A Unified Approach For Mean Square Stability of Continuous-Time Markovian Jumping Linear Systems With Additive Disturbances[1]

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Abstract

Necessary and sufficient conditions for mean square stability(MSS) of continuous-time linear systems subject to Markovian jumps in the parameters and additive disturbances are established. We consider two scenarios regarding the additive disturbances: the one in which the system is driven by a Wiener process, and the one characterized by functions in $L_2^m(\mathbb{R}^+)$, which is the usual scenario for the H_∞ approach. For both cases it is shown that MSS is equivalent to asymptotic wide sense stationarity (AWSS), to the spectrum of an augmented matrix lying in the open left half plane, and to the existence of a solution for a certain Lyapunov equation. Furthermore, it is proved that the Lyapunov equation can be written down in two equivalent forms with each one providing an easier-to-check sufficient condition. It is also shown that MSS is equivalent to the state $x(t)$ belonging to L_2^n whenever the disturbances are in $L_2^m(\mathbb{R}^+)$. These results provide, *inter alia*, a flexible theory, in a unified basis, for MSS of continuous-time Markovian jump linear systems.

1 Introduction

In recent years, there has been a steadly rising level of activity with linear systems which are subject to abrupt changes in their structures. Most of the literature considers the case in which the abrupt changes are modeled by a Markov chain, namely, linear systems with Markovian jump parameters(LSMJP). These changes arises quite often in practice and may be due, for instance, to component and/or interconnection failures, *inter alia*. This is to be found, for instance, in robotic manipulator systems, in aircraft control systems, large scale flexible structures for space stations(such as antenna, solar arrays, etc.), flexible manufacturing systems, on which an actuator or a sensor failure is a quite common occurence. Without any intention of being exhaustive here, we mention [4]-[9], [11], [12] and [14]-[17], as a small sample of works dealing with different aspects of control problems. We mention also [2],[10], [16][18](and references therein), as works dealing with applications of this class.

Stability questions for LSMJS(discrete and continuous-time) has given rise to a fairly amount of papers surrounding this subject(see, for instance,[1], [3], [5], [8], [7], [11], [13], [14], [15], [17] and the references therein). In ([13], [15], [17]), necessary and sufficient conditions for mean square stability (MSS) were obtained for the continuous-time noise free case. A common feature in these papers, is that the MSS criteria is expressed as the maximal real part of an augmented matrix being less than zero, and in fact this number was shown to be the mean square Lyapunov exponent of the system. To some extent, the discrete-time counterpart of this approach, is given in [5] (including a second order independent wide sense stationary noise and a criteria for almost sure stability). In [12] necessary and sufficient conditions for MSS of the discrete-time noise free case were obtained in terms of the existence of a solution of a Lyapunov equation. The continuous-time counterpart of this result is derived in [8], including a study on the relationship among various moment and sample path stability. Almost sure stability are examined, for the noise free case, in [1], [3], [5], [7] and [15]. A comprehensive historical account on earlier works can be found in [7] and [14].

This paper deals with mean square stability for continuous-time LSMJS. In order to more fully characterize system uncertainties , we consider in our study additive disturbances. We deal with two scenarios: the one in which the disturbances are characterized via a Wiener process, and the one characterized by any function in $L_2^m(\mathbb{R}^+)$. Using a completely different technique from previous works for continuous-time LSMJP, based on operator theory (see, e.g.,[5]), we unify and systematize the recent development in the field, besides tracing a parallel with the classical LQ-problem. It is

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shown that MSS is equivalent to the maximal real part of the eigenvalues of an augmented matrix being less than zero or to the existence of a solution of a Lyapunov equation. The first criterion(based on the eigenvalues) translates clearly the intuitive idea that unstable modes of operation do not necessarily compromise the global stability of the system. In fact it can be shown that stability of all modes of operation is neither necessary nor sufficient for global stability of the system (see also [11]). The eigenvalues criteria shows clearly the connection between MSS and the probability of visits to the unstable modes. A cursory examination of the augmented matrix reveals that a balance between the modes and the transition probability matrix is essential for MSS. For the case of one mode operation (no jumps in the parameters)our criteria reconcile to a well known stability results for continuous-time linear systems. It is also shown that the Lyapunov equation can be written in two equivalent forms and each of these forms provides an easier-to-check sufficient condition. In addition, it is proved that MSS is equivalent to asymptotic wide sense stationary stability(AWSS). Furthermore, it is shown that the state $x(t) \in L_2^n$ for $L_2^m(\mathbb{R}^+)$-disturbances.

An outline of the content of this paper is as follows. In Section 2 we provide the bare essential of notational conventions and some preliminaries. The model and problem statement are described in Section 3. Mean square stability for the homogeneous case(including Lyapunov equations)is treated in Section 4. Section 5 accounts for the case with additive disturbances. For all the cases, it is shown that MSS is equivalent to asymptotic wide sense stationarity (AWSS), to the spectrum of an augmented matrix lying in the open left half plane, and to the existence of a solution for a certain Lyapunov equation.

