Robust H_∞ FIR Filtering for Uncertain Time-Varying Nonlinear Systems

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Abstract

This paper investigates the robust nonlinear H_∞ filter with FIR (Finite Impulse Response) structure for nonlinear discrete time-varying uncertain systems represented by the state-space model having parameter uncertainty. Firstly, the discrete-time nonlinear H_∞ FIR filter without parameter uncertainty is derived by using the equivalence relationship between the FIR filter and the recursive filter, which corresponds to the standard nonlinear H_∞ filter. Secondly, the robust H_∞ FIR filter is proposed for the discrete-time nonlinear uncertain systems. It is also derived from the equivalence relationship between the robust nonlinear H_∞ FIR filter and the robust nonlinear H_∞ filter proposed by de Souza *et al.*[3].

1 INTRODUCTION

Over the past several years, the problem of the nonlinear H_∞ filtering has been studied by a number of authors [1,4,5]. There are two commonly used approaches for providing solutions to nonlinear H_∞ control and filtering problems. One is based on the dissipativity theory and the differential game theory. Another is based on the nonlinear version of the classical Bounded Real Lemma as developed by Willems [6] and Hill and Moylan [7]. However, the nonlinear H_∞ filters proposed so far are mainly limited to time-invariant systems. Therefore they can not be applied to general time-varying systems on the infinite horizon since one of two Riccati differential equations required to solve the problem can not be computed on the infinite horizon.

This paper deals with the issue of the robust nonlinear H_∞ filtering problem for discrete time-varying systems with the parameter uncertainties on the infinite horizon. The basic idea of the current paper is to formulate the robust nonlinear H_∞ filtering problem on the discrete-time moving horizon and to adopt the FIR (Finite Impulse Response) filter structure. The estimator of the current paper is rather a one-step-ahead predictor than a filter.

FIR filters are widely used in the signal processing area, and they were utilized in the estimation problem as the optimal FIR filters. Since the optimal FIR filters use the finite observations only over a finite preceding time interval, they can overcome the divergence problem and have the built-in BIBO (Bounded Input/Bounded Output) stability and the robustness to the numerical problems such as coefficient quantization errors and round-off errors, which are well known properties of the FIR structure in signal processing area. Also note that IIR (Infinite Impulse Response) or recursive filter structure requires the initial conditions on each horizon, which is an impractical assumption, but that FIR filter structure does not requires the initial conditions. The optimal FIR filters are, however, presented so far not in the H_∞ setting but in the minimum variance formulation.

The nonlinear H_∞ filter proposed is to be called hereafter as the *robust nonlinear H_∞ FIR filter* in the sense that it is a nonlinear H_∞ filter with the FIR structure for uncertain systems. It will be shown that the nonlinear H_∞ FIR filter always has a solution if the standard nonlinear H_∞ filter exists on the finite horizon. It is noted that the nonlinear filter proposed works on the time-varying nonlinear systems with time-varying parametric uncertainties, and that this point will be one of the main contributions of the current paper.

For the case when there is no parameter uncertainty in the system, we are concerned with designing a nonlinear H_∞ FIR filter such that the induced l_2 operator norm of the mapping from the noise signal to the estimation error is within a specified bound. It is shown that this problem can be solved via one Riccati equation. We also consider the design of nonlinear filters which guarantee a prescribed H_∞ performance in the presence of parametric uncertainties. In this situation, the solution is to be obtained in terms of two Riccati

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equations.

This paper is organized as follows: The robust nonlinear H_∞ FIR filtering problem is formulated in Section 2. The nominal nonlinear H_∞ FIR filtering problem is solved in Section 3, and the robust nonlinear H_∞ FIR filter is proposed in Section 4. Conclusions are summarized in Section 5.

2 PROBLEM FORMULATION

Consider the uncertain nonlinear time-varying system of the form

$$\begin{aligned} x_{k+1} &= (A_k + \triangle A_k)x_k + G_k g(x_k) + B_k w_k & (1)\\ y_k &= (C_k + \triangle C_k)x_k + H_k h(x_k) + D_k w_k & (2)\\ z_k &= L x_k \ , & (3) \end{aligned}$$

where $x_k \in \Re^n$ is the state vector with the initial state x_0 unknown, $w_k \in \Re^q$ is a noise signal which belongs to $\ell_2[0,\infty)$, $y_k \in \Re^m$ is the measurement, $z_k \in \Re^p$ is a linear combination of state variables to be estimated, $g(\cdot) : \Re^n \to \Re^{n_g}$ and $h(\cdot) : \Re^n \to \Re^{n_h}$ are known nonlinear vector functions and $A_k, B_k, C_k, D_k, G_k, H_k$ and L_k are known real time-varying matrices of appropriate dimensions that describe the nominal system together with $g(\cdot)$ and $h(\cdot)$. The matrices $\triangle A_k$ and $\triangle C_k$ represent time-varying parameter uncertainties in A_k and C_k, respectively. These uncertainties are assumed to be of the following structure

