Structured Uncertainty Analysis of Robust Stability for Spatially Distributed Systems

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Abstract

This paper considers one of the fundamental issues in design and analysis of sampled multidimensional systems - that of uncertainty modeling and robust stability analysis. This paper extends methods of structured uncertainty analysis (μ-analysis) towards spatially distributed system with dynamical and spatial coordinates. The main contribution with respect to earlier work in this area is in clarification of stability issues for multidimensional systems with noncausal coordinates. Here stability is understood in a broad sense and includes decay (localization) of system response along noncausal spatial coordinates. The presented framework allows to address such practically important issues as robustness of dynamical stability and spatial localization of multidimensional closed-loop feedback system response and boundary effects in a unified way.

1 Introduction

Linear sampled multidimensional systems have been studied in many applications. Presently, control applications for systems incorporating large actuator and sensor arrays are becoming increasingly important. This work was primarily motivated by the control system applications, but the mathematical analysis presented herein should be applicable in image processing and numerical methods for partial differential equations (PDE). This paper pursues one of the fundamental issues in design and analysis of sampled multidimensional systems - that of robust stability analysis. There is a need to establish fundamental robust stability analysis concepts for practical analysis of multidimensional systems similar to the concepts established for practical multivariable control design [4].

Array signal processing has well-established theory and applications. At the same time, applied approaches to control of large distributed actuator and sensor arrays are much less developed. In an array control system, the system state, measurement, and control change in time and depend on spatial coordinates. This paper focuses on Linear Spatially Invariant (LSI) systems. Modal decomposition of such system can be obtained through spatial Fourier transform; see [1] for discussion. Spatial frequency analysis of array control systems is in many respects similar to usual frequency domain analysis of Linear Time Invariant (LTI) dynamical systems. The multidimensional frequency domain analysis is used in image processing [3] and stability analysis of finite-difference PDE solvers [11].

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In the frequency domain analysis of sampled multidimensional systems, multiple complex Laplace variables are introduced to represent the shift operators in time and along spatial coordinates, e.g., see [3]. Thus, study of the dynamical properties for such systems is replaced by the complex analysis of the resulting multivariable functions. A very efficient mathematical approach for dealing with a practically important class of rational multivariable complex functions is based on Linear Fractional Transformation (LFT) representation [14]. The LFT approach is very attractive for this work because it allows for convenient incorporating of structured uncertainty models into the analysis [14].

Multidimensional systems, possibly with uncertainties are considered in [2, 5, 14]. These are Roesser systems [10] that are causal in all variables and can be represented in the form

$$\begin{bmatrix} x_0(t+1,k_1,\ldots,k_n) \\ x_1(t,k_1+1,\ldots,k_n) \\ \vdots \\ x_n(t,k_1,\ldots,k_n+1) \end{bmatrix} = Ax(t,k_1,\ldots,k_n)+Bu, \quad (1)$$

$$x = [x_0^T \; x_1^T \; \ldots \; x_n^T]^T, \quad (2)$$

$$y(t,k_1,\ldots,k_N) = Cx(t,k_1,\ldots,k_N)+Du, \quad (3)$$

where $u = u(t,k_1,\ldots,k_N)$, y, x, x_0, ..., x_n are vector valued multidimensional functions of integer arguments and A, B, C, D are constant matrices of appropriate sizes. An output y of a Roesser system can be computed from the input u by propagating an initial condition in positive directions along each coordinate. The stability and robustness analysis approaches for such systems are conceptually straightforward extensions of the approaches for standard dynamical systems depending on time only.

Unfortunately, in many (perhaps most) important practical applications multidimensional systems are not causal in spatial coordinates and Roesser models are not applicable. An LFT-based approach to analysis and control design for noncausal multidimensional systems was proposed in [5] and is further developed in a number of follow-on papers including [6, 7]. These papers consider non-causal pseudo state-space models of the form

$$\begin{bmatrix} x_0(t+1,k_1,\ldots,k_N) \\ x_1(t,k_1+1,\ldots,k_N) \\ x_{-1}(t,k_1-1,\ldots,k_N) \\ \vdots \\ x_{-n}(t,k_1,\ldots,k_N-1) \end{bmatrix} = Ax(t,k_1,\ldots,k_N)+Bu, \quad (4)$$

$$x = [x_0^T \; x_1^T \; x_{-1}^T \; \ldots \; x_{-n}^T]^T, \quad (5)$$

$$y(t,k_1,\ldots,k_N) = Cx(t,k_1,\ldots,k_N)+Du, \quad (6)$$

where y, x, x_0, ..., x_{-n}, $u = u(t,k_1,\ldots,k_N)$ are vector-valued functions and A, B, C, D are constant matrices of appropriate sizes. The system (4)–(6) can be represented as an LFT of a constant matrix with a frequency structure that includes discrete Laplace variables for the spatial coordinates and their inverses. In [5, 6, 7] the stability and other dynamic properties of such multidimensional systems are studied as algebraic properties of its multivariable transfer function given by the LFT.