2 Notation and Preliminaries

For $\mathbb{X}$ and $\mathbb{Y}$ complex Banach spaces we set $\mathbb{B}(\mathbb{X},\mathbb{Y})$ for the Banach space of all bounded linear operator of $\mathbb{X}$ into $\mathbb{Y}$, with the uniform induced norm represented by $\|.\|$. For simplicity we shall set $\mathbb{B}(\mathbb{X}) := \mathbb{B}(\mathbb{X},\mathbb{X})$.If $\mathbb{X}$ is a Hilbert space then $< .;. >$ will stand for the inner product, and for $\mathcal{T} \in \mathbb{B}(\mathbb{X})$, $\mathcal{T}^\star$ will indicate the adjoint operator of $\mathcal{T}$. As usual, $\mathcal{T} \geq 0$ ($\mathcal{T} > 0$)will mean that the operator $\mathcal{T} \in \mathbb{B}(\mathbb{X})$ is positive semi-definite(positive definite), respectively. In particular, we shall denote by $\mathbb{C}^n$ the n-dimensional complex Euclidean spaces and by $\mathbb{B}(\mathbb{C}^n,\mathbb{C}^m)$ the normed bounded linear space of all $m \times n$ complex matrices, with $\mathbb{B}(\mathbb{C}^n) := \mathbb{B}(\mathbb{C}^n,\mathbb{C}^n)$ and $\mathbb{B}(\mathbb{C}^n)^+ := \{L \in \mathbb{B}(\mathbb{C}^n); L = L^* \geq 0\}$. In this case, the superscripts - , $\prime$, and * will denote complex conjugate, transpose and conjugate transpose, respectively. Either the uniform induced norm in $\mathbb{B}(\mathbb{C}^n)$ or the standard Euclidean norm in $\mathbb{C}^n$ is represented by $\|.\|$. We also use $\mathbb{R}^+$ to denote the interval $[0,\infty)$. For $D_i \in \mathbb{B}(\mathbb{C}^n)$, $i = 1,...,N$, $diag(D_i)$ is a Nn x Nn matrix where the matrices D_i are put together corner-to-corner diagonally, with all other entries being zero. We refer to I_ℓ as the ℓ x ℓ identity matrix and to $L_2^n(\mathbb{R}^+)$ as the space of functions $f : [0,\infty) \rightarrow \mathbb{C}^n$ such that each component $f^i(.)$ is in the standard L_2 $(\mathbb{R}^+)$ space of Lebesgue square integrable functions. Similarly, L_2^n is the space of square integrable stochastic process. In addition, we denote by $I\!\!R_e\{\lambda_i(\mathcal{T})\}$ the real part of the eigenvalue $\lambda_i(\mathcal{T})$ of the operator $\mathcal{T}$ and write generically $I\!\!R_e\{\lambda(\mathcal{T})\} < 0$ if all its eigenvalues have real part less than zero. Furthermore, the linear functional $tr(.) : \mathbb{B}(\mathbb{C}^n) \rightarrow \mathbb{C}$ will stand for the trace operator and we define by $L \otimes K \in \mathbb{B}(\mathbb{C}^{sn},\mathbb{C}^{rm})$, the Kronecker product for any $L \in \mathbb{B}(\mathbb{C}^s,\mathbb{C}^r)$ and $K \in \mathbb{B}(\mathbb{C}^n,\mathbb{C}^m)$, recalling that for $L \in \mathbb{B}(\mathbb{C}^n)$ and $K \in \mathbb{B}(\mathbb{C}^m)$ the Kronecker sum is defined as $L \oplus K := L \otimes I_m + I_n \otimes K \in \mathbb{B}(\mathbb{C}^{nm})$.

Set $\mathbb{H}^{n,m}$ the linear space made up of all N-sequences of complex matrices $V = (V_1,...,V_N)$ with $V_i \in \mathbb{B}(\mathbb{C}^n,\mathbb{C}^m)$, $i = 1,...,N$ and, for simplicity, set $\mathbb{H}^n := \mathbb{H}^{n,n}$. For $V = (V_1,...,V_N) \in \mathbb{H}^{n,m}$, we consider the following norms in $\mathbb{H}^{n,m}$

$$\|V\|_1 := \sum_{i=1}^{N} \|V_i\| \;\; and \;\; \|V\|_2 := \left(\sum_{i=1}^{N} tr\left(V_i^* V_i\right)\right)^{1/2} \tag{1}$$