$$\begin{bmatrix} \triangle A_k \\ \triangle C_k \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} F_k E, \qquad (4)$$

where F_k is an unknown real time-varying matrix satisfying

$$F_k^T F_k \le I, k = 0, 1, 2, ... \qquad (5)$$

and H_1, H_2 and E are known real constant matrices of appropriate dimensions that specify how the elements of the nominal matrices A_k and C_k are affected by the uncertain parameters in F_k.

Assumption 1.

(a) $[D_k \ \ H_2 \ \ H_k]$ *is of full row rank*;
(b) $D_k B_k^T = 0$;
(c) $g(0) = 0$;
(d) *There exist known constant matrices* V_1 *and* V_2 *such that for any* x_1 *and* $x_2 \in \Re^n$,

$$\begin{aligned} \|g(x_1) - g(x_2)\| &\le \|V_1(x_1 - x_2)\| \\ \|h(x_1) - h(x_2)\| &\le \|V_2(x_1 - x_2)\| \quad . \end{aligned}$$

□

Assumption 1(a) and 1(b) means that the robust H_∞ FIR filtering problem is 'nonsingular'. Observe that if the parameter uncertainty in the output matrix disappears, i.e. $H_2 = 0$, Assumption 1(a) reduces to $D_k D_k^T > 0$, which is a standard assumption in the H_∞ FIR filtering problem for the nominal system.

Observe that discrete-time nonlinear models of the form of Eqs. (1)-(2) can be used to represent many important physical systems. The parameter uncertainty in the linear terms can be regarded as the variation of the operating points of the nonlinear system.

In this section we are concerned with designing a nonlinear causal filter $\mathcal{F}$ with FIR structure for estimating z_k with a guaranteed performance in a H_∞ sense, using the measurements $\mathcal{Y}_{k-1} = \{y_j, j = 0, 1, 2, \ldots, k-1\}$ and where no *a priori* estimate of the initial state of Eq. (1) is assumed. Letting $\hat{z}_k$ denote the estimate of z_k, the filter is required to guarantee a uniformly small estimation error $z_k - \hat{z}_k$, for any $w \in \ell_2[0,\infty)$ and $x_0 \in \Re^n$. Then the *robust nonlinear* H_∞ *FIR filtering problem* is formulated as follows:

Given the system of Eqs.(1)-(3) and a prescribed level of noise attenuation $\gamma > 0$ *on each horizon* $[k-N, k]$, *find a causal filter* $\mathcal{F}$ *such that the filtering error dynamics is globally uniformly asymptotically stable and* $\| \ z - \hat{z} \ \|^2 < \gamma\{\| \ w \ \|_{N2}^2 + x_0^T R x_o\}$ *for any non-zero* $(x_0, w) \in \Re^n \oplus \ell_2[0,\infty)$ *and for all uncertainties satisfying Eqs.(4)-(5), where* $\|z-\hat{z}\|^2 = (z-\hat{z})^T(z-\hat{z})$, $R = R^T > 0$ *is a given weighting matrix for* x_0 *and* $\| \cdot \|_{N2}$ *denotes the usual* l_2*-norm on the horizon* $[k-N, k]$. *In the sequel,* $\| \cdot \|_{N2}$ *represents the usual* l_2*-norm on the horizon* $[0, N]$ *for the simplicity.*

Provided that there is no parameter uncertainty in the system, *i.e.*, $\triangle A_k = 0$ and $\triangle C_k = 0$ for all k in the above formulation, the problem reduces to the *nonlinear* H_∞ *FIR filtering problem*, which corresponds to the nonlinear H_∞ filtering problem.

Note that the performance index in the above problem statements is a worst-case performance measure and can be viewed as a generalization of the standard H_∞ performance measure to deal with unknown initial state. The weighting matrix, R, is a measure of the uncertainty in x_0 relative to the uncertainty in w. A 'large' value of R indicates that the initial state is likely to be very close to zero.

We end this section by recalling a version of the bounded real lemma for linear discrete time-varying systems which will be used in the derivation of a solution to the above filtering problems.

