems of the form (1) and generates the first continuous clf under the assumption of asymptotic controllability to the origin (cf. [17]). A continuous clf is

generated in [1] under the assumption of asymptotic controllability to a closed (not necessarily compact) set. In [5] and [16], a standard regularization technique from nonsmooth analysis is used to show that, at least for compact attractors, these continuous clfs can be converted to a family of locally Lipschitz clfs having the infinitesimal decrease property on compact sets disjoint from the target set. In [4], this type of family of locally Lipschitz clfs is generated directly from the solution to a family of optimization problems, again for the case of asymptotic controllability to the origin. In [11], such a family of locally Lipschitz clfs is combined into a single locally Lipschitz clf, answering a longstanding question on the existence of same.

Our result, which extends these previous results, has as its central proof component a recent result (which appeared in [9]) on the existence of a locally Lipschitz weak converse Lyapunov function for locally Lipschitz differential inclusions having a weakly globally asymptotically stable set. Our approach is to convert the control system into a differential inclusion (which is the approach also taken in [4] and [11]) and then use the result on the existence of a converse theorem for the differential inclusion to get the promised control-Lyapunov function.

The remainder of the paper is organized as follows: In section 2 we present background material necessary for the sequel. Section 3 presents our main result on the existence of a locally Lipschitz clf. Section 4 shows one approach for designing a feedback control law from a locally Lipschitz clf. Section 5 discusses the control algorithm's robustness to measurement noise. The proof of the main result is given in section 6.

2 Preliminaries

We will require the following lemma regarding the minimum of the inner product over compact sets, which easily follows from [7, §5, Lemma 8].

Lemma 1 *Given a compact $\mathcal{V} \subseteq \mathbb{R}^n$ and constants $c \in \mathbb{R}^n$, $\gamma \in \mathbb{R}$, if $\min_{v \in \text{co}\mathcal{V}} \langle c, v \rangle \leq \gamma$, then $\min_{v \in \mathcal{V}} \langle c, v \rangle \leq \gamma$.*

Definition 1 *The Dini subderivate of a locally Lipschitz function $V : \mathcal{O} \mapsto \mathbb{R}$ (\mathcal{O} open) at $x \in \mathcal{O}$ in the*

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direction $v \in \mathbb{R}^n$ is defined as

$$DV(x; v) := \liminf_{\varepsilon \rightarrow 0^+} \frac{V(x + \varepsilon v) - V(x)}{\varepsilon}.$$

See [6, pg. 136] for the definition of the Dini sub-derivate for general functions and the equivalence to the above definition for locally Lipschitz functions.

We let $|\cdot|$ denote the Euclidean norm on \mathbb{R}^n , i.e., $|x| = \sqrt{\langle x, x \rangle}$. We let $\overline{B}_n(x, r)$ denote the closed ball in \mathbb{R}^n of radius r centered at x , i.e., $\overline{B}_n(x, r) := \{\xi \in \mathbb{R}^n : |\xi - x| \leq r\}$. We define $\overline{B}_n := \overline{B}_n(0, 1)$ where 0 denotes the origin in \mathbb{R}^n . Recall that a function $\alpha : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ belongs to class- \mathcal{K}_∞ ($\alpha \in \mathcal{K}_\infty$) if it is continuous, zero at zero, strictly increasing, and unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ belongs to class- \mathcal{KL} if, for each $t \geq 0$, $\beta(\cdot, t)$ is nondecreasing and $\lim_{s \rightarrow 0^+} \beta(s, t) = 0$, and, for each $s \geq 0$, $\beta(s, \cdot)$ is nonincreasing and $\lim_{t \rightarrow \infty} \beta(s, t) = 0$.

Definition 2 A set-valued map $F : \mathbb{R}^n \rightarrow$ subsets of \mathbb{R}^n is locally Lipschitz if $\forall x \in \mathbb{R}^n$ there exists a neighborhood \mathcal{U} of x and $L > 0$ such that $x_1, x_2 \in \mathcal{U}$ implies $F(x_1) \subseteq F(x_2) + L|x_1 - x_2|\overline{B}_n$. We say that this property is uniform in distance to the closed set \mathcal{A} if, for any $\ell > 0$ the above neighborhood can be defined as $\mathcal{U} := \{x \in \mathbb{R}^n : |x|_{\mathcal{A}} \leq \ell\}$.