It is easy to verify that $\mathbb{H}^{n,m}$ equipped with any of the above norms is a Banach space and, in fact, $(\mathbb{H}^{n,m}, \|.\|_2)$ is a Hilbert space, with inner product given, for $S = (S_1,...,S_N)$ and $V = (V_1,...,V_N)$ in $\mathbb{H}^{n,m}$, by

$$\langle V; S \rangle = \sum_{i=1}^{N} tr\left(V_i^* S_i\right). \tag{2}$$

For $V = (V_1,...,V_N) \in \mathbb{H}^{n,m}$ we shall write $V^* = (V_1^*,...,V_N^*) \in \mathbb{H}^{m,n}$ and say that $V \in \mathbb{H}^n$ is hermitian if $V = V^*$. We define $\mathbb{H}^{n*} := \{V = (V_1,...,V_N) \in \mathbb{H}^n; V_i = V_i^*, i = 1,...,N\}$. and $\mathbb{H}^{n+} := \{V = (V_1,...,V_N) \in \mathbb{H}^{n*}; V_i \geq 0, i = 1,...,N\}$ and shall write, for $V = (V_1,...,V_N) \in \mathbb{H}^n$ and $S = (S_1,...,S_N) \in \mathbb{H}^n$, that $V \geq S$ if $V - S = (V_1 - S_1,...,V_N - S_N) \in \mathbb{H}^{n+}$, and that $V > S$ if $V_i - S_i > 0$ for $i = 1,...,N$.

Define now the operators φ and $\hat{\varphi}$ in the following way: for $V = (V_1,...,V_N) \in \mathbb{H}^{n,m}$, with $V_i = (v_{i1}....v_{in}) \in \mathbb{B}(\mathbb{C}^n,\mathbb{C}^m)$, $v_{ij} \in \mathbb{C}^m$

$$\varphi(V_i) := \begin{bmatrix} v_{i1} \\ . \\ . \\ . \\ v_{in} \end{bmatrix} \quad and \quad \hat{\varphi}(V) := \begin{bmatrix} \varphi(V_1) \\ . \\ . \\ . \\ \varphi(V_N) \end{bmatrix},$$

with $\varphi(V_i) \in \mathbb{C}^{mn}$ and $\hat{\varphi}(V) \in \mathbb{C}^{Nmn}$. Furthermore, for $v_i := \varphi(V_i)$, and $\mathcal{V} := \hat{\varphi}(V)$,$i = 1, ..., N$, we define

$$\hat{\varphi}^{-1}\left(\begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_N \end{bmatrix}\right) := \left[\hat{\varphi}_1^{-1}(\mathcal{V}), ..., \hat{\varphi}_N^{-1}(\mathcal{V})\right]$$
$$= \left[\varphi^{-1}(v_1), ..., \varphi^{-1}(v_N)\right].$$

3 The Models and Problem Statement

Let be a complete probability space carrying its natural filtration $\{\mathcal{F}_t, t \in \mathbb{R}^+\}$, as usual augmented by all null sets in the $I\!P$–completion of $\mathcal{F}$, and the following statistically mutually independent objects:

(0.1) An m-dimensional Wiener process $W = \{(w(t), \ \mathcal{F}_t), t \in \mathbb{R}^+\}$ with incremental covariance operator Rdt.

(0.2) A homogeneous Markov process $\theta = \{(\theta_t, \ \mathcal{F}_t), t \in \mathbb{R}^+\}$, with right continuous trajectories and taking values on the finite set $\mathcal{S} := \{1, 2, ..., N\}$.We assume also that

$$P(\theta_{t+h} = j | \theta_t = i) = \begin{cases} \lambda_{ij}h + o(h), & i \neq j \\ 1 + \lambda_{ii}h + o(h), & i = j \end{cases} \quad (3)$$

where $[(\lambda_{ij})]$ is the stationary NxN transition rate matrix of $\{\theta\}$ with $\lambda_{ij} \geq 0$, $i \neq j$ and $\lambda_i = -\lambda_{ii} = \sum_{j:\ j \neq i} \lambda_{ij} < \infty$, i.e., the process is supposed to be conservative. The notation $o(h)$ denotes an infinitesimal of higher order than h, i.e., $\lim_{h \downarrow 0} \frac{o(h)}{h} = 0$ (a function $f(.)$ is said of order $o(h)$ if $\lim_{h \downarrow 0} \frac{f(h)}{h} = 0$). We define $p_{ij}(t) := I\!P(\theta_{t+s} = j | \theta_s = i)$, $i, j = 1, ..., N$ and denote $p_i(t) := I\!P(\theta_t = i)$, for any $i \in \mathcal{S}$. Notice that, in this setting, $P_t := (p_1(t), ..., p_N(t))'$, satisfies the Kolmogorov forward differential equation $dP_t/dt = \Lambda P_t$; $P_0 = \mathrm{P}$, $\mathrm{t} \in \mathbb{R}^+$, where $\Lambda := [(\lambda_{ij})]'$. In addition, we assume that $\{(\theta_t, \ \mathcal{F}_t), t \in \mathbb{R}^+\}$ has initial distribution $\{v(i); i = 1, ..., N\}$.