Consider the following linear time-varying system

$$x_{k+1} = A_k x_k + B_k w_k \qquad (6)$$

$$z_k \;=\; C_k x_k \, , \tag{7}$$

where $x_k \in \Re^n$ is the state vector with the initial state x_0 being unknown, $w_k \in \Re^q$ is the input which belongs to $\ell_2[0,\infty)$, $z_k \in \Re^p$is the measurement, and A_k, B_k and C_k are known bounded real time-varying matrices. Also, we define the following worst-case performance measure for the system of Eqs. (6)-(7):

$$J(z,w,x_0,R) = \sup_{(x_0,w)\neq 0} \left\{ \left[\frac{\|z\|^2}{\|w\|_{N2}^2 + x_0^T R x_0} \right]^{1/2} \right\},$$

where $R = R^T > 0$ is a given weighting matrix for the initial state and $0 \neq (x_0, w) \in \Re^n \oplus \ell_2[0,N]$. Then, we have the following result.

Lemma 1. [2] Consider the system of Eqs. (6)-(7) and let $\gamma > 0$ be a given scalar. Then, the following statements are equivalent for the moving horizon $[0,N]$:

(a) The system of Eq. (6) is exponentially stable and $J(z,w,x_0,R) < \gamma$;
(b) There exists a bounded time-varying matrix $Q_k = Q_k^T \geq 0$, $\forall\, k \geq 0$, satisfying $I - \gamma^{-2} C_k Q_k C_k^T > 0$, $\forall\, k \geq 0$, and such that

$$A_k Q_k A_k^T - Q_{k+1} + \gamma^{-2} A_k Q_k C_k^T (I - \gamma^{-2} C_k Q_k C_k^T)^{-1} \\ \cdot C_k Q_k A_k^T + B_k B_k^T = 0, \;\; Q_0 = R^{-1},$$

and the system

$$x_{k+1} = [A_k + \gamma^{-2} A_k Q_k C_k^T (I - \gamma^{-2} C_k Q_k C_k^T)^{-1} C_k] x_k$$

is exponentially stable;
(c) There exists a bounded time-varying matrix $P_k = P_k^T > 0$, $\forall\, k \geq 0$, satisfying $I - \gamma^{-2} B_k^T P_{k+1} B_k > 0$, $\forall\, k \geq 0$, and such that

$$A_k^T P_{k+1} A_k - P_k + \gamma^{-2} A_k^T P_{k+1} B_k (I - \gamma^{-2} B_k^T P_{k+1} B_k)^{-1} \\ \cdot B_k^T P_{k+1} A_k + C_k^T C_k < 0, \quad P_0 < \gamma^2 R.$$

$\bigtriangledown\bigtriangledown\bigtriangledown$

Observe that when the initial state of Eq. (6) is zero, the performance index $J(z,w,x_0,R)$ becomes the usual H_∞ performance measure, namely

$$J(z,w) = \sup_{0 \neq w} \left\{ \frac{\|z\|}{\|w\|_{N2}} \right\}.$$

The index of performance $J(z,w)$ can be viewed as the limit of $J(z,w,x_0,R)$ as the smallest eigenvalue of R approaches infinity. In this case it happens that $Q_0 = 0$ in the statement (b) of Lemma 1 while the requirement $P_0 < \gamma^2 R$ in the statement (c) will become superfluous.

Firstly, in the current paper, the *nonlinear H_∞ FIR filtering problem* will be solved, and then the *robust nonlinear H_∞ FIR filtering problem* is to be dealt with. It is noted that the problem does not need the assumption of stabilizability or detectability of the system since it is formulated on the finite moving horizon.

3 H_∞ FIR NONLINEAR FILTERS

In the sequel we shall provide a solution to both the problems of nominal and robust H_∞ filtering with FIR structure using a Riccati equation approach.

We first present a performance analysis result for the system of Eqs. (1) and (3).

Theorem 1. Consider the system of Eqs. (1) and (3) satisfying Assumption 1. Given a scalar $\gamma > 0$ and an initial state weighting matrix $R = R^T > 0$ then, the system of Eq. (1) is globally uniformly asymptotically stable and

$$\|z\|^2 < \gamma\{\|w\|_{N2}^2 + x_0^T R x_0\}$$

for any non-zero $(x_0, w) \in \Re^n \oplus \ell_2[0,N]$ and for all $\triangle A_k$ satisfying Eqs. (4)-(5) if there exist a scalar $\varepsilon > 0$ and a bounded time-varying matrix $Q_k = Q_k^T > 0$, $\forall\, k \geq 0$ satisfying $I - \gamma^{-2} B_1^T Q_{k+1} B_1 > 0$, $\forall\, k \geq 0$, and such that

$$A_k^T Q_{k+1} A_k - Q_k + \gamma^{-2} A_k^T Q_{k+1} B_1 (I - \gamma^{-2} B_1^T Q_{k+1} B_1)^{-1} \\ \cdot B_1^T Q_{k+1} A_k + L_k^T L_k + \varepsilon^2 E^T E + V_1^T V_1 < 0, \quad Q_0 < \gamma^2 R,$$

where

$$B_1 = [B_k \quad \frac{\gamma}{\varepsilon} H_1 \quad \gamma G_k]. \tag{8}$$

Proof : It can be easily established similarly to the proof of Theorem 4.2 in [10]. $\bigtriangledown\bigtriangledown\bigtriangledown$

In the case when there is no parameter uncertainty in Eq. (1), Theorem 1 reduces to the following corollary.