For causal multidimensional systems realized by Roesser model (1) ambiguity between the transfer function and one of its realizations "is benign and convenient and can be always resolved from the context" [14, Section 10.2, p. 257]. Unfortunately this is not the case for non-causal multidimensional systems. First, it is important to note that unlike (1)–(3) the equations (4)–(6) cannot in fact be considered as a realization of a non-causal multidimensional system. Some of the update schemes corresponding to an "equivalent" model of the form (4)–(6) can be stable, other unstable in the non-causal coordinates. This is similar to the mathematical theory of PDE where key issues of existence, uniqueness and well-posedness of solution need to be resolved before analytical techniques, such as ones based on operator transforms, can be used.

In practice, it is usually known whether a plant or controller realization are spatially stable. The key issue then is to find whether a closed-loop including the multidimensional plant and the controller is stable, despite the unceratinties present in the loop. This paper proposes an extension of Structured Singular Value analysis (μ-analysis) to handle this issue.

The main contributions of this work are: (i) in clarification of robust stability issues for multidimensional systems; (ii) definition of practical analysis approaches for such systems based on straightforward extension of existing μ-analysis tools; and (iii) integration of the modeling error caused by boundaries into the analysis framework. The issue of the boundary effects was brought up in the recent literature on the subject, but no convenient analysis approach has been proposed so far.

2 Analysis for one spatial dimension

To clarify the issues with the modeling and realization for non-causal spatial system consider the simplest special case of (4), where only one spatial coordinate is present, $N = 1$, and there is no dependence on time t.

Consider a one-dimensional non-causal LSI system. It can be described as a noncausal convolution of the input u with a pulse response h

$$y(k) = \sum_{q=-\infty}^{\infty} h(k-q)u(q) \tag{7}$$

For modeling and analysis purposes, an infinite summation over the actuator index is considered in (7). The analysis to follow assumes that the pulse response $h(x_1)$ in (7) corresponds to a stable system. The stability here is understood in the Bounded Input Bounded Output (BIBO) sense meaning that for any input u with finite l_2 norm, the output y in (7) is also required to have a finite l_2 norm.

The BIBO stability requires the pulse response to be absolutely summable [3]: $\sum |h(k)| < \infty$. For an absolutely summable pulse response h, a transfer function of the distributed system can be calculated as a 2-sided discrete Laplace transform (z-transform) of the pulse response

$$\hat{h}(\lambda) = \sum_{k=-\infty}^{\infty} h(k)\lambda^{-k}, \tag{8}$$

where λ is a complex Laplace variable that corresponds to the unit shift operator along the coordinate x_1. A theory of two-sided z-transform can be found, for instance in [9].

The expansion (8) is a Lorain series and it converges uniformly inside a ring $\rho_1 < |\lambda| < \rho_2$. Since the response is absolutely summable, $\rho_1 < 1 < \rho_2$, so that the convergence ring is not empty and contains the unit circle. Denote $r = \min(\rho_1^{-1}, \rho_2)$. Then the transfer function is analytical in the ring $r^{-1} < |\lambda| < r$, and r defines spatial response localization degree. This is because the spatial pulse response rolls off at least as fast as r^{-k} where k is the distance from the pulsed actuator.

An important class of spatially distributed systems is given by transfer functions that are rational in λ. Non-causal Finite Impulse Response (FIR) models yield a special case of rational transfer functions. Recall that a causal rational transfer function can be always represented in an LFT form [14]. Such LFT representation is equivalent to a familiar state-space causal realization of the transfer function. For a causal system the state-space realization allows computing the output sequence $y(k)$ from the input sequence $u(k)$ by iterating the state-space equation forward in time. The poles of the transfer function define dynamical stability of the causal system. If all poles are inside unit circle, the system pulse response decays with time. If there is a pole outside the unit circle, the response will grow exponentially.