(0.3) A random variable x: $\Omega \rightarrow \mathbb{R}^n$ with $E[\|x\|^2] < \infty$, where E[.] denotes mathematical expectation.

We deal with three types of linear systems with Markovian jump parameters. First, in order to bring to bear some basic results in its more general form and put MSS in a unified basis, we consider the homogeneous system:

$$\dot{x}(t) = A(\theta_t)x(t), \quad x(0) = x, \ \ \theta_0 = v, \ \ t \in \mathbb{R}^+. \quad (4)$$

We consider next the class of dynamical systems modelled by the following stochastic differential equation:

$$\dot{x}(t) = A(\theta_t)x(t) + B(\theta_t)w(t), \ \ x_0 = x, \ \ \theta_0 = v, \ \ t \in \mathbb{R}^+, \quad (5)$$

where the additive disturbance, $w(.)$, is modelled by:

(0.1b) $\{w(t); t \in \mathbb{R}^+\}$ is any $L_2^m(\mathbb{R}^+)-function$, which is the usual scenario for the H_∞ approach. In addition, we shall consider also the class of dynamical systems modelled by the following Itô's stochastic differential equation:

$$dx(t) = A(\theta_t)x(t)dt + B(\theta_t)dw(t), \quad x(0) = x, \ \ \theta_0 = v, \ (6)$$

$t \in \mathbb{R}^+$, where we require, in addition to (0.1) − (0.3), that:

(0.4) $\theta = \{(\theta_t, \ \mathcal{F}_t), t \in \mathbb{R}^+\}$ is a recurrent Markov process with initial distribution $\{v(i); i = 1, ..., N\}$.We recall that, in this setting, it is a standard result of Markov chain theory that there exist limiting probabilities $\{\pi_i;\ i = 1, ..., N\}$, which do not depend on the initial distribution, with $\{\sum_{i=1}^N \pi_i = 1\}$, and satisfy the inequalities:

$$\max_i |p_{ij}(t) - \pi_j| \leq Ce^{-Dt}, \quad |p_j(t) - \pi_j| \leq Ce^{-Dt}, \quad (7)$$

for suitable positive constants C and D.

Furthermore, $A(.)$ and $B(.)$ are such that $A(\theta_t) = A_j$ and $B(\theta_t) = B_j$ for $\theta_t = j$, $j \in \mathcal{S}$, with A_j ; $B_j, j = 1, ..., N$ being constant matrices in $\mathbb{B}(\mathbb{C}^n)$ and $\mathbb{B}(\mathbb{C}^m, \mathbb{C}^n)$, respectively .

In addition, define for $t \in \mathbb{R}^+$:

$$q(t) := E(x(t)) \in \mathbb{C}^n, \quad (8)$$
$$\mathcal{Q}(\tau, t) := E(x(t+\tau)x(t)^*) \in \mathbb{B}(\mathbb{C}^n), \quad (9)$$
$$Q(t) := \mathcal{Q}(0, t) \in \mathbb{B}(\mathbb{C}^n)^+ \quad (10)$$
$$q_i(t) := E\big(x(t)1_{\{\theta_t = i\}}\big) \in \mathbb{C}^n \quad (11)$$
$$Q_i(t) := E\big(x(t)x(t)^*1_{\{\theta_t = i\}}\big) \in \mathbb{B}(\mathbb{C}^n)^+ \quad (12)$$
$$\mathcal{Q}_i(s, t) := E\big(x(t+s)x(t)^*1_{\{\theta_{t+s} = i\}}\big) \in \mathbb{B}(\mathbb{C}^n) (13)$$

where $1_{\{.\}}$ stands for the Dirac measure. Set also

$$\hat{q}(t) := \begin{bmatrix} q_1(t) \\ \cdot \\ \cdot \\ \cdot \\ q_N(t) \end{bmatrix} \in \mathbb{C}^{Nn}, \quad \begin{array}{l} \hat{Q}(t) = (Q_1(t), ..., Q_N(t)) \\ \hat{\mathcal{Q}}(s, t) = (\mathcal{Q}_1(s, t), ..., \mathcal{Q}_N(s, t)). \end{array}$$

We define:

Definition 1 *A linear system with Markovian jump parameter is mean square stable (MSS), if for any initial condition* $x(0)$ *and initial distribution* v*, there exist* $q \in \mathbb{C}^n$ *and* $Q \in \mathbb{B}(\mathbb{C}^n)^+$ *independent of* x_0*, such that:*

(a)$\|q(t) - q\| \rightarrow 0$ *and (b)*$\|Q(t) - Q\| \rightarrow 0$*, as* $t \rightarrow \infty$

Definition 2 *A linear system with Markovian jump parameter is asymptotically wide sense stationary (AWSS) if for any initial condition* x_o *and initial distribution* v*, there exist* $q \in \mathbb{C}^n$ *and* $\mathcal{Q}(\tau) \in \mathbb{B}(\mathbb{C}^n)$ *independent of* x_0*, such that:*

(a)$\|q(t) - q\| \mapsto 0$ *and (b)*$\|\mathcal{Q}(\tau, t) - \mathcal{Q}(\tau)\| \mapsto 0$*, as* $t \mapsto \infty$.