Corollary 1. Consider the system of Eqs. (1) and (3) with $\triangle A_k \equiv 0$ and satisfying Assumption 1. Given a scalar $\gamma > 0$ and an initial state weighting matrix $R = R^T > 0$ then, the system of Eq. (1) is globally uniformly asymptotically stable and

$$\| \, z \, \|_2 < \gamma\{\| \, w \, \|_2^2 + x_0^T R x_0\}^{\frac{1}{2}}$$

for any non-zero $(x_0, w) \in \Re^n \oplus \ell_2[0,\infty)$ if there exist a scalar $\varepsilon > 0$ and a bounded time-varying matrix $Q_k = Q_k^T > 0$, $\forall\, k \geq 0$, satisfying $I - \gamma^{-2} \bar{B}_1^T Q_{k+1} \bar{B}_1 > 0$, $\forall\, k \geq 0$, and such that

$$A_k^T Q_{k+1} A_k - Q_k + \gamma^{-2} A_k^T Q_{k+1} \bar{B}_1 (I - \gamma^{-2} \bar{B}_1^T Q_{k+1} \bar{B}_1)^{-1} \\ \cdot \bar{B}_1^T Q_{k+1} A_k + L_k^T L_k + V_1^T V_1 < 0, \quad Q_0 < \gamma^2 R,$$

where $\bar{B}_1 = [B_k \quad \gamma G_k]$. $\square$

Note that when the initial state of the system of Eq. (1) is known to be zero, the time-varying matrix Q_k in Theorem 1 and Corollary 1 may be replaced by a constant matrix $Q = Q^T > 0$. Furthermore, the condition $Q < \gamma^2 R$ will no longer be required as an initial state which is certain to be zero corresponds to choosing a 'very large' value of R.

We now present a solution to the nominal nonlinear H_∞ FIR filtering problem for the system of Eqs. (1)-(3).

Theorem 2. Consider the system of Eqs.(1)-(3) with $\triangle A_k \equiv 0$ and $\triangle C_k \equiv 0$, and satisfying Assumption 1. Given a scalar $\gamma > 0$ and an initial state weighting matrix $R = R^T > 0$, the nominal nonlinear H_∞ FIR filtering problem is solvable if there exists a bounded time-varying matrix $S_k = S_k^T \geq 0$, $\forall\ k \geq 0$, satisfying $I - \gamma^{-2}\tilde{L}S_k\tilde{L}^T > 0$, $\forall\ k \geq 0$, and such that $S_0 = R^{-1}$,

$$\begin{aligned} S_{k+1} &= A_k S_k A_k^T - (A_k S_k \tilde{C}_k^T + \tilde{B}\tilde{D}_1^T)(\tilde{C}S_k\tilde{C}^T \\ &\quad + \tilde{R})^{-1}(\tilde{C}S_k A_k^T + \tilde{D}_1\tilde{B}^T) + \tilde{B}\tilde{B}^T, \end{aligned} \tag{9}$$

and the system

$$\rho_{k+1} = A_{1k}\rho_k \tag{10}$$

is exponentially stable, where

$$\begin{aligned} A_{1k} &= A_k - (A_k S_k \tilde{C}_1^T + \tilde{B}\tilde{D}_1^T)(\tilde{R} + \tilde{C}_1 S_k \tilde{C}^T)^{-1}\tilde{C}, \\ \tilde{L}^T\tilde{L} &= L_k^T L_k + V^T V, \qquad V = [V_1^T \quad V_2^T]^T, \\ \tilde{B} &= [B_k \quad \gamma G_k \quad 0], \qquad \tilde{D} = [D_k \quad 0 \quad \gamma H_k], \\ \tilde{C} &= \begin{bmatrix} C_k \\ \gamma^{-1}\tilde{L} \end{bmatrix}, \qquad \tilde{D}_1 = \begin{bmatrix} \tilde{D} \\ 0 \end{bmatrix}, \\ \tilde{R} &= \begin{bmatrix} \tilde{D}\tilde{D}^T & 0 \\ 0 & -I \end{bmatrix}. \end{aligned}$$

