Unknown input reduced order observer for singular bilinear systems with bilinear measurements

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Abstract

In this paper, a method to design a reduced order UIO [1] for bounded control inputs SBS [2] with bilinear measurements and subjected unknown disturbances is investigated. The design of this UIO is divided into two parts. The first one consists in solving some algebraic constraints to obtain an observation error which is decoupled from the unknown disturbances. In the second part, an LMI [3] is solved to ensure the exponential stability of the reconstruction error for all admissible control inputs and unmeasurable disturbances.

Keywords

Singular bilinear systems, Bilinear measurements, Unknown input reduced order observer, Exponential Convergence, Linear matrix inequality.

1 Introduction

In the three past decades, there has been increasing interest in the state observer design of bilinear systems due to the fact that many important physical processes, such as chemical, biomedical or thermal processes, may be appropriately modeled as bilinear systems when linear models are inadequate [1]. Hara and Furuta [2] presented a design procedure of minimum order estimator which works only for a class of bilinear systems with linearisable error dynamics. A less restrictive approach, based on the Lyapunov stability theorem, such that the observation error may be bilinear, is proposed by Funahashi [3] and Tibken *et al.* [4, 5]. The observers cited above require strong structural constraints. These constraints have been alleviated by Derese *et al.* [6] and Wang and Kao [7] by including the knowledge of the control input bounds in the observer design. Derese *et al.* [6] considered the optimisation of the feedback amplification (only on the linear part) by the use of an algebraic Riccati equation. Wang and Kao [7] presented a design algorithm based on Lyapunov stability for both continuous-time and discrete-time. All of these observers do not require the uniform observability of the bilinear system. Besançon and Hammouri [8] have characterised a class of non-uniformly observable non-linear systems for which there exists an observer with an estimation error decaying to zero irrespective of the control inputs, only the bounds of these inputs must be known. They proposed an observer synthesis including the above-mentioned observers as special cases. Kalman like observers have been applied to the state estimation of observable bilinear systems (see [9] and the references therein) for which the observer gain depends on the solution of an input dependent differential Riccati equation.

In practice, the system's dynamics is often affected by unmeasurable disturbances which can be modelled as unknown inputs. Based on results in [2], Hać [10] and Saif [11] considered the design of UIO for bilinear systems in which the error estimation dynamics is linear and without an a priori knowledge about the disturbance dynamics. Zasadzinski *et al.* [12] proved that, under suitable transformation, the design of the UIO proposed in [10, 11] is equivalent to the design of an UIO for linear systems. These existence conditions are rather restrictive and these UIO work only for a class of bilinear systems with linearisable observation error dynamics. The results presented in [12] have been extended to SBS in [13]. In [14], Ying *et al.* proposed an UIO for bilinear systems requiring the knowledge of the disturbance dynamics. The design of the above-mentioned UIO require strong structural constraints, so their practical implementation is quite involved.

Considerable attention has been focused on the state observation problems of singular linear systems without unmeasurable disturbances [15, 16, 17, 18, 19] or with unmeasurable disturbances [20, 21, 22]. Various approaches as singular value decomposition or extensions of Luenberger-like observers to singular systems have been proposed in the literature. Observer design for singular non-linear systems has been studied by Kaprelian and Turi [23] using an extended linearisation technique. Boutayeb and Darouach [24] generalised the results in Kaprelian and Turi [23] to the case where the matrix E is rectangular.

This paper is devoted to the design of an exponential reduced order UIO for SBS with bilinear measurements

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[1] UIO : unknown input observer

[2] SBS : singular bilinear system

[3] LMI : linear matrix inequality

and subjected to bounded control inputs and unmeasurable disturbances. It must be noticed that all previously cited works on non-linear observers consider that the measurement equation is linear. The case where the matrix E is rectangular is considered. The reduced UIO observer design problem is formulated in §2 and solved in §3. The design of this UIO is divided into two parts given in §3.1 and 3.2. The first one consists in solving some algebraic constraints to obtain an observation error which is decoupled from the unknown disturbances. An existence condition for this decoupling is given in terms of a rank condition. The second part consists in converting the UIO design problem into a robust stabilisation one with structured uncertainties, then an LMI approach is used to solve a bilinear Lyapunov inequality to ensure the exponential stability of the reconstruction error dynamics for all admissible control inputs and for all unmeasurable disturbances. The design procedure of the reduced order UIO is summarised in §4.

2 Problem statement

In this paper, we consider th following SBS with bilinear measurements and subjected to control bounded inputs and unmeasurable disturbances

$$E\dot{x} = \widetilde{A}^0 x + \sum_{i=1}^{m} u^i \widetilde{A}^i x + Bu + D_1 w \tag{1a}$$

$$v = C_1 x + D_{12} u - D_{12} r \tag{1b}$$

$$y = C^0 x + \sum_{i=1}^{m} u^i C^i x + D_2 w \tag{1c}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ the control inputs (measured), $w \in \mathbb{R}^q$ the unmeasurable disturbances and $y \in \mathbb{R}^p$ the measurements. In this paper, the following notations are used : u^i is the i^{th} coordinates of vector u and $\widetilde{A}^i$ is the matrix associated to u^i in the SBS (1).