A function $x : [0, T] \mapsto \mathbb{R}^n$ ($T > 0$) is said to be a *solution* of the differential inclusion $\dot{x} \in F(x)$ if it is absolutely continuous and satisfies, for almost all $t \in [0, T]$, $\dot{x}(t) \in F(x(t))$. A function $x : [0, T] \mapsto \mathbb{R}^n$ ($0 < T \leq +\infty$) is said to be a *maximal solution* of the differential inclusion if it does not have an extension which is a solution belonging to \mathbb{R}^n , i.e., either $T = \infty$ or there does not exist a solution $y : [0, T_+] \mapsto \mathbb{R}^n$ with $T_+ > T$ such that $y(t) = x(t)$ for all $t \in [0, T]$. In what follows, we use $\phi(\cdot, x)$ to denote a solution of $\dot{x} \in F(x)$ starting at x . We denote by $\mathcal{S}[0, T](x)$, or $\mathcal{S}[0, T](x)$, the set of maximal solutions starting at x that are defined on the compact time interval $[0, T]$, or $[0, T)$.

3 Main Result

In this section we state our main result that uniform asymptotic controllability to a set implies the existence of a locally Lipschitz control Lyapunov function. In what follows, we take \mathcal{U} to be a locally compact metric space with a unique zero element, “0”, and, by abuse of notation, $|u| := d(u, 0)$. We define the closed unit ball in the metric space \mathcal{U} as $\overline{B}_{\mathcal{U}} := \{\xi \in \mathcal{U} : d(\xi, 0) \leq 1\}$.

Definition 3 Let $\mathcal{A} \subseteq \mathbb{R}^n$ be a closed, nonempty set and let $\sigma : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ be nondecreasing. We say that (1) is uniformly globally asymptotically controllable (UGAC) to \mathcal{A} with $\mathcal{U} \cap \sigma$ controls if there exists a function $\beta \in \mathcal{KL}$ such that: for each $x \in \mathbb{R}^n$ there exist a measurable, essentially bounded function $u : [0, \infty) \mapsto$

\mathcal{U} and a solution $\phi(\cdot, x, u)$ of $\dot{x} = f(x, u(t))$ satisfying

$$\begin{aligned} |\phi(t, x, u)|_{\mathcal{A}} &\leq \beta(|x|_{\mathcal{A}}, t), \\ |u(t)| &\leq \sigma(|\phi(t, x, u)|_{\mathcal{A}}), \quad \text{a.a. } t \geq 0. \end{aligned} \quad (2)$$

Remark 1 We note that $\mathcal{U} \cap \sigma$ is an abuse of notation. It is shorthand for allowing controls from the set $\{u \in \mathcal{U} : |u| \leq \sigma(|x|_{\mathcal{A}})\}$ for each $x \in \mathbb{R}^n$. ■

The following lemma, which is stated without proof due to space constraints, asserts the equivalence of the usual definition of UGAC (such as in [13, Defn 2.2] or [15]) and that of UGAC with $\mathcal{U} \cap \sigma$ controls. The usual definition of UGAC limits the control based on the size of the initial condition of the state, whereas for UGAC with $\mathcal{U} \cap \sigma$ controls we limit the control through the size of the trajectory.

Lemma 2 The system (1) is UGAC to \mathcal{A} with $\mathcal{U} \cap \sigma$ controls if and only if $\exists \beta_c \in \mathcal{KL}$ and $\sigma_c : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ nondecreasing s.t.: $\forall x \in \mathbb{R}^n$ there exist a measurable, essentially bounded function $u : [0, \infty) \mapsto \mathcal{U}$ and a maximal solution $\phi(t, x, u(t))$ of (1) s.t. $\|u\|_\infty \leq \sigma_c(|x|_{\mathcal{A}})$ and $|\phi(t, x, u(t))|_{\mathcal{A}} \leq \beta_c(|x|_{\mathcal{A}}, t)$.