Moreover, if the above conditions hold, a suitable nonlinear filter is given by

$$\begin{aligned} \hat{x}_{(k+1)} &= A_k\hat{x}_k + G_k g(\hat{x}_k) \\ &\quad + K_k[y_k - C_k\hat{x}_k - H_k h(\hat{x}_k)], && (11) \\ \hat{z}_k &= L_k\hat{x}_k, && (12) \end{aligned}$$

where

$$\begin{aligned} K_k &= (A_k\hat{S}_k C_k^T + \tilde{B}\tilde{D}^T)(C_k\hat{S}_k C_k^T + \tilde{D}\tilde{D}^T)^{-1} && (13) \\ \hat{S}_k &= S_k + \gamma^{-2}S_k\tilde{L}^T(I - \gamma^{-2}\tilde{L}S_k\tilde{L}^T)^{-1}\tilde{L}S_k. && (14) \end{aligned}$$

Proof : Firstly, note that the condition $I - \gamma^{-2}\tilde{L}S_k\tilde{L}^T > 0$, $\forall\ k \geq 0$, together with Assumption 1 guarantee the non-singularity of the matrix $\tilde{R} + \tilde{C}S_k\tilde{C}^T$, $\forall\ k \geq 0$. Letting $\tilde{x}_k = x_k - \hat{x}_k$ and $e_k = z_k - \hat{z}_k$, it follows from Eqs.(1)-(3) (setting $\triangle A_k \equiv 0$ and $\triangle C_k \equiv 0$) end Eqs. (11)-(12) that

$$\begin{aligned} \tilde{x}_{k+1} &= (A_k - K_kC_k)\tilde{x}_k + (G_1 - K_kH_1)\xi(x_k, \hat{x}_k) \\ &\quad + (B_k - K_kD_k)w_k && (15) \\ e_k &= L\tilde{x}_k, && (16) \end{aligned}$$

where

$$\xi(x_k, \hat{x}_k) = \begin{bmatrix} g(x_k) - g(\hat{x}_k) \\ h(x_k) - h(\hat{x}_k) \end{bmatrix}$$

and

$$G_1 = [G_k \quad 0]\,, \quad H_1 = [0 \quad H_k]\,.$$

Note that by Assumption 1,

$$\parallel \xi(x_k, \hat{x}_k) \parallel \leq \parallel V\tilde{x}_k \parallel .$$

It can be shown from Eq. (9) that $Q_k = \gamma^{-2}S_k$ is such that $I - \tilde{L}Q_k\tilde{L}^T > 0$, $\forall\ k \geq 0$, and satisfies

$$\begin{aligned} &(A_k - K_kC_k)Q_k(A_k - K_kC_k)^T - Q_{k+1} + (A_k \\ &-K_kC_k)Q_k\tilde{L}^T(I - \tilde{L}Q_k\tilde{L}^T)^{-1}\tilde{L}Q_k(A_k - K_k \\ &\cdot C_k)^T + \gamma^{-2}\tilde{B}_1\tilde{B}_1^T = 0, \quad Q_0 = \gamma^{-2}R^{-1}, \end{aligned} \tag{17}$$

where

$$\tilde{B}_1 = [(B_k - K_kD_k) \quad \gamma(G_1 - K_kH_1)].$$

Also, it is easy to verify that the state matrix A_{1k} of the system of Eq. (10) can be rewritten as

$$A_{1k} = (A_k - K_kC_k)[I + Q_k\tilde{L}^T(I - \tilde{L}Q_k\tilde{L}^T)^{-1}\tilde{L}].$$

Since the system of Eq.(10) is exponentially stable, in view of Lemma 1 and Corollary 1, Eq. (17) implies that the estimation error dynamics of Eqs. (15) and (16) is globally uniformly asymptotically stable and

$$\|e\|^2 < \gamma\{\|w\|_{N2}^2 + x_0^T R x_0\}$$

for any non-zero $(x_0, w) \in \Re^n \oplus \ell_2[0, N]$. $\bigtriangledown \bigtriangledown \bigtriangledown$

When the initial state of the system of Eq. (1) is known to be zero, or when a stationary filter design is concerned, Theorem 1 can be specialized as follows.

