On the Finite Length DFT of Input-Output Signals of Multivariable Linear Systems[1]

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Abstract

The exact relation between the DFT of the input and the DFT of the output of a finite dimensional MIMO linear system is established for the case of an arbitrary excitation signal and finite number of signal samples. The derived expression also includes the colored noise case.

1 Introduction

Frequency domain identification encompass fitting linear dynamical models to frequency domain data, i.e. samples of the Fourier transform of the input and output data [6]. These samples are most often derived by employing the Discrete Fourier Transform (DFT) to a record of time domain known input signals and measured output signals. If a periodic input signal is applied to a linear system and the length of the measurement interval, N, is an integer multiple of the period time of the input signal, it is well known [6] that the DFT of the noise free stationary periodic output and the DFT of the periodic input are (in the discrete time case) exactly related as

$$Y_N(\omega_k) = G(e^{i\omega_k})U_N(\omega_k). \tag{1}$$

Here $G(e^{i\omega})$ is the frequency response function of the system and $Y_N(\omega_k)$ and $U_N(\omega_k)$ is the N point DFT of the output and input signals respectively and $\omega_k = \frac{2\pi k}{N}$, $k = 0,1,\ldots,N-1$. If the condition of periodic excitation and stationarity is violated (1) does not hold any more.

This paper presents the exact frequency domain expression, relating the system transfer function and the DFT of a finite record of input and output signals of a multivariable linear system when the input signal is arbitrary. Here the case of discrete time systems is covered and we refer to [3, 4] for the case of continuous time systems. The derived expressions can be used to improve the quality of the estimates when identifying systems in the frequency domain. It is shown that a finitely parametrized transfer function exactly captures the frequency domain effects of a non-periodic non-stationary excitation situation. Using a stochastic setting a parametric frequency domain model is derived in parallel which captures the influence of colored noise and can be used for a finite data pseudo maximum-likelihood frequency domain identification criterion, see [4] for details.

This work extends the results presented in [5] where the scalar case was considered.

2 Preliminaries

The N-point discrete Fourier transform (DFT) of the signal $\{s(t)\}_{t=0}^{N-1}$ is defined as

$$S_N(\omega) \triangleq \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} s(t)e^{-i\omega t} \tag{2}$$

where $\omega \in [-\pi,\pi]$ is the normalized angular frequency in radians per second.

Consider a white noise signal $e(t)$ which is independent identically distributed (i.i.d.) with zero mean and variance $\Lambda < \infty$. Denote by $E_N(\omega_k)$ the DFT of the noise signal $\{e(t)\}_{t=0}^{N-1}$. Hence the mean is $\mathbf{E}\{E_N(\omega_k)\} = 0$ and the second moment is given by

$$\mathbf{E}\{E_N(\omega_k)E_N^*(\omega_s)\} = \begin{cases} \Lambda, & \omega_k = \omega_s \\ 0, & \omega_k \neq \omega_s \end{cases} \tag{3}$$

where $\mathbf{E}\{\cdot\}$ denotes expectation, and X^* denotes the conjugate transpose of X. If $e(t)$ is drawn from a normal distribution then the real and complex parts of $E_N(\omega_k)$ also will have a normal distribution. In this case $E_N(\omega_k)$ will be *complex normal* for $k = 1,2,\ldots,N-1$ and $\omega_k \neq \pi$, see [1]. Furthermore the DFT for different frequencies are statistically independent. If $e(t)$ is i.i.d. with finite variance $\Lambda < \infty$ then by the Central Limit Theorem [2] $E_N(\omega_k)$ for $\omega_k \notin \{0,\pi\}$ will converge in distribution to a complex normal random variable as $N \to \infty$.

3 Discrete time systems

A discrete time system of finite order admits a state-space realization

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + Ke(t), \quad x(0) = x_0 \\ y(t) &= Cx(t) + Du(t) + e(t) \end{aligned} \tag{4}$$

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where $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^p$ is the output, $e(t) \in \mathbb{R}^p$ is i.i.d. zero mean noise with covariance Λ and A, B, C, D, K are real matrices of appropriate dimensions. We assume x_0 to be non-stochastic and hence independent of $e(t)$. The system and noise transfer functions are given by

$$G(z) = D + C(zI - A)^{-1}B, \quad H(z) = I + C(zI - A)^{-1}K \tag{5}$$

Consider the system described by (4) and assume N points of the input and output signals are available. The history of the input up to time $t < 0$ is unknown but its impact on the future is captured by the state at time zero, $x(0) = x_0$. Assume $\det(e^{i\omega_k}I - A)$ is non-zero for all $k = 0, 1, \ldots, N-1$.