Definition 4 Let $\sigma : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ nondecreasing. We say a locally Lipschitz function $V : \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$ is a control-Lyapunov function with $\mathcal{U} \cap \sigma$ controls (clf with $\mathcal{U} \cap \sigma$ controls) for the system (1) if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}})$, and V satisfies the weak infinitesimal decrease property
$$\min_{w \in \overline{\text{co}}f(x, \mathcal{U} \cap \sigma(|x|_{\mathcal{A}})\overline{B}_{\mathcal{U}})} DV(x; w) \leq -V(x), \quad \forall x \in \mathbb{R}^n.$$

Remark 2 The use of the minimum is justified here, and throughout the paper, in place of an infimum by virtue of the fact that the set over which the infimum is taken is compact and the function is continuous i.e., $DV(x; \cdot)$ is locally Lipschitz for all $x \in \mathbb{R}^n$ when $V(\cdot)$ is locally Lipschitz (see [6, Exercise 3.4.1a]). ■

Our result will require the following technical assumption which is essentially [1, Definition 1.5].

Assumption 1 Given a closed and non-empty set \mathcal{A} and $\sigma : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ nondecreasing, we say that the function $f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ satisfies the boundedness assumption with respect to \mathcal{A} and $\mathcal{U} \cap \sigma$ controls if, $\forall r_1, r_2 \in \mathbb{R}_{>0}$, there exists $M_{r_1, r_2} > 0$ such that

$$\sup_{\{|x|_{\mathcal{A}} \leq r_1, |u| \leq \sigma(r_2)\}} |f(x, u)| \leq M_{r_1, r_2}.$$

We can now state our main result:

Theorem 1 Suppose (1) satisfies the boundedness assumption and is UGAC to the set \mathcal{A} with $\mathcal{U} \cap \sigma$ controls. Furthermore, assume that the set-valued map

$F(x) := \{z \in \mathbb{R}^n : z = f(x, u), u \in \mathcal{U} \cap \sigma(|x|_{\mathcal{A}})\overline{\mathcal{B}}_{\mathcal{U}}\}$ is such that $F(x)$ is nonempty and compact $\forall x \in \mathbb{R}^n$ and $F(\cdot)$ is locally Lipschitz. Then there exists a locally Lipschitz clf with $\mathcal{U} \cap \sigma$ controls for (1).

Remark 3 Two examples of regularity conditions on $f(\cdot, \cdot)$ which would give rise to a locally Lipschitz $F(\cdot)$ are as follows:

1. Let $\sigma \equiv +\infty$ and $f(x, \cdot)$ be measurable $\forall x \in \mathbb{R}^n$ and $f(\cdot, u)$ be locally Lipschitz uniformly in $u \in \mathcal{U}$. Then $F(\cdot)$ is locally Lipschitz.
2. Consider $\mathcal{U} = \mathbb{R}^m$ and let $\sigma(\cdot)$ be locally Lipschitz (and nondecreasing) and $f(\cdot, \cdot)$ be locally Lipschitz. Then $F(\cdot)$ is locally Lipschitz. ■

Analogous to our terminology for set-valued maps, we say that a function $g : \mathbb{R}^n \mapsto \mathbb{R}^n$ is locally Lipschitz, uniform in distance to the closed set \mathcal{A} if, for any $\ell > 0$ and a closed set $\mathcal{U} := \{x \in \mathbb{R}^n : |x|_{\mathcal{A}} \leq \ell\}$ there exists $L > 0$ such that, for every $x_1, x_2 \in \mathcal{U}$ we have $|g(x_1) - g(x_2)| \leq L|x_1 - x_2|$.

For control design purposes, we are also interested in the following result which states when the clf is locally Lipschitz uniformly in distance to the set \mathcal{A} .

Theorem 2 *Suppose the assumptions of Theorem 1 hold. Furthermore, assume that $F(\cdot)$ is locally Lipschitz, uniform in distance to the set \mathcal{A} . Then there exists a locally Lipschitz clf with $\mathcal{U} \cap \sigma$ controls which is locally Lipschitz, uniform in distance to the set \mathcal{A} .*

Remark 4 The examples given in Remark 3 for generating a locally Lipschitz set-valued map $F(\cdot)$ extend easily to the case of generating a set-valued map $F(\cdot)$ which is locally Lipschitz, uniform in distance to the set \mathcal{A} . This is done by requiring the corresponding Lipschitz property on f to be uniform in distance to the set \mathcal{A} . ■

4 Control Construction

Using the clf of Theorem 2, we can construct a (discontinuous) feedback stabilizer that, when implemented with a sample and hold strategy, guarantees semiglobal practical asymptotic stability of the set \mathcal{A} and robustness to small disturbances and measurement noise. Results of this type have already been presented in [5] and [16] for the case where \mathcal{A} is compact, and a construction is given in [4] that applies to the case of noncompact sets \mathcal{A} . Our construction resembles the constructions used in both of these references. In comparison to the construction in [4], we use proximal aiming to a point that minimizes the clf in a ball around the current point rather than proximal aiming to a sublevel set of the clf.