Theorem 3. Consider the system of Eqs. (1)-(3) with $x_0 = 0$, $\triangle A_k \equiv 0$ and $\triangle C_k \equiv 0$, and satisfying Assumption 1. Given a scalar $\gamma > 0$ and an initial state weighting matrix $R = R^T > 0$, the nominal nonlinear H_∞ FIR filtering problem is solvable if there exists a stabilizing solution $S = S^T \geq 0$ to the algebraic Riccati equation

$$\begin{aligned} S &= A_k S_k A_k^T - (A_k S\tilde{C}^T + \tilde{B}\tilde{D}_1^T)(\tilde{C}S\tilde{C}^T + \tilde{R})^{-1} \\ &\quad \cdot (\tilde{C}SA_k^T + \tilde{D}_1\tilde{B}^T) + \tilde{B}\tilde{B}^T \end{aligned} \tag{18}$$

such that $I - \gamma^{-2}\tilde{L}S\tilde{L}^T > 0$. Moreover, if the above conditions hold, a suitable nonlinear filter is given by Eq. (11)-(12), where the filter gain of Eq. (13) is constant. $\bigtriangledown \bigtriangledown \bigtriangledown$

It should be pointed out that in Theorems 2 and 3 no stability requirement is imposed on the system of Eq.(1). We also observe that, when there are no nonlinear terms in the system of Eqs. (1) - (2), i.e. $g(\cdot) \equiv 0$ and $h(\cdot) \equiv 0$, the result of Theorem 3 will reduce to the H_∞ FIR linear filter.

4 ROBUST H_∞ FIR NONLINEAR FILTERS

Next, we solve the robust nonlinear H_∞ FIR filtering problem. To this end, we shall make a further assumption on the system of Eq. (1).

Assumption 2.

The nominal state matrix A_k is non-singular. □

Theorem 4. Consider the uncertain system of Eqs. (1) - (3) satisfying Eqs. (4)-(5) and Assumptions 1-2. Let $\nu > 0$ be an arbitrary small scalar.
Given a scalar $\gamma > 0$ and an initial state weighting matrix $R = R^T > 0$, the robust H_∞ FIR filtering problem is solvable if for some scalar $\varepsilon > 0$, the following conditions hold:

(a) There exists a stabilizing solution $P = P^T \geq 0$ to the algebraic Riccati equation:

$$\begin{aligned} A_k^T P A_k - P + \gamma^{-2} A_k^T P B_1 (I - \gamma^{-2} B_1^T P B_1)^{-1} \\ \cdot B_1^T P A_k + E_1^T E_1 + \nu I = 0 \end{aligned} \tag{19}$$

such that $I - \gamma^{-2} B_1^T P B_1 > 0$, and $P < \gamma^2 R$, where B_1 is as in Eq. (17) and

$$E_1^T E_1 = \varepsilon^2 E^T E + V_1^T V_1; \tag{20}$$

(b) There exists a bounded time-varying matrix $S_k = S_k^T \geq 0$, $\forall\ k \geq 0$, satisfying $I - \gamma^{-2} \hat{L} S_k \hat{L} > 0$, $\forall\ k \geq 0$, and such that $S_0 = (R - \gamma^{-2} P)^{-1}$,

$$\begin{aligned} S_{k+1} &= \hat{A} S_k \hat{A}^T - (\hat{A} S_k \hat{C}_1^T + \hat{B} \hat{D}_1^T)(\hat{C}_1 S_k \hat{C}_1^T \\ &\quad + \hat{R})^{-1} (\hat{C}_1 S_k \hat{A}^T + \hat{D}_1 \hat{B}^T) + \hat{B} \hat{B}^T \end{aligned} \tag{21}$$

and the system

$$\rho_{k+1} = A_{2k} \rho_k \tag{22}$$

is exponentially stable, where

$$\begin{aligned} A_{2k} &= \hat{A} - (\hat{A} S_k \hat{C}_1^T + \hat{B} \hat{D}_1^T)(\hat{C}_1 S_k \hat{C}_1^T + \hat{R})^{-1} \hat{C}_1 \\ \hat{L}^T \hat{L} &= L_k^T L_k + V^T V, \quad V = [V_1^T \quad V_2^T]^T, && (23) \\ \hat{C}_1 &= \begin{bmatrix} \hat{C} \\ \gamma^{-1} \hat{L} \end{bmatrix}, \qquad \hat{D}_1 = \begin{bmatrix} \hat{C} \\ \gamma^{-1} \hat{L} \end{bmatrix}, \\ \hat{R} &= \begin{bmatrix} \hat{D} \hat{D}^T & 0 \\ 0 & -I \end{bmatrix}, && (24) \\ \hat{A} &= A_k + \triangle A_e && (25) \\ &= A_k + \gamma^{-2} \bar{B} \bar{B}^T (P^{-1} - \gamma^{-2} B_1 B_1^T)^{-1} A_k, \\ \hat{C} &= C_k + \triangle C_e && (26) \\ &= C_k + \gamma^{-2} \bar{D} \bar{B}^T (P^{-1} - \gamma^{-2} B_1 B_1^T)^{-1} A_k, \\ \hat{B} &= \left[\ \bar{B} M \quad \gamma G_k \quad 0\ \right], && (27) \\ \hat{D} &= \left[\ \bar{D} M \quad 0 \quad \gamma H_k\ \right], && (28) \\ \bar{B} &= \left[B_k \ \ \frac{\gamma}{\epsilon} H_1 \right], \quad \bar{D} = \left[D_k \ \ \frac{\gamma}{\epsilon} H_2 \right], && (29) \\ M &= \left[I + \gamma^{-2} \bar{B}^T (P^{-1} - \gamma^{-2} B_1 B_1^T) \bar{B} \right]^{-\frac{1}{2}}. && (30) \end{aligned}$$