In this paper we shall be essentially interested in studying MSS conditions for the above systems, in a unified basis, including showing that MSS is equivalent to the state $x(t)$ belonging to L_2^n whenever the disturbances are in $L_2^m(\mathbb{R}^+)$.

4 The Homogeneous Case

In this section necessary and sufficient conditions for MSS of the homogeneous case are established. It is required that either the real part of all the eigenvalues of an augmented matrix being less than zero or that there exist a unique solution of a Lyapunov equation. Furthermore, it is proved that the Lyapunov equation can be written down in two equivalent forms with each one providing an easier-to-check sufficient condition.

We consider first the homogeneous equation (4), restated here for convenience,

$$dx(t) = A(\theta_t)x(t)dt, \quad x(0) = x, \quad \theta_0 = v, \quad t \in \mathbb{R}^+.$$

In what follows we shall be using the notations:

$$\begin{array}{ll} F := \Lambda' \otimes I_n + diag(A_i); & V := \Lambda' \otimes I_{n^2}; \\ G := diag(I_n \oplus A_i); & H := diag(_i \oplus A_i); \\ \mathcal{A} := V + H; & \mathcal{B} := V + G. \end{array} \tag{14}$$

We define also the following operators in $\mathbb{B}(\mathbb{H}^n)$:

$$\begin{array}{l} \mathcal{E}(.) = (\mathcal{E}_1(.), ..., \mathcal{E}_N(.)) \ and \ \mathcal{F}(.) = (\mathcal{F}_1(.), ..., \mathcal{F}_N(.)) \\ \mathcal{L}(.) = (\mathcal{L}_1(.), ..., \mathcal{L}_N(.)) \ and \ \mathcal{T}(.) = (\mathcal{T}_1(.), ..., \mathcal{T}_N(.)), \end{array} \tag{15}$$

where, for $P = (P_1, ..., P_N) \in \mathbb{H}^n$ and $i = 1, ..., N$,

$$\begin{array}{l} \mathcal{E}_i(P_1, ..., P_N) := \sum_{j=1}^N P_j \in \mathbb{B}(\mathbb{C}^n); \\ \mathcal{F}_i(P_1, ..., P_N) := A_i P_i + \sum_{j=1}^N P_j \in \mathbb{B}(\mathbb{C}^n); \\ \mathcal{L}_i(P_1, ..., P_N) := A_i P_i + P_i A_i^* + \sum_{j=1}^N P_j \in \mathbb{B}(\mathbb{C}^n); \\ \mathcal{T}_i(P_1, ..., P_N) := A_i^* P_i + P_i A_i + \sum_{j=1}^N P_j \in \mathbb{B}(\mathbb{C}^n). \end{array} \tag{16}$$

The next proposition provides differential equations to compute the first and second moments of the state variable of (4).

Proposition 1 *For $t \in \mathbb{R}^+$ and $j = 1, ..., N$, we have for (4):*

$$\begin{array}{ll} (a) & \dot{q}_i(t) = A_i q_i(t) + \mathcal{E}_i(q_1(t), ..., q_N(t)) \\ (b) & \dot{Q}_i(t) = \mathcal{L}_i(Q_1(t), ..., Q_N(t)) \end{array} \tag{17}$$

Proof: See [9].

Corollary 1

$$(a) \quad \dot{\hat{q}}(t) = F\hat{q}(t); \quad and \quad (b) \quad \dot{\hat{Q}}(t) = \mathcal{L}(\hat{Q}(t)). \tag{18}$$

The following result is germane to our MSS approach.

Proposition 2 *For any $Q \in \mathbb{H}^{n+}$ and $\mathcal{Q} \in \mathbb{H}^n$, we have:*

$$\begin{array}{ll} (a) & \hat{\varphi}(\mathcal{L}(Q)) = \mathcal{A}\hat{\varphi}(Q); \quad (b)\hat{\varphi}(\mathcal{T}(Q)) = \mathcal{A}^*\hat{\varphi}(Q); \\ (c) & \hat{\varphi}(\mathcal{F}(\mathcal{Q})) = \mathcal{B}\hat{\varphi}(\mathcal{Q}) \end{array} \tag{19}$$

Proof: See [9].