This permits a very concise statement of the control synthesis algorithm.

For $V : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\ell_1, \ell_2 \in \{-\infty\} \cup \mathbb{R}$ s.t. $\ell_1 < \ell_2$, we define $\mathcal{V}(\ell_1, \ell_2) := \{x \in \mathbb{R}^n : \ell_1 \leq V(x) \leq \ell_2\}$. We denote $\mathcal{V}(-\infty, \ell_2)$ by $\mathcal{V}(\ell_2)$.

The following assumptions, under which we construct our feedback law, all follow directly from Theorem 2. However, these assumptions are somewhat weaker as it simplifies the exposition. Specifically, as we use a sample and hold strategy to implement our feedback control, we are only concerned with the semiglobal practical qualities of the clf.

4.1 Assumptions

Suppose $\sigma(\cdot)$ is nondecreasing and we are given $\ell_1 < \ell_2$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$, $\varepsilon_4 > 0$, $c > 0$, $L_V > 0$, $L_f > 0$, $\widetilde{M} > 0$ such that

1. for all $x_1, x_2 \in \mathcal{V}(\ell_1, \ell_2 + \varepsilon_2) + \varepsilon_3\overline{\mathcal{B}}_n$, and all $u \in \mathcal{U} \cap \sigma(\max\{|x_1|_{\mathcal{A}}, |x_2|_{\mathcal{A}}\} + \varepsilon_4)\overline{\mathcal{B}}_{\mathcal{U}}$
 - (a) $|V(x_1) - V(x_2)| \leq L_V|x_1 - x_2|$,
 - (b) $|f(x_1, u) - f(x_2, u)| \leq L_f|x_1 - x_2|$,
 - (c) $\min_{w \in \overline{\sigma}f(x, \mathcal{U} \cap \sigma(|x|_{\mathcal{A}})\overline{\mathcal{B}}_{\mathcal{U}})} DV(x; w) \leq -2c$,
2. $f(\cdot, u)$ continuous and bounded in norm by \widetilde{M} on $\mathcal{V}(\ell_2 + \varepsilon_2) + \varepsilon_3\overline{\mathcal{B}}_n$, $\forall u \in \mathcal{U} \cap \sigma(|x|_{\mathcal{A}} + \varepsilon_4)\overline{\mathcal{B}}_m$.

Note again that, with appropriate values for the constants, these assumptions are all satisfied by the clf of Theorem 2.

4.2 Control design

For the control system

$$\dot{x} = f(x, u) + d, \quad u \in \mathcal{U} \cap \sigma(|x|_{\mathcal{A}} + \varepsilon_4)\overline{\mathcal{B}}_{\mathcal{U}} \quad (3)$$

we define a (discontinuous) control law as follows:

1. Let $r \in \left(0, \min\left\{\frac{\varepsilon_2}{L_V}, \varepsilon_3, \varepsilon_4, \frac{c}{L_f L_V}\right\}\right)$.
2. For each $x \in \mathcal{V}(\ell_1, \ell_2 + \varepsilon_2)$,
 - (a) let $s \in \overline{\mathcal{B}}_n(x, r)$ s.t. $V(s) \leq V(\xi) \forall \xi \in \overline{\mathcal{B}}_n(x, r)$;
 - (b) let $\alpha \in \mathcal{U} \cap \sigma(|x|_{\mathcal{A}} + r)\overline{\mathcal{B}}_{\mathcal{U}}$ such that $\langle x - s, f(x, \alpha) \rangle \leq -\frac{c}{L_V}|x - s|$
(§ 4.5 explains why such an α exists).
3. $\forall x \notin \mathcal{V}(\ell_1, \ell_2 + \varepsilon_2)$ let $\alpha \in \mathcal{U} \cap \sigma(|x|_{\mathcal{A}})\overline{\mathcal{B}}_{\mathcal{U}}$ be arbitrary.
4. Take $u = \alpha(x)$.