Moreover, if conditions (a) and (b) are satisfied, a suitable nonlinear filter is given by

$$\begin{aligned} x_{e(k+1)} &= \hat{A} \hat{x}_k + G_k g(\hat{x}_k) \\ &\quad + K_k [y_k - \hat{C} \hat{x}_k - H_k h(\hat{x}_k)] && (31) \\ \hat{z}_k &= L_k \hat{x}_k, && (32) \end{aligned}$$

where

$$K_k = (\hat{A} \hat{S}_k \hat{C}^T + \hat{B} \hat{D}^T)(\hat{C} \hat{S}_k \hat{C}^T + \hat{D} \hat{D}^T)^{-1} \tag{33}$$

$$\hat{S}_k = S_k + \gamma^{-2} S_k \hat{L}^T (I - \gamma^{-2} \hat{L} S_k \hat{L}^T)^{-1} \hat{L} S_k. \tag{34}$$

Proof : It can be easily established similarly to the proof of Theorem 9 in [8]. $\nabla\nabla\nabla$.

The arbitrary small scalar $\nu > 0$ is introduced in order to guarantee that the stabilizing solution of Eq. (19) is positive definite. In the case when $E_1^T E_1 > 0$, or the pair $(A,\ E_1)$ has no unobservable modes in the closed unit disk, ν can be set to zero.

We observe that the existence of a matrix P satisfying condition (a) of Theorem 4 will guarantee the global uniform asymptotic stability of the uncertain system of Eq. (1) for all uncertainties satisfying Eqs. (4)-(5). Note that due to the existence of parameter uncertainty in Eq. (1), the requirement of global asymptotic stability of Eq. (1) is needed in order to ensure the boundedness of the estimation error dynamics for all admissible uncertainties.

It should be noted that the result of Theorem 4 does not recover that of Theorem 2 when the uncertainties $\triangle A_k$ and $\triangle C_k$ disappear. The reason for this is because when parameter uncertainty exists an asymptotic stability requirement has to be imposed on the system of Eq. (1), which in turns gives rise to the Eq. (19) of Theorem 4.

Note that similarly to the robust linear H_∞ FIR filter [12], the robust nonlinear H_∞ FIR filter of Eqs. (31)-(32) can be rewritten as

$$\begin{aligned} \hat{x}_{(k+1)} &= (A_k + \triangle A_{worst}) \hat{x}_k + G_k g(\hat{x}_k) + B_k \hat{w}_k \\ &\quad + K_k [y_k - (C_k + \triangle C_{worst}) \hat{x}_k \\ &\quad - H_k h(\hat{x}_k) - D_k \hat{w}_k] \\ \hat{z}_k &= L_k \hat{x}_k, \end{aligned}$$

where

$$\begin{aligned} \triangle A_{worst} &= \epsilon^{-2} H_1 H_1^T (P^{-1} - \gamma^{-2} B_1 B_1^T)^{-1} A_k \\ \triangle C_{worst} &= \epsilon^{-2} H_1 H_2^T (P^{-1} - \gamma^{-2} B_1 B_1^T)^{-1} A_k \\ \hat{w}_k &= \gamma^{-2} B^T (P^{-1} - \gamma^{-2} B_1 B_1^T)^{-1} A_k \hat{x}_k. \end{aligned}$$

In the above, $\triangle A_{worst}$ and $\triangle C_{worst}$ can be regarded as the worst-case parameter uncertainty of $\triangle A_k$ and $\triangle C_k$, respectively, whereas $\hat{w}_k$ can be interpreted as the estimated worst-case noise signal. It should also be noted that the filter gain of Eq. (33) also depends on the known structural matrices of the parameter uncertainty, namely E, H_1 and H_2.