These well-known facts are recited here just to make the point that for a non-causal system things work differently. First, a non-causal rational transfer function cannot be always represented as an LFT of a constant matrix with the frequency structure λI (or $\lambda^{-1}I$), where I is a unity matrix. One counter example is given by a non-causal FIR system. However, a non-causal rational transfer function (8) can be represented as an LFT of a constant matrix with the frequency structure $\Lambda = \text{block diag}\{\lambda^{-1}I_1, \lambda I_2\}$, where I_1 and I_2 are unity matrices of dimensions N_1 and N_2. For one-dimensional case in question, such LFT would yield a pseudo state-space model of the form (4), where $N = 1$ and the causal variable t is absent.

$$\begin{bmatrix} x_1(k+1) \\ x_2(k-1) \end{bmatrix} = Ax(k) + Bu(k), \tag{9}$$

$$y(k) = Cx(k) + Du(k) \tag{10}$$

$$x(k) = [x_1^T(k) \;\; x_2^T(k)]^T,$$

$$\hat{h}(\lambda) = D + C\Lambda(I - \Lambda A)^{-1}B, \tag{11}$$

where $x_1(k)$ and $x_2(k)$ are vectors of the dimensions N_1 and N_2, $y(k)$ and $u(k)$ are vectors of dimensions m and n respectively, and A, B, C, D are constant matrices of appropriate sizes. These matrices and the frequency structure $\Lambda = \text{block diag}\{\lambda{-}1I_1, \lambda I_2\}$, define an LFT representation (11) of the transfer function (8).

Unlike the causal case, the non-causal pseudo state space model (9) does not imply a unique way of computing the output sequence $y(k)$ from the input sequence $u(k)$ by iterating a state-space equation. For a causal system the realization and the state-space equations are one and the same. For a non-causal system, the notion of realization should includes its implementation: defining a particular way of computing solution for the pseudo state space equations (9). The poles of the transfer function (11) are uniquely defined by the model (9). Yet, without defining a method of computing a solution to (9) in the system realization it is impossible to determine whether these poles correspond to a growing or decaying response of the system.

The mentioned issues with noncausal systems can be clearly illustrated by the following simple example system of the form (9)

$$x(k+1) = ax(k) + u(k), \qquad |a| > 1, \qquad y(k) = x(k), \qquad (12)$$

where $x(k)$ is a scalar and the update is performed from left to right. The system (12) can be formally represented through a transfer function $\lambda/(1-a\lambda)$ that is analytical inside the ring $a^{-1} < |\lambda| < a$. At the same time, this system does not have a summable pulse response and is unstable. If the system is described by the same difference equation, but it is re-written such that the update is performed from right to left (in the anti-causal direction), the system will have an exponentially decaying anti-causal pulse response.

For a one-dimensional non-causal system considered in this section, a realization with a BIBO stable implementation can be built from a non-causal transfer function if the transfer function is nonsingular for $|\lambda| = 1$. This can be done by using a Wiener-Hopf factorization approach. To this end, the transfer function (11) is presented in one of the two alternative forms

$$\hat{h}(\lambda) = \Phi_+(\lambda) \cdot \Phi_-(\lambda^{-1}), \qquad (13)$$
$$\hat{h}(\lambda) = \Psi_+(\lambda) + \Psi_-(\lambda^{-1}), \qquad (14)$$

where $\Phi_+(\zeta)$, $\Phi_-(\zeta)$, $\Psi_+(\zeta)$, and $\Psi_-(\zeta)$ are proper analytical continuation functions that are each analytical outside the unit circle. For a rational transfer function $\hat{h}(\lambda)$, a Wiener-Hopf factorization can be obtained by sorting apart the poles inside and outside the unit circle. A BIBO stable implementation of the system realization can be build as a cascade connection of a causal system that can be realized in accordance with standard causal theory from the transfer function $\Phi_+(\lambda)$ and an anti-causal system realized from $\Phi_-(\lambda^{-1})$. An alternative implementation can be built as a parallel connection of the causal and anticausal systems defined by the transfer functions $\Psi_+(\lambda)$ and $\Psi_-(\lambda^{-1})$ respectively. Unfortunately, there is no known way of extending this approach towards a general non-causal multidimensional rational transfer function, even if only one noncausal coordinate and a causal coordinate (time) are present.