4.3 Closed-loop results

We let $T_1 > 0$ be such that

$$\frac{cT_1}{8L_V} \leq r - \sqrt{r^2 - \frac{rcT_1}{2L_V}}. \quad (4)$$

Such a value exists since the derivative w.r.t. T_1 of the function on the right-hand side evaluated at $T_1 = 0$ is equal to $\frac{c}{4L_V}$. We define $M := \widetilde{M} + \frac{c}{2L_V}$, $a_1 := M^2 L_f$, $a_2 := M(M + rL_f)$, $a_3 := \frac{cr}{4L_V}$, and

$$T^* := \min \left\{ T_1, \frac{\ell_2 - \ell_1}{L_V M}, \frac{\varepsilon_3}{M}, \frac{\sqrt{a_2^2 + 4a_1 a_3} - a_2}{2a_1} \right\}.$$

We note that $T^* > 0$ and $T^* \rightarrow 0$ as $r \rightarrow 0$. This is evident from the last term that defines T^* .

Theorem 3 *Suppose $u = \alpha(x)$ is implemented by sampling and holding with holding period $T \in (0, T^*]$. Then $\forall x_0 \in \mathcal{V}(\ell_2)$, all $d(\cdot)$ s.t. $\|d\|_\infty \leq \frac{c}{2L_V}$ and $\forall t \geq 0$, the resulting solutions satisfy*

$$V(x(t)) \leq \max \left\{ V(x_0) - \frac{c^2 \max\{t - T, 0\}}{8L_V M}, \ell_1 + MT \right\} + L_V r.$$

Proof outline: For a given solution $x(\cdot)$ and $T > 0$, we define, for all nonnegative integers i such that $x(iT)$ is defined, $x_i := x(iT)$, $\alpha_i = \alpha(x_i)$, and $s_i \in \overline{\mathcal{B}}_n(x_i, r)$ s.t. $V(s_i) \leq V(\xi) \forall \xi \in \overline{\mathcal{B}}_n(x_i, r)$. In particular, $\forall x_i \in \mathcal{V}(\ell_1, \ell_2 + \varepsilon_2)$, s_i corresponds to the selection used to define the control law on $\mathcal{V}(\ell_1, \ell_2 + \varepsilon_2)$ and α_i is the constant control input over the i th time interval.

4.4 Dynamical systems description

It can be shown that the solutions of the closed-loop system are constrained by the following inequalities:

1. ‘Initial condition’ constraint:

$$V(s_i) \leq V(x_i) \quad (5)$$

2. ‘Output’ constraint:

$$\left\{ \begin{array}{l} x_i \in \mathcal{V}(\ell_1, \ell_2 + \varepsilon_2) \\ t \in [iT, (i+1)T] \end{array} \right\} \Rightarrow V(x(t)) \leq V(s_i) + L_V r \quad (6)$$

$$\left\{ \begin{array}{l} x_i \in \mathcal{V}(\ell_1) \\ t \in [iT, (i+1)T] \end{array} \right\} \Rightarrow V(x(t)) \leq \ell_1 + L_V MT \quad (7)$$

3. ‘State’ constraint:

$$x_i \in \mathcal{V}(\ell_1, \ell_2 + \varepsilon_2) \Rightarrow V(s_{i+1}) \leq V(s_i) \quad (8)$$

$$x_i, x_{i+1} \in \mathcal{V}(\ell_1, \ell_2 + \varepsilon_2) \Rightarrow V(s_{i+1}) \leq V(s_i) - \frac{c^2 T}{8L_V M}. \quad (9)$$

Due to space constraints, we only briefly describe the intuition for why these assertions are true. The inequality (5) follows directly from the definition of s_i . The inequality (7) follows from the bound on the derivative of solutions which follows from the assumed bound on f . The inequalities (6) and (8) follow from the fact that $x_i \in \mathcal{V}(\ell_1, \ell_2 + \varepsilon)$ implies $x(\cdot)$ approaches s_i for $t \in [iT, (i+1)T]$, the latter being a consequence of the control construction. The condition (9) is a nontrivial consequence of the spacing between s_{i+1} and s_i , which follows from how much closer $x((i+1)T)$ is to s_i than $x(iT)$ is to s_i (a result of the control construction), and the Dini condition on V .