Let $Y_N(\omega)$, $U_N(\omega)$ and $E_N(\omega)$ denote the N-point DFT of the output, input and noise signals respectively. Then for $\omega_k = \frac{2\pi k}{N}$, $k = 0, 1 \ldots, N-1$ the following equation holds

$$Y_N(\omega_k) = G(e^{i\omega_k})U_N(\omega_k) + H(e^{i\omega_k})E_N(\omega_k) + T(e^{i\omega_k})\frac{1}{\sqrt{N}} \tag{6}$$

where

$$\begin{aligned} T(z) &= zC(zI - A)^{-1}(I - A^N)(x_0 - x_p) \\ x_p &= (I - A^N)^{-1} \sum_{t=0}^{N-1} A^t \begin{bmatrix} B & K \end{bmatrix} \begin{bmatrix} u(N-1-t) \\ e(N-1-t) \end{bmatrix} \end{aligned} \tag{7}$$

A full proof can be found in [4]. The explicit form of $T(z)$ enables the possibility of estimating it along with the transfer function $G(z)$. This is beneficial when the data record is short and when a non-periodic excitation signal is used. Furthermore note that $G(z)$ and $T(z)$ share the same left matrix factor $C(zI - A)^{-1}$ and only the difference $x_0 - x_p$ needs to be modeled which requires n extra parameters besides the ones required to model $G(z)$ (and $H(z)$). The effect of $T(z)$ can be described as an extra input with a constant DFT for all frequencies.

Note that the relation (6) holds also for unstable systems. In such a case the size of the transient term will grow exponentially as N increases and it is essential to include it in any open loop estimation scheme.

3.1 Stochastic properties

Let us now return to discuss the impact of the noise. In (6) the terms $H(z)$ and $T(z)$ depend on the noise signal $e(t)$. Let us introduce

$$T^d(z) = zC(zI - A)^{-1}(I - A^N)(x_0 - x_p^d) \tag{8}$$

$$x_p^d = (I - A^N)^{-1} \sum_{t=0}^{N-1} A^t Bu(N-1-t). \tag{9}$$

$$T^s(z) = -zC(zI - A)^{-1} \sum_{t=0}^{N-1} A^t Ke(N-1-t) \tag{10}$$

where T^d and T^s denote the deterministic and stochastic part of T respectively. Hence $T(z) = T^d(z) + T^s(z)$. Since $T^s(z)$ is linear in $e(t)$ we have

$$\mathbf{E}\{T(z)\} = T^d(z)$$

Recall that $E_N(\omega)$ has a zero mean value and hence for all ω_k we obtain

$$\mathbf{E}\{Y_N(\omega_k)\} = G(e^{i\omega_k})U(\omega_k) + T^d(e^{i\omega_k})\frac{1}{\sqrt{N}} \tag{11}$$

Furthermore using (3), (6) and (8)-(11) we have

$$\begin{aligned} &\mathbf{E}\left\{(Y_N(\omega_k) - \mathbf{E}\{Y_N(\omega_k)\})(Y_N(\omega_s) - \mathbf{E}\{Y_N(\omega_s)\})^*\right\} \\ &= \begin{cases} H(e^{i\omega_k})\Lambda H^*(e^{i\omega_k}) + \frac{\xi(\omega_k,\omega_k)}{N}, & \omega_k = \omega_s \\ \frac{\xi(\omega_k,\omega_s)}{N}, & \omega_k \neq \omega_s \end{cases} \end{aligned} \tag{12}$$

If all eigenvalues of A has a modulus less than one there exists a constant $c < \infty$ such that for all ω_k, ω_s $\|\xi(\omega_k, \omega_s)\| \leq c$ holds. Asymptotically, as the number of data points tends to infinity, the terms $T^d(e^{i\omega_k})\frac{1}{\sqrt{N}}$ and $\xi(\omega_k, \omega_s)/N$ tend to zero. Then $Y_N(\omega_k)$ becomes complex normally distributed with mean $G(e^{i\omega_k})U(\omega_k)$ and variance $H(e^{i\omega_k})\Lambda H^*(e^{i\omega_k})$ and independent between frequencies. For a non-parametric treatment and relaxed assumptions we refer to [1].

4 Conclusion

The relation between the DFT of input-output signals of multivariable discrete time linear systems has been derived for arbitrary input signals and finite measurement interval. In parallel the influence of colored noise on the DFT is covered.

References

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