3 Model of multidimensional system

The goal of this section is to establish mathematical models for subsequent analysis. Consider a model of a discrete-time spatially distributed system controlled by a N-dimensional array of actuator and sensor units. Each unit has n control inputs and m measurement outputs. The control, measurement, and dynamical state coordinates of such system can be described as functions of the discrete time t (an integer sample number) and integer spatial coordinates $k_1, \ldots, k_N$ (corresponding to the unit number along each of the array dimensions). In a 3-D physical space, an actuator array cannot have more than $N = 3$ dimensions. In the existing applications of array control, $N = 1$ (e.g., linear actuator arrays in paper or printing machines), or $N = 2$ (actifely controlled reflectors, imaging applications).

In what follows, vector or matrix-valued functions $x(t, k_1, \ldots, k_N)$ of integer time and spatial coordinates will be considered. The following notations for the norms will be used: $|x|$ will denote a Euclidean norm of a vector or an operator norm (maximal singular value) of a matrix; $\|x\|_2$ will denote the l_2 norm of the function (multidimensional sequence) $x(t, k_1, \ldots, k_N)$

$$\|x\|_2^2 = \sum_{t=0}^{\infty} \sum_{k_1,\ldots,k_N=-\infty}^{\infty} |x(t, k_1, \ldots, k_N)|^2 \qquad (15)$$

The multidimensional system in question is assumed to be LTI/LSI and can be modeled by a relationship between the control input $u(t, k_1, \ldots, k_N)$ and the measurement output $y(t, k_1, \ldots, k_N)$. This relationship can be described by a multi-dimensional convolution of the input u with a pulse response H

$$y(t, k_1, \ldots, k_N) = \sum_{\tau=0}^{\infty} \sum_{q_1,\ldots,q_N=-\infty}^{\infty} H(t-\tau, k_1 - q_1, \ldots, k_N - q_N; \Delta) u(\tau, q_1, \ldots, q_N), \qquad (16)$$

where $H(t, k_1, \ldots, k_N; \Delta) \in \Re^{m,n}$ and the system is assumed to contain an unknown part - an uncertainty. In (16), $\Delta \in \mathbf{\Delta}$ describes a realization of uncertainty, where $\mathbf{\Delta}$ is the uncertainty set. The uncertainty Δ and the uncertainty set $\mathbf{\Delta}$ will be defined further on.

The analysis to follow studies stability of the system (16) in the Bounded Input Bounded Output (BIBO) sense. This means for any input u such that $\|u\|_2 < \infty$ the output y is such that $\|y\|_2 < \infty$. The BIBO stability requires the pulse response to be absolutely summable:

$$\sum_{t=0}^{\infty} \sum_{x_1,\ldots,x_N=-\infty}^{\infty} |H(t, x_1, \ldots, x_N; \Delta)| \;<\; \infty, \qquad (17)$$

where as mentioned above $|H| = \bar{\sigma}(H)$ is the point-wise operator norm of the matrix-valued pulse response H. For an absolutely summable pulse response H (17), a transfer function $\hat{H} = \hat{H}(z, \lambda_1, \ldots, \lambda_1; \Delta)$ of the distributed system can be calculated as a multi-dimensional discrete Laplace transform (z-transform) of the pulse response

$$\hat{H} = \sum_{\tau, x_1, \ldots, x_N} H(\tau, k_1, \ldots, k_N; \Delta) z^{-\tau} \lambda_1^{-k_1} \ldots \lambda_N^{-k_N}, \qquad (18)$$

where z, λ_1, ..., λ_N are complex Laplace variables. These variables correspond to the unit shift operators in time and along each of the N spatial coordinates respectively. For an absolutely summable response (17), the expansion (18) is guaranteed to converge for any complex numbers z, λ_1, ..., λ_N in the domain

$$\mathbf{\Lambda_1} = \{z, \lambda_1, \ldots, \lambda_N \in \boldsymbol{C} : |z| \geq 1, |\lambda_1| = 1, \ldots, |\lambda_N| = 1\} \quad (19)$$

The following useful enhancement of the above BIBO stability definition and of the domain (19) will be further used in this paper.