In what follows, we derive bounds on $V(x(\cdot))$ by considering two regions of the state-space; $\mathcal{V}(\ell_1, \ell_2 + \varepsilon_2)$ and $\mathcal{V}(\ell_1)$. That is, we examine strips (defined by distance to the closed set \mathcal{A}) and the region below these strips (or closer to the closed set \mathcal{A}) to obtain bounds consisting of the maximum of two quantities. The constraints (5), (7)-(8) give that

$$x_i \in \mathcal{V}(\ell_2 + \varepsilon_2) \Rightarrow V(s_{i+1}) \leq \max\{V(s_i), \ell_1 + L_V MT\}. \quad (10)$$

For $x_0 \in \mathcal{V}(\ell_2)$, (5) and (10) yield

$$V(s_i) \leq \max\{V(x_0), \ell_1 + L_V MT\}.$$

Using (6) for $x_0 \in \mathcal{V}(\ell_1, \ell_2 + \varepsilon_2)$ and (7) for $x_0 \in \mathcal{V}(\ell_1)$ we obtain, for $x_0 \in \mathcal{V}(\ell_2)$ and all $t \geq 0$

$$V(x(t)) \leq \max\{V(s_i) + L_V r, \ell_1 + L_V MT\} \quad (11)$$

$$\begin{aligned} &\leq \max\{\max\{V(x_0), \ell_1 + L_V MT\} + L_V r, \\ &\quad \ell_1 + L_V MT\} \quad (12) \\ &\leq \ell_2 + \varepsilon_2, \end{aligned}$$

where the last line follows from $\ell_1 + L_V MT \leq \ell_2$, and $L_V r \leq \varepsilon_2$. In particular, the last line implies that we remain in the region where our assumptions are valid (i.e., $\{x \in \mathbb{R}^n : V(x) \leq \ell_2 + \varepsilon_2\}$).

It follows from (12) that if $x_j \in \mathcal{V}(\ell_1)$ then

$$V(x(t)) \leq \ell_1 + L_V MT + L_V r, \quad \forall t \geq jT.$$

Therefore, for sample values, x_i , in a certain set ($\{x_i \in \mathbb{R}^n : V(x_i) \leq \ell_1\}$), the Lyapunov function evaluated along the remaining trajectory is bounded as above. This demonstrates the second bound of Theorem 3.

Finally, it follows from (5), (6), and (9) that if $x_k \in \mathcal{V}(\ell_1, \ell_2 + \varepsilon_2)$, $\forall k \in \{0, j-1\}$ then, $\forall t \in [0, jT]$,

$$V(x(t)) \leq V(x_0) + L_V r - \frac{c^2 \max\{t - T, 0\}}{8L_V M}.$$

So, for sample values in a certain set ($\{x_i \in \mathbb{R}^n : \ell_1 \leq V(x) \leq \ell_2 + \varepsilon_2\}$), the Lyapunov function evaluated along the trajectory satisfies the above bound. This demonstrates the first portion of the bound in Theorem 3. Therefore, considering all sample values such that $V(x_i) \leq \ell_2$ for all positive integers i , we have the bound required by Theorem 3.

4.5 Existence of Control Selection

Since $x \in \mathcal{V}(\ell_1, \ell_2 + \varepsilon_2)$, we have that $s \in \mathcal{V}(\ell_1, \ell_2 + \varepsilon_2) + \varepsilon_3 \overline{\mathcal{B}}_n$. Let $w \in \overline{\text{co}}f(s, \mathcal{U} \cap \sigma(|s|_{\mathcal{A}}) \overline{\mathcal{B}}_{\mathcal{U}})$ be s.t.

$$DV(s; w) \leq -2c \quad (13)$$

Let $\varepsilon > 0$ be small enough so that there exists $z \in \partial \overline{\mathcal{B}}_n(x, r)$ (i.e., z is on the boundary of $\overline{\mathcal{B}}_n(x, r)$) so that $z - (s + \varepsilon w)$ is parallel to $x - s$ and pointing in the same direction, i.e.,

$$\left\langle z - (s + \varepsilon w), \frac{x - s}{|x - s|} \right\rangle = |z - s - \varepsilon w|. \quad (14)$$