Proposition 3 *If $I\!\!R_e\{\lambda(\mathcal{A})\} < 0$ then $I\!\!R_e\{\lambda(F)\} < 0$.*

Proof: See [9].

Our main MSS results will follow from the next propositions:

Proposition 4 *If $I\!\!R_e\{\lambda(\mathcal{A})\} < 0$, then for every $S = (S_1, ..., S_N)$, $S_i \in \mathbb{B}(\mathbb{C}^n)$, $i = 1, ..., N$, there exist a unique $G = (G_1, ..., G_N)$, $G_i \in \mathbb{B}(\mathbb{C}^n)$, $i = 1, ..., N$, such that*

$$\mathcal{L}_i(G) + S_i = 0 \qquad , i = 1, ..., N. \tag{20}$$

Moreover, for $i = 1, ..., N$

a) $G_i = -\hat{\varphi}_i^{-1}\big(\mathcal{A}^{-1}\hat{\varphi}(S_1, ..., S_N)\big)$; b) $\hat{\varphi}(G) = e^{\mathcal{A}t}\hat{\varphi}(S)dt$; c) $S_i = S_i^*$ iff $G_i = G_i^*$;

d) $S_i \geq 0$ implies $G_i \geq 0$; e) $S_i > 0$ implies $G_i > 0$.

Proof: See [9].

Proposition 5 *If there are $G_i > 0 \in \mathbb{B}(\mathbb{C}^n)$, $i = 1, ..., N$ such that*

$$\mathcal{T}_i(G) + S_i = 0 \qquad , i = 1, ..., N$$

for some $S_i > 0 \in \mathbb{B}(\mathbb{C}^n)$, then $I\!\!R_e((\mathcal{A}^*)) < 0$.

Proof: See [9].

Proposition 6 *If $I\!\!R_e\{\lambda(\mathcal{A})\} < 0$ then system (4) is MSS according to Definition 1. Furthermore, $q = 0$ and $Q = 0$.*

Proof: See [9].

Proposition 7 *If system (4) is MSS according to Definition 1(b) then $I\!\!R_e\{\lambda(\mathcal{A})\} < 0$.*

Proof: See [9].

The next result shows a MSS result in the spirit of the classical linear case.It shows that MSS stability is equivalent to the spectrum of an augmented matrix, $\mathcal{A}$, lying in the open left half plane.

Theorem 1 *System(4) is MSS according to Definition 1 if and only if $I\!\!R_e\{\lambda(\mathcal{A})\} < 0$.*

Proof: Follows from Propositions (6) and (7).

We show now equivalent forms of Lyapunov equation and Lyapunov inequality.

Theorem 2 *The following assertions are equivalent to MSS of system (4):*

(a) For some $G_j > 0 \in \mathbb{B}(\mathbb{C}^n)$, $j = 1, ..., N$, we have

$$\mathcal{L}_i(G) < 0, \quad i = 1, ..., N.$$

(b) For any $S_i > 0 \in \mathbb{B}(\mathbb{C}^n)$, $i = 1, ..., N$, there are unique $G = (G_1, ..., G_1)$, $G_i > 0 \in \mathbb{B}(\mathbb{C}^n)$, $i = 1, ..., N$, such that

$$\mathcal{L}_i(G) + S_i = 0, \quad i = 1, ..., N.$$

Moreover

$$G_i = \hat{\varphi}_i^{-1}\big(\mathcal{A}^{-1}\hat{\varphi}(S_1, ..., S_N)\big), \quad i = 1, ..., N.$$

Proof: See [9].

We conclude this subsection by showing that from Theorem 2 we can derive some easier-to-check conditions for MSS of (4).

Corollary 2 *Conditions (i) and (ii) below are equivalent:*

(i) $\exists \quad \alpha_j > 0, \quad j = 1,...,N,$ *such that* $\alpha_i \lambda_{max}(A_i + A_i^*) + \sum_{j=1}^N \lambda_{ij}\alpha_j < 0.$

(ii) $\exists \quad \alpha_j > 0, \quad j = 1,...,N,$ *such that* $\alpha_i \lambda_{max}(A_i + A_i^*) + \sum_{j=1}^N \lambda_{ji}\alpha_j < 0,$

where $\lambda_{max}(\mathcal{T})$*:=max{* λ*:* λ *is an eigenvalue of the operator* $\mathcal{T}$*}. Moreover, if the above conditions(one of them)are satisfied then system (4) is MSS.*

Proof: See [9].

Corollary 3 *If for some real number* $\delta_i > 0$*,* $i = 1,...,N$ *one of the following conditions below is satisfied*

(1) $\lambda_{max}\Big[(A_i + A_i^* + \frac{1}{\delta_i}\sum_{j:\ j\neq i} \lambda_{ij}\delta_j)I_n\Big] < -\lambda_{ii}$

(2) $\lambda_{max}\Big[(A_i + A_i^* + \frac{1}{\delta_i}\sum_{j:\ j\neq i} \lambda_{ji}\delta_j)I_n\Big] < -\lambda_{ii},$

then system (4) is MSS. Moreover these conditions are weaker than those in Corollary (2).