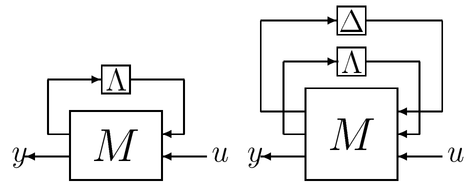

Figure 1: LFT models

Definition 1 *A pulse response (16), (17) will be called to have a spatial decay rate r ($r > 1$) and dynamical decay rate α, ($\alpha \geq 1$) if it is: (i) absolutely summable; (ii) decays at least as fast as α^{-t} with time uniformly in the spatial coordinates; and (iii) decays at least as fast as $r^{-|l|}$, where l is the Euclidean distance from the pulsed actuator along the spatial coordinates, uniformly in time.*

As discussed in more detail further, the spatial decay rate r allows to specificy an acceptable influence of the boundary conditions. In what follows, α and r are considered as fixed design specification parameters. The transfer function expansion (18) for an absolutely summable response with a spatial decay rate r and dynamical decay rate α is guaranteed to converge (is analytical) for any complex numbers z, λ_1, ..., λ_N in the domain

$$\boldsymbol{\Lambda_{\alpha,r}} = \{z, \lambda_1, \ldots, \lambda_N \in \boldsymbol{C} : \\ |z| \geq \alpha, r^{-1} \leq |\lambda_{1,\ldots,N}| \leq r, r \geq 1, \alpha \geq 1\}, \qquad (20)$$

One of this paper goals is to provide robust stability analysis for a multidimensional closed-loop feedback system. In addition to a plant of the form (16), the feedback loop would include a controller that computes input based on the past plant output. Such controller is a dynamical system that can be modeled in the form similar to (16):

$$u(t, k_1, \ldots, k_N) = \sum_{\tau=1}^{\infty} \sum_{q_1,\ldots,q_N=-\infty}^{\infty} \\ C(t-\tau, k_1-q_1, \ldots, k_N-q_N) y(\tau, q_1, \ldots, q_N), \quad (21)$$

where $C(t, k_1, \ldots, k_N) \in \Re^{n,m}$ is the controller pulse response. The summation in time in (21) starts from $\tau = 1$, not from $\tau = 0$. This means the controller (21) is assumed to be strictly causal - no feedthrough is allowed in the feedback term. This is satisfied in real-life systems, where the control input is computed during the sample time and can be only based on the information available at the past sample. The assumption of the controller being strictly causal is important in the analysis of the next section.

Consider now an LFT model corresponding to difference equations of the form (4)–(6), see [5] for more discussion. It is further assumed that these difference equations are satisfied for the system (16). The LFT model is shown schematically in Figure 3 (left). The constant matrix M in the diagram has the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

The frequency structure Λ in the diagram has the form

$$\Lambda = \text{diag}\left\{z^{-1}I_n, \lambda_1 I_{p_1}, \lambda_1^{-1} I_{n_1}, \ldots, \lambda_N I_{p_N}, \lambda_N^{-1} I_{n_N}\right\}, \quad (22)$$

where the dimensions are as appropriate. This model does not yet explicitly take into account model uncertainty present in (16). Consider the following uncertainty structure that is as usual in the Structured Singular Value analysis - μ-analysis.

$$\Delta = \text{diag}\left\{\delta_1 I_{r_1}, \ldots, \delta_S I_{r_S}, \delta_{S+1} I_{q_1}, \ldots, \delta_{S+Q} I_{q_Q}, \\ \Delta_1, \ldots, \Delta_F\right\}, \quad (23)$$

where I_j is a unity matrix of the size j, δ_s are real or complex scalars, and Δ_f are square complex matrix blocks. A more detailed discussion of the uncertainty description (23) and explanation of its physical meaning can be found in the textbook [14].

The LFT model including the uncertainty is shown schematically in Figure 3 (right). The two upper blocks in the feedback loop include the uncertainty description (23) and frequency structure (22). The transfer function (18) with the uncertainties (23) can be presented in the LFT form

$$\hat{P}(z, \lambda_1, \ldots, \lambda_1; \Delta) = M_{22} + M_{21}\bar{\Delta}\left(I - M_{11}\bar{\Delta}\right)^{-1}, \\ \bar{\Delta} = \text{block diag}\{\Delta, \Lambda\}, \quad (24)$$

where the submatrices M_{ij} of the matrix M provide a partitioning compatible with (23), (22) and Figure 3. As usual, it is assumed that the uncertainties (23) have been scaled such that they belong to a set

$$\boldsymbol{\Delta} = \{\delta_1, \ldots, \delta_S \in \Re, \delta_{S+1}, \ldots, \delta_{S+Q} \in \boldsymbol{C}, \Delta_j \in \boldsymbol{C}^{m_j, m_j} : \\ |\delta_k| \leq 1, \bar{\sigma}(\Delta_j) \leq 1\} \quad (25)$$

It is important to emphasize that there are some differences in the above modeling of multidimensional systems compared to the standard modeling in μ-analysis of dynamical systems. In (22), both spatial Laplace variables λ_k and their inverses are included as separate indeterminants. This is done for instance because a non-causal FIR response modeling requires using both positive and negative powers of the spatial Laplace variables.