We note that $\left\langle z - s, \frac{x - s}{|x - s|} \right\rangle = \mathcal{O}(\varepsilon^2)$. We also note that, by the above Dini condition, $s + \varepsilon w$ lies in the region where the assumptions are valid. So we have, using $V(z) \geq V(s)$,

$$\begin{aligned} -\varepsilon \left\langle \frac{x - s}{|x - s|}, w \right\rangle &= |z - s - \varepsilon w| + \mathcal{O}(\varepsilon^2) \\ &\geq \frac{V(z) - V(s + \varepsilon w)}{L_V} + \mathcal{O}(\varepsilon^2) \\ &\geq \frac{V(s) - V(s + \varepsilon w)}{L_V} + \mathcal{O}(\varepsilon^2) \end{aligned}$$

or

$$\frac{V(s + \varepsilon w) - V(s)}{L_V \varepsilon} \geq \left\langle \frac{x - s}{|x - s|}, w \right\rangle + \mathcal{O}(\varepsilon).$$

Taking the liminf as $\varepsilon \rightarrow 0$ and using (13), we get

$$-\frac{2c}{L_V} \geq \frac{DV(s; w)}{L_V} \geq \left\langle \frac{x - s}{|x - s|}, w \right\rangle.$$

Appealing to Lemma 1, this implies that $\exists \alpha \in \mathcal{U} \cap \sigma(|s|_{\mathcal{A}}) \overline{\mathcal{B}}_{\mathcal{U}} \subset \mathcal{U} \cap \sigma(|x|_{\mathcal{A}} + r) \overline{\mathcal{B}}_{\mathcal{U}}$ s.t.

$$-\frac{2c}{L_V} \geq \left\langle \frac{x - s}{|x - s|}, f(s, \alpha) \right\rangle.$$

Now using the Lipschitz property for f and the condition $r \leq \frac{c}{L_V L_f}$, we have

$$\left\langle \frac{x - s}{|x - s|}, f(x, \alpha) \right\rangle \leq -\frac{2c}{L_V} + L_f r \leq -\frac{c}{L_V}.$$

5 Robustness to measurement noise

In this section we will demonstrate that our control design is robust with respect to small measurement errors. That is, if we implement our control using a corrupted measurement $x + n$ rather than with the true state x , the trajectory of the controlled system will still approach the attractor \mathcal{A} . Similar results are established in [16] and [4].

Consider the system

$$\dot{x} = f(x, \alpha(x_i + n_i)) \quad (15)$$

where n_i represents samples of a bounded noise function $n : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^n$. We construct a fake noise function $n_L(\cdot)$ that is globally Lipschitz and matches $n(\cdot)$ at sampling instances. If N is a bound for $|n(\cdot)|$ and T is the sampling period, then $n_L(\cdot)$ can be constructed so that it is bounded by N and its Lipschitz constant is $2N/T$. We perform a coordinate change, $z = x + n_L$, in order to write

$$\dot{z} = f(z + n_L, \alpha(z_i)) + \dot{n}_L. \quad (16)$$

Exploiting the Lipschitz constant for f , this system can be written in the form

$$\dot{z} = f(z, \alpha(z_i)) + d, \quad (17)$$

where $|d| \leq N(L_f + 2/T)$. Therefore the result of Theorem 3 applies if we insist that $N \leq \frac{Tc}{2L_V(2+L_fT)}$.

6 Proof of Main Result

6.1 A preliminary result for inclusions

The proof of our main result relies on a recent weak converse Lyapunov function for weak asymptotic stability for differential inclusions which appeared in [9]. We present (a slightly simplified version of) that result now. First we need definitions of weak asymptotic stability and weak converse Lyapunov function, analogous to Definitions 3 and 4.

Definition 5 For $\dot{x} \in F(x)$, the closed set $\mathcal{A} \subset \mathbb{R}^n$ is weakly uniformly globally asymptotically stable (weakly-UGAS) if $\exists \beta \in \mathcal{KL}$ s.t., $\forall x \in \mathbb{R}^n$, $\exists \phi \in \mathcal{S}(x)$ defined on $[0, \infty)$ satisfying $|\phi(t, x)|_{\mathcal{A}} \leq \beta(|x|_{\mathcal{A}}, t)$, $\forall t \geq 0$.

Definition 6 A function $V : \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$ is called a locally Lipschitz weak converse Lyapunov function for $\dot{x} \in F(x)$ w.r.t. \mathcal{A} if $\exists \mathcal{K}_{\infty}$ functions α_1 and α_2 s.t. $\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}})$, and for all $x \in \mathbb{R}^n$ $\min_{w \in F(x)} DV(x; w) \leq -V(x)$.