Also, it is easy to see that the coefficient matrices of the filter of Eqs. (31)-(32) can be rewritten in the following form :

$$\begin{aligned}
\hat{A} &= A_k + \gamma^{-2}\bar{B}(I - \gamma^{-2}\bar{B}^T W \bar{B})^{-1}\bar{B}^T W A_k \\
\hat{B} &= \begin{bmatrix} \bar{B}(I - \gamma^{-2}\bar{B}^T W \bar{B})^{-\frac{1}{2}} & \gamma G_k & 0 \end{bmatrix} \\
\hat{C} &= C_k + \gamma^{-2}\bar{D}(I - \gamma^{-2}\bar{B}^T W \bar{B})^{-1}\bar{B}^T W A_k \\
\hat{D} &= \begin{bmatrix} \bar{D}(I - \gamma^{-2}\bar{B}^T W \bar{B})^{-\frac{1}{2}} & 0 & \gamma H_k \end{bmatrix},
\end{aligned}$$

where $W = P + PG_k(I - G_k^T P G_k)^{-1} G_k^T P$.

Observe the similarity of the above coefficient matrices with those for the robust linear H_∞ FIR filter.

When the initial state of the system of Eq. (1) is known to be zero, or when a stationary filter design is concerned, the Riccati difference equation of Eq. (21) will be replaced by an algebraic Riccati equation, i.e. condition (b) of Theorem 4 is replaced by the following

(b') There exists a stabilizing solution $S = S^T \geq 0$ to the algebraic Riccati equation

$$\begin{aligned}
S = \hat{A}S\hat{A}^T - S - (\hat{A}S\bar{C}^T + \hat{B}\bar{D}^T)(\bar{R} + \bar{C}S\bar{C}^T)^{-1} \\
(\bar{C}S\hat{A}^T + \bar{D}\hat{B}^T) + \hat{B}\hat{B}^T
\end{aligned}$$

such that $I - \gamma^{-2}\hat{L}S\hat{L}^T > 0$.

Furthermore, the requirement for $P < \gamma^2 R$ in Eq. (19) will no longer be needed and the filter gain of Eq. (33) will be constant.

Also, we note that, when there are no nonlinear terms in the system of Eqs. (1)-(2), i.e. $G_k = 0$, $H_k = 0$, $g(\cdot) \equiv 0$ and $h(\cdot) \equiv 0$, and subject to zero initial state, the filter of Eqs. (31)-(32) will recover the robust linear H_∞ FIR filter [12].

5 CONCLUSIONS

In this paper the robust nonlinear H_∞ FIR filter has been proposed for nonlinear discrete time-varying systems with parameter uncertainty. Firstly, the discrete-time H_∞ FIR filter is obtained for the nonlinear system without the parametric uncertainty. Secondly, the discrete-time robust nonlinear H_∞ FIR filter for the uncertain nonlinear system is derived by modifying the system model. This result can also be viewed as an extension of Kwon *et al.*[11] which treat linear time-varying systems with parameter uncertainty.

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References

[1] H. Kwakernaak, "A polynomial approach to minimax frequency domain optimization of multivariable feedback systems", *Int. J. Control*, vol. 44, pp. 117-156, 1986.

[2] L. Xie, C. E. de Souza and Y. Wang, "Robust control of discrete-time uncertain dynamical system", *Automatica*, vol. 29, pp. 1133-1137, 1993.x

[3] C. E. de Souza, L. Xie and Y. Wang, "H_∞ filtering for a class of uncertain nonlinear systems", *Systems and Control Letters*, vol. 20, pp. 419-429, 1993.

[4] M. J. Grimble, "H_∞ design of optimal linear filters", *Linear Circuits, Systems and Signal processing: Theory and Application*, North-Holland, Amsterdam, pp. 533-540, 1988.

[5] D. J. Limbeer and U. Shaked, "New results in H_∞ filtering", *Proc. 9th Int. Symp. on MTNS*, Kobe, Japan, pp. 317-322, 1991.

[6] J. C. Willems, "Dissipative dynamical systems Part I General theory", *Arch. Rational Mech. Anal.*, vol. 45, pp. 321-351, 1972.

[7] D. J. Hill and P. J. Moylan, "Dissipative dynamical systems: basic input-output and state properties," *J. Franklin Inst.*, vol. 309, pp. 327-357, 1980.

[8] C. E. de Souza and L. Xie, "Robust H_∞ filtering", *Control and Dynamic Systems*, vol. 65, pp. 323-377, 1994.

[9] C. E. de Souza and L. Xie, "On the discrete-time bounded real lemma with application in the characterization of static state feedback H_∞ controllers", *System and Control Letters*, vol. 18, pp. 61-71, 1992.

[10] L. Xie, C. E. de Souza and M. Fu, "H_∞ estimation for discrete-time linear uncetain systems", *Int. J. Robust and Nonlinear Control*, vol. 18, pp. 111-123, 1991.

[11] O. K. Kwon, C. E. de Souza and H. S. Ryu, "Robust H_∞ FIR filter for discrete-time uncertain systems", *Proc. 35th CDC*, Kobe, Japan, 4819-4824, 1996.