4 Structured singular value analysis

A robust stability condition on a transfer function have been introduced in [5] and follow-up papers [6, 7]. According to this condition, the matrix inverted in (24) is required to be nonsingular for all frequency structure variables (22) in the domain (19), $\Lambda \in \boldsymbol{\Lambda_1}$, and the uncertainty parameters (23) within the set (25), $\Delta \in \boldsymbol{\Delta}$. The example of Section 2 shows that this condition, though necessary, is in fact not sufficient for a multidimensional system BIBO stability, unless additional assumptions are made. A sufficient condition is discussed further in this section.

The approach taken in this paper is based on a continuous dependence of the degree of stability in the system on the uncertainty and frequency structure parameters. Instead of resolving the problem of implementing a realization a practically acceptable knowledge about the initial spatial stability is assumed. The following fact will be central in establishing our results

Proposition 1 *Consider an uncertain multidimensional system (16) where pulse response is such that $A^{-t}H(t, k_1, \ldots, k_N; \Delta)$ is known to be absolutely summable* *in the sense (17) for some $A > 0$. Let the system satisfy difference equations that allow to present it through a formal transfer function in the LFT form (24). Assume further that the formal transfer function $\hat{P}(z, \lambda_1, \ldots, \lambda_1; \Delta)$ (24) is analytical in the set $\mathbf{\Lambda_{\alpha,r}}$ (20) for any uncertainty Δ (23) in (24). Then the system (16) is BIBO stable for any uncertainty in the considered set (25), has a spatial decay rate r and dynamical decay rate α.*

Proof. Since the formal transfer function $\hat{P}(z, \lambda_1, \ldots, \lambda_1; \Delta)$ is analytical in $\mathbf{\Lambda_{\alpha,r}}$, it can be expanded as

$$\hat{P}(z, \lambda_1, \ldots, \lambda_1; \Delta) = \sum_{\tau=1}^{\infty} \sum_{k_1,\ldots,k_N=-\infty}^{\infty} P(\tau, k_1, \ldots, k_N; \Delta) z^{-\tau} \lambda_1^{-k_1} \ldots \lambda_N^{-k_N}, \quad (26)$$

and the expansion converges uniformly in $\mathbf{\Lambda_{\alpha,r}}$. Now consider the transfer function expansion (18) computed for the pulse response H in (16). In accordance with the assumption, the expansion (18) converges in a subset of $\mathbf{\Lambda_{\alpha,r}}$: for $|z^{-1}| < A^{-1}$, and $|\lambda_j| = 1$. The formal transfer function $\hat{P}$ coincides with $\hat{H}$ on this subset. Therefore $H(\tau, k_1, \ldots, k_N; \Delta) = P(\tau, k_1, \ldots, k_N; \Delta)$, the transfer function expansion (18) coincides with (26) and converges in $\mathbf{\Lambda_{\alpha,r}}$. In accordance with Definition 1, this guarantees that the proposition statement is true. QED

Proposition 1 requires an *a priory* knowledge that the pulse response H of the system does not grow faster than exponentially in time and is summable in space coordinates as a pre-requisite for the stability analysis. By itself, the result of Proposition 1 is not too useful in practical stability analysis because in order to decide about BIBO stability of a multidimensional system it requires fairly detailed information about spatial stability of this system to be available in the first place. Proposition 1, however, helps in obtaining the following useful resut for closed-loop spatially distributed array systems.

Proposition 2 *Consider a closed-loop dynamical system that consists of a multidimensional plant (16) and a multidimensional controller (21) in a feedback configuration. Assume that for an open loop plant (16) the pulse response is such that $A_1^{-t} H(t, k_1, \ldots, k_N; \Delta)$ is known to be absolutely summable in the sense (17) for some $A_1 > 0$. Assume further that the pulse response of the controller is such that $A_2^{-t} G(t, k_1, \ldots, k_N)$ is known to be absolutely summable in the sense (17) for some $A_2 > 0$. Under these conditions, if a formal closed-loop transfer function is analytical in the set $\mathbf{\Lambda_{\alpha,r}}$ (20) for any uncertainty Δ (23) in (25), then the closed-loop system is BIBO stable for any uncertainty in the considered set (25), has a spatial decay rate r, and dynamical decay rate α.*