Our result on the existence of a weak converse Lyapunov function under the assumption of weak asymptotic stability uses the following technical assumption which parallels Assumption 1.

Assumption 2 Given the set-valued map $F(\cdot)$ from \mathbb{R}^n to subsets of \mathbb{R}^n , for each $r > 0$ there exists $M_r > 0$ such that $|x|_{\mathcal{A}} \leq r$ implies $\sup_{\omega \in F(x)} |\omega| \leq M_r$.

A version of the main result in [9] is given below.

Theorem 4 Suppose Assumption 2 holds, $F(x)$ is nonempty, compact and convex $\forall x \in \mathbb{R}^n$, and $F(\cdot)$ is locally Lipschitz. If the closed set \mathcal{A} is weakly-UGAS for $\dot{x} \in F(x)$ then \exists a locally Lipschitz weak converse Lyapunov function for $\dot{x} \in F(x)$ w.r.t. \mathcal{A} .

Remark 5 We note that the decrease condition in [9] is not stated in terms of the Dini subderivate (which we do here), but is phrased as an inner product condition on the gradient almost everywhere. However, the condition on the Dini subderivate was established in the proof given in [9]. ■

The next result can be established with simple modifications to the proof of Theorem 4 given in [9].

Theorem 5 *Suppose the assumptions of Theorem 4 hold. Furthermore, assume that $F(\cdot)$ is locally Lipschitz, uniform in distance to the set \mathcal{A} . Then \exists a locally Lipschitz weak converse Lyapunov function for $\dot{x} \in F(x)$ w.r.t. \mathcal{A} where the local Lipschitz property is uniform in distance to the set \mathcal{A} .*

We are now in a position to prove our main result.

6.2 Proof of Theorems 1 and 2

We first prove Theorem 1. By assumption $F(x)$ is compact $\forall x \in \mathbb{R}^n$ and, from Assumption 1, it satisfies Assumption 2 which implies that $\overline{\text{co}}F(\cdot)$ satisfies Assumption 2. Since $F(\cdot)$ is locally Lipschitz, the set-valued map $\overline{\text{co}}F(\cdot)$ is also locally Lipschitz (see [3, §1.1, Prop. 6]). Furthermore, $\overline{\text{co}}F(x)$ is also nonempty, compact, and convex $\forall x \in \mathbb{R}^n$. Therefore $\overline{\text{co}}F(x)$ satisfies all of the assumptions of Theorem 4 $\forall x \in \mathbb{R}^n$.

Next we show that the differential inclusion $\dot{x} \in \overline{\text{co}}F(x)$ is such that the set \mathcal{A} is weakly-UGAS. Let $\beta \in \mathcal{KL}$, $\sigma(\cdot)$ nondecreasing, and $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ a solution of $\dot{x} = f(x, u)$ satisfying (2) come from the assumption of UGAC to \mathcal{A} with $\mathcal{U} \cap \sigma$ controls. By construction, for almost all $t \geq 0$,

$$\overbrace{\phi(t, x, u)} \in \overline{\text{co}}F(\phi(t, x, u)).$$

Therefore, \mathcal{A} is weakly-UGAS for $\dot{x} \in \overline{\text{co}}F(x)$.

Next, by Theorem 4 there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $V : \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$ locally Lipschitz on \mathbb{R}^n such that $\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}})$, and $V(\cdot)$ satisfies the weak infinitesimal decrease property

$$\min_{w \in \overline{\text{co}}F(x)} DV(x; w) \leq -V(x), \quad \forall x \in \mathbb{R}^n.$$

From the definition of $F(\cdot)$ it follows that

$$\min_{w \in \overline{\text{co}}f(x, \mathcal{U} \cap \sigma(|x|_{\mathcal{A}}) \overline{\mathcal{B}}_m)} DV(x; w) \leq -V(x), \quad \forall x \in \mathbb{R}^n.$$

and, therefore, $V(\cdot)$ is a locally Lipschitz clf with $\mathcal{U} + \sigma$ controls for (1).

The proof for Theorem 2 follows exactly as above except we appeal to Theorem 5 instead of Theorem 4 so that the Lipschitz property of our clf is uniform in distance to the set \mathcal{A} . ■

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