Proof: See [9].

5 The L_2^m $(\mathbb{R}^+)$ and Jump Diffusion Case

We consider in this section two scenarios regarding the additive disturbance: the one as in equation (5), characterized by functions in L_2^m $(\mathbb{R}^+)$, and that in equation (6), a jump diffusion, where the noise is characterized via a Wiener process. For both cases, under suitable conditions, it is shown that MSS is equivalent to asymptotically wide sense stationarity(AWSS). In addition, it is shown that the state $x(t) \in L_2^n$ for $L_2^m(\mathbb{R}^+)$-disturbances.

5.1 The L_2^m $(\mathbb{R}^+)$ Disturbance Case

We consider in this subsection the class of dynamical systems modelled by the following stochastic equation:

$$\dot{x}(t) = A(\theta_t)x(t) + B(\theta_t)w(t), \quad x(0) = x, \ \ \theta_0 = v, \ \ t \in \mathbb{R}^+,$$

where the additive disturbance $\{w(t); t \in \mathbb{R}^+\}$ is any $L_2^m(\mathbb{R}^+)$-function. Let us begin defining the following auxiliary objects:

$$\hat{u}(t) := \begin{bmatrix} u_1(t) \\ . \\ . \\ . \\ u_N(t) \end{bmatrix}; \quad \begin{array}{l} \Omega(t) = (\Omega_1(t), ..., \Omega_N(t)); \\ \Delta(s,t) = (\Delta_1(s,t), ..., \Delta_N(s,t)); \\ u_i(t) := B_i w(t) p_i(t) \in \mathbb{C}^n; \end{array}$$

and

$$\begin{array}{l} \Omega_i(t) := B_i w(t) q_i^*(t) + q_i(t) w^*(t) B_i^* \\ \Delta_i(s,t) := B_i w(t+s) \sum_{j=1}^N p_{ji}(s) q_j^*(t) \\ p_{ji}(s) := \mathbb{P}(\theta_s = i | \theta_0 = j). \end{array}$$

Proposition 8 *For* $t \in \mathbb{R}^+$ *and j=1,...,N, we have:*

$$\begin{array}{ll} (a) & \dot{q}_i(t) = A_i q_i(t) + \mathcal{E}_i(q_1(t), ..., q_N(t)) + u_i(t); \\ (b) & \dot{Q}_i(t) = \mathcal{L}_i(Q_1(t), ..., Q_N(t)) + \Omega_i(t) \\ (c) & \dot{\mathcal{Q}}_i(s,t) = A_i \mathcal{Q}_i(s,t) + \mathcal{E}_i(\mathcal{Q}_1(s,t), ..., \mathcal{Q}_N(s,t)) \\ & \qquad + \Delta_i(s,t) \end{array} \tag{21}$$

Proof: See [9].

Corollary 4 *We have also the following:*

$$\begin{array}{l} (a)\ \dot{\hat{q}}(t) = F\hat{q}(t) + \hat{u}(t); \quad (b)\ \dot{\hat{Q}}(t) = \mathcal{L}(\hat{Q}(t)) + \Omega(t); \\ (c)\ \dot{\hat{\mathcal{Q}}}(s,t) = \mathcal{F}(\hat{\mathcal{Q}}(s,t)) + \Delta(s,t); \\ (d)\ \hat{\varphi}(\Omega(t)) = diag(\mathcal{X}_i(t))\hat{\varphi}(\mathcal{W}(t)); \\ (e)\ \hat{\varphi}(\Delta(s,t)) = diag(\mathcal{Y}_i(s,t))\hat{\varphi}(\mathcal{W}(t)) \end{array}$$

where $\mathcal{X}_i(t) := q_i(t) \otimes B_i + B_i \otimes q_i(t)$*;* $\mathcal{Y}_i(s,t) := \Big((\sum_{j=1}^N p_{ji}(s) q_j^*(t)) \otimes B_i\Big)$ *and* $\mathcal{W}(t) := (w(t), ..., w(t)) \in \mathbb{B}\big(\mathbb{C}^N, \mathbb{C}^m\big)$*.*

Proposition 9 *If* $\mathbb{R}_e\{\lambda(\mathcal{A})\} < 0$ *then system (5) is MSS according to Definition 1, with* $q = 0$ *and* $Q = 0$*.*

Proof: See [9].

Proposition 10 *If system (5) is MSS according to Definition 1(b) then* $\mathbb{R}_e\{\lambda(\mathcal{A})\} < 0$*.*

Proof: See [9].