Proof. Let us first prove that the closed-loop response H_c is such that $A_c^{-t} H_c(t, k_1, \ldots, k_N; \Delta)$ is absolutely summable for some $A_c > 0$. Note that in accordance with (21) the controller transfer function can be represented in the form $\hat{G}(\cdot) = z^{-1}\hat{C}(\cdot)$, where $\hat{C}(\cdot)$ is analytical for $|z^{-1}| < A_2^{-1}$ and $|\lambda_j| = 1$. Consider the closed-loop transfer function $\hat{H}_c(z, \lambda_1, \ldots, \lambda_1; \Delta)$ computed from the closed-loop pulse response $H_c(\tau, k_1, \ldots, k_N; \Delta)$. In accordance with the Proposition assumptions, the transfer function $\hat{H}_c$ can be represented for $z \to \infty$, and $|\lambda_j| = 1$, $j = 1, \ldots, N$ as

$$\hat{H}_c = [I + z^{-1}\hat{H}\hat{C}]^{-1}\hat{H} \quad (27)$$

Consider the system on the unit circle $|\lambda_j| = 1$, $j = 1, \ldots, N$. The expansion (27) converges uniformly in z, $\lambda_1, \ldots, \lambda_1$,, and Δ for $|z^{-1}| < A_c^{-1}$, where $A_c > 0$ is such that the $|I + z^{-1}\hat{H}\hat{C}| > 0$. This follows from [8, Theorem 2, Section 2.4]. To demonstrate that such A_c exists, note that because of the uniform summability condition, the following bounds hold: $|\hat{H}(z, \lambda_1, \ldots, \lambda_1; \Delta)| < C_1$ for $|z| > A_1$ and $|\hat{C}(z, \lambda_1, \ldots, \lambda_1; \Delta)| < C_2$ for $|z| > A_2$. Thus, for $|z| > A_c = \max(A_1,\ A_2,\ 2(C_1C_2)^{-1})$ $|I + z^{-1}\hat{H}\hat{C}| \geq 1 - A_c|\hat{H}|\cdot|\hat{C}| > 1/2$.

The uniform convergence of the expansion (27) in the domain $|z^{-1}| < A_c^{-1}$, $|\lambda_j| = 1$, $j = 1, \ldots, N$, means that $A_c^{-t} H_c(t, k_1, \ldots, k_N; \Delta)$ is absolutely summable. Therefore the conditions of Proposition 1 hold for the closed-loop transfer function. Thus, the conclusions of Proposition 1 hold for this function as well. QED

The essential meaning of Proposition 2 is that a closed loop consisting of a multidimensional plant and multidimensional controller is stable provided that (i) both the plant and the controller are spatially stable (they may be unstable dynamically), (ii) the controller does not have a feedthrough term, and (iii) the multidimensional closed-loop transfer function is analytical in the stability domain. The condition (ii) implicitly stated in (21) is not very limiting in practice because all it requires is that the control action at each time sample influences the measurements at the next sample and not the same sample measurement. This always holds for practical digital controllers.

The results of Propositions 1 and 2 clear way for analysis of robust stability of multidimensional systems in the LFT framework. This analysis can be performed by computing the Structured Singular Value (SSV) with respect to the uncertainty structure (23) similar to standard μ-analysis. Standard software packages such as μ-tools are commercially available for SSV computations. As demonstrated below, existing μ-analysis tools can be readily extended towards multidimensional spatially distributed systems. It will be shown that the uncertainty associated with the boundary effects can also be handled in the introduced framework.

Consider the LFT diagram in Figure 3 describing the transfer function (24). For each $N+1$-tuple of complex numbers z, λ_1, ..., λ_N, consider the LFT with the uncertainty structure Δ in (23). The structured singular value with respect to the uncertainty Δ can be defined as usual [14]

$$\mu_\Delta\left(M(z, \lambda_1, \ldots, \lambda_N)\right) = \frac{1}{\min\left\{\bar{\sigma}(\Delta) : \det[I - M(z, \lambda_1, \ldots, \lambda_N)\Delta] = 0,\ \text{for}\ \Delta \in \mathbf{\Delta}\right\}} \quad (28)$$