Proposition 11 *System (5) is MSS according to Definition 1 if and only if it is AWSS according to Definition 2. Furthermore,* $\mathcal{Q}(s) = 0$*.*

Proof: See [9].

Theorem 3 *System (5) is MSS according to Definition 1 if and only if* $\mathbb{R}_e\{\lambda(\mathcal{A})\} < 0$*.*

Proof: Follows from Proposition 9 and 10.

We conclude this section with a stronger result which guarantees that if $w(t) \in L_2^m(\mathbb{R}^+)$, then $x(t) \in L_2^n$.

Theorem 4 $\mathbb{R}_e\{\lambda(\mathcal{A})\} < 0$ *if and only if* $x(t) \in L_2^n$ *for every* $w(t) \in L_2^m(\mathbb{R}^+)$*.*

Proof: See [9].

5.2 The Jump Diffusion Case

In this subsection we deal with MSS issues for the class of systems described in Section 3 by eq. (6), i.e.,

$$dx(t) = A(\theta_t)x(t)dt + B(\theta_t)dw(t), \quad x_0 = x, \ \ \theta_0 = v,$$

$t \in \mathbb{R}^+$, under Assumptions (0.1)-(0.4).

The next proposition provides differential equations to compute the first and second moments of the state variable.

Proposition 12 *For* $t \in \mathbb{R}^+$ *and j=1,...,N, we have:*

$$\begin{array}{ll} (a) & \dot{q}_i(t) = A_i q_i(t) + \mathcal{E}_i(q_1(t), ..., q_N(t)) \\ (b) & \dot{Q}_i(t) = \mathcal{L}_i(Q_1(t), ..., Q_N(t)) + B_i R B_i^* \mathbb{P}(\theta_t = i) \\ (c) & \dot{\mathcal{Q}}_i(s,t) = A_i \mathcal{Q}_i(s,t) + \mathcal{E}_i(\mathcal{Q}_1(s,t), ..., \mathcal{Q}_N(s,t)) \end{array}$$

Proof: See [9].

Let now $\hat{R}(t) := (R_1(t), ..., R_N(t)) \in \mathbb{H}^n$, with $R_i(t) := B_i R B_i^* p_i(t) \in \mathbb{B}(\mathbb{C}^n)$. Then:

Corollary 5 *equation (a)* $\dot{\hat{q}}(t) = F\hat{q}(t)$; *(b)* $\dot{\hat{Q}}(t) = \mathcal{L}(\hat{Q}(t)) + \hat{R}(t)$; *(c)* $\dot{\hat{\mathcal{Q}}}(s,t) = \mathcal{F}(\hat{\mathcal{Q}}(s,t))(21)$

Proposition 13 *If* $\mathbb{R}_e\{\lambda(\mathcal{A})\} < 0$ *then system (6) is MSS according to Definition 1, with* $q = 0$ *and* $Q = \sum_{i=1}^N \hat{\varphi}_i^{-1}\big(\mathcal{A}^{-1}\hat{\varphi}(\mathcal{R}_1, ..., \mathcal{R}_N)\big)$*, where* $\mathcal{R}_i{:=}\bar{R}_i\pi_i$ *and* $\bar{R}_i{:=}B_iRB_i^*$*; i=1,...N.*

Proof: See [9].

Proposition 14 *If system (6) is MSS according to Definition 1(b) then* $\mathbb{R}_e\{\lambda(\mathcal{A})\} < 0$.

Proof: See [9].

Proposition 15 *System (6) is MSS according to Definition 1 if and only if it is AWSS according to Definition 2. Furthermore,* $\mathcal{Q}(s) = \sum_{i=1}^N \hat{\varphi}_i^{-1}\big(e^{\mathcal{B}s}\mathcal{A}^{-1}\hat{\varphi}(\mathcal{R}_1, ..., \mathcal{R}_N)\big)$.

Proof: See [9].

Theorem 5 *System (6) is MSS according to Definition 1 if and only if* $\mathbb{R}_e\{\lambda(\mathcal{A})\} < 0$.

Proof: Follows from Propositions 13 and 14.

Finally, we conclude with the following unifying result.

Theorem 6 *The following assertions are equivalents:*

(a) $\mathbb{R}_e\{\lambda(\mathcal{A})\} < 0$. *(b) System (4) is MSS according to Definition 1.*
(c) System (4) is MSS according to Definition 1(b).
(d) System (6) is MSS according to Definition 1.
(e) System (6) is MSS according to Definition 1(b).
(f) System (6) is AWSS according to Definition 2.
(g) System (5) is MSS according to Definition 1.
(h) System (5) is MSS according to Definition 1(b).
(i) System (5) is AWSS according to Definition 2.

Proof: It follows, essentially, from Propositions 7, 10, 11, 14, 15 and Theorems 1, 3, 5.

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