where $\mathbf{\Delta}$ is as defined in (25). It will be further assumed that the transfer function (24) satisfies conditions of Proposition 1 or Proposition 2. Consider the inner loop in the right diagram in Figure 3. This inner loop includes the frequency structure Λ and defines a transfer function $M(z, \lambda_1, \ldots, \lambda_N)$, the same as in (28). The diagram in Figure 3 can be interpreted as a closed-loop consisting of the system $M(z, \lambda_1, \ldots, \lambda_N)$ and the uncertainty Δ in the feedback loop. The structured singular value for this loop can be defined as an inverse robust stability margin $\alpha_{\max}$ of the system with respect to the uncertainty $\mathbf{\Delta}$ (23), where the 'stability' is understood as the system being stable in time and having a spatial decay rate r in accordance with Definition 1.

If the conditions of Proposition 1 or Proposition 2 hold, the system BIBO stability and the decay rates can be determined from the algebraic properties of the formal transfer function. In this case the structured singular value can be defined as

$$\mu_{\Delta,\Lambda_{\alpha,r}}(M) = \frac{1}{\alpha_{\max}} = \sup_{\Lambda_{\alpha,r}} \mu_\Delta \left(M(z, \lambda_1, \ldots, \lambda_N) \right), \qquad (29)$$

where $\mathbf{\Lambda_{\alpha,r}}$ is the set (20).

The following formulas give a constructive way for computing the SSV (29)

Proposition 3 *The structured singular value (28), (29) can be computed as*

$$\mu_{\Delta,\Lambda_{\alpha,r}}(M) = \sup_{\omega,\nu_1,\ldots,\nu_N \in [0,2\pi]} \mu_{\Delta,\alpha,r}(M;\omega,\bar{\nu}), \quad (30)$$

$$\mu_{\Delta,\alpha,r}(M;\omega,\bar{\nu}) = \min_{\rho_n=\{r^{-1},r\}} \mu_\Delta \left(M(\alpha^{-1}e^{i\omega}, \rho_1 e^{i\nu_1}, \ldots, \rho_N e^{i\nu_N}) \right), \quad (31)$$

where $\bar{\nu} = [\nu_1, \ldots, \nu_N]^T$ the minimum is computed over all combinations of the factors ρ_n, $n = 1, \ldots, N$, with each factor taking one of the two values r or r^{-1}.

The proof of Proposition 3 is based on the Zero Exclusion Principle and is similar to the standard proof for the structured singular value computation [14, Lemma 11.1, Section, 11.2]. Standard μ-tools software can be used in computing (31).

In analysis of controlled dynamical systems, the SSV is commonly computed on a grid of the dynamical frequencies ω. In computing (30), (31), a multidimensional grid of dynamical and spatial frequencies has to be considered. The μ plots used for description of multivariable dynamical systems, here change into $N+1$ dimensional μ hyper-surfaces (31). Such representation and computations are acceptable, because in present-day practical applications of array control there are only $N = 1$ or $N = 2$ spatial coordinates. At the same time, the computers presently are 1000 times or more faster than 20 years ago when SSV was first introduced and used in computations on one-dimensional frequency grids. In the future, 3-D applications of array control might appear, but then available computing power will increase further. The main difficulty with the multi-dimensional SSV plots could be visualization and interpretation of the results, rather than the computational performance. Similar to frequency gridding commonly used in the standard SSV analysis of dynamical systems, multidimensional frequency gridding seems to be a practically reasonable approach to take in many practical applications.

For modeling and analysis purposes, the summation over an infinite spatial domain is considered in (16). In reality, the spatial domain in question is always bounded, though it may be very large. Therefore there is usually a need for considering edge effects that (16) does not address. The boundary effects can be proved to be contained in a boundary layer provided the system response taking into account the boundary effects is BIBO stable and decays sufficiently fast in the spatial coordinates. The last condition might not hold in case a boundary instability developes in the system. A detailed analysis of the boundary layer stability is beyond the scope of this paper and warrants a separate study.

Note that the definition of the structured singular value used in (29) somewhat differs from the standard definition in spirit because it requires the system to have a given spatial decay rate and dynamical decay rate. These decay rates guarantee that the influence of the initial and boundary conditions on the multidimensional system response decay as fast as specified. In particular, for a given spatial decay rate r the effects of the boundary conditions can be guaranteed to be contained in a boundary layer with a characteristic width $\log r$ near the boundaries of the spatial domain. Through the parameter r, the impact of the boundary effects can be explicitly included into the control design and analysis tradeoff.

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