We assume that the control inputs are bounded, i.e. $u \in \Omega \subset \mathbb{R}^m$ where the polytope Ω is given by

$$\Omega := \left\{ u \in \mathbb{R}^m \mid u^i_{\min} \leqslant u^i \leqslant u^i_{\max} \text{ for } i = 1, \ldots, m \right\}. \tag{2}$$

Without loss of generality, the following assumptions are made on matrices E, D_1 and D_2

$$r_1 = \operatorname{rank} E \leqslant \min(r, n) \text{ with } E \in \mathbb{R}^{r \times n}, \tag{3a}$$

$$q = \operatorname{rank} \begin{bmatrix} D_1^T & D_2^T \end{bmatrix}^T \leqslant p, \tag{3b}$$

$$p = \operatorname{rank} \begin{bmatrix} C^0 & \ldots & C^m & D_2 \end{bmatrix}. \tag{3c}$$

These assumptions are often verified for physical processes.

By using changes of basis in the measurement and disturbance spaces respectively, the SBS (1) is decomposed in an equivalent one in the observation point of view in order to extract the parts of the measurement vector which are disturbance-free and linear. Then y in (1c) is decomposed as follows :

1. y_1 which is bilinear and not disturbance-free,
2. y_{21} which is bilinear and disturbance-free,
3. y_{22} which is linear and disturbance-free.

To obtain a disturbance-free measurement vector, y is decomposed in two parts as follows

$$y_1 = C_1^0 x + \sum_{i=1}^{m} u^i C_1^i x + w_1 \tag{4a}$$

$$y_2 = C_2^0 x + \sum_{i=1}^{m} u^i C_2^i x \tag{4b}$$

where V_1 and V_2 are non-singular such that $V_1^T V_1 = I$ and $V_1^T D_2 V_2 = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$, with $V_1^T y = \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right]$, $V_1^T C^i = \left[\begin{smallmatrix} C_1^{iT} & C_2^{iT} \end{smallmatrix}\right]^T$ and $V_2^{-1} w = \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]$. Since y_{22} has a bilinear part, a row compression on C_{22}^i is used in order to extract the linear part of the disturbance-free measurement y_2. This can be made by using a non-singular matrix V_3 such that $V_3 V_3^T = I$ and $V_3^T \left[\begin{smallmatrix} C_{21}^1 & \cdots & C_{21}^m \end{smallmatrix}\right] = \left[\begin{smallmatrix} C_{21}^1 & \cdots & C_{21}^m \\ 0 & \cdots & 0 \end{smallmatrix}\right]$ where $\left[\begin{smallmatrix} C_{21}^1 & \cdots & C_{21}^m \end{smallmatrix}\right]$ is of full row rank. Then measurement equation (4b) is decomposed as follows

$$y_{21} = C_{21}^0 x + \sum_{i=1}^{m} u^i C_{21}^i x \tag{5a}$$

$$y_{22} = C_{22}^0 x \tag{5b}$$

where $V_3^T y_2 = \left[\begin{smallmatrix} y_{21} \\ y_{22} \end{smallmatrix}\right]$ and $V_3^T C_2^0 = \left[\begin{smallmatrix} C_{21}^{0\,T} & C_{22}^{0\,T} \end{smallmatrix}\right]^T$.

Using (4) and (5), and inserting (4a) in (1a), the SBS (1) is equivalent to the following SBS in the observation point of view

$$E\dot{x} = A^0 x + \sum_{i=1}^{m} u^i A^i x + Bu + D_{11} y_1 + D_{12} w_2 \tag{6a}$$

$$y_1 = C_1^0 x + \sum_{i=1}^{m} u^i C_1^i x + w_1 \tag{6b}$$

$$y_{21} = C_{21}^0 x + \sum_{i=1}^{m} u^i C_{21}^i x \tag{6c}$$

$$y_{22} = C_{22}^0 x \tag{6d}$$

where $A^i = \widetilde{A}^i - D_{11} C_1^i$ for $i = 0, \ldots, m$ and $D_1 V_2 = \left[\begin{smallmatrix} D_{11} & D_{12} \end{smallmatrix}\right]$. We have the following vector dimensions in the SBS (6) : $w_1 \in \mathbb{R}^{q_1}$, $w_2 \in \mathbb{R}^{q_2}$ $(q = q_1 + q_2)$, $y_1 \in \mathbb{R}^{q_1}$, $y_{21} \in \mathbb{R}^{\overline{p}}$ and $y_{22} \in \mathbb{R}^v$ $(p = q_1 + \overline{p} + v)$. Using (3b) and (3c), matrices D_{12} and C_{22}^0 are of full column and full row rank, respectively.

The aim of this paper is to design a reduced order UIO for the SBS (6) with the following structure

$$\begin{aligned} \dot{z} \quad &= \quad N(u) z + L_1^0 y_1 + L_{21}^0 y_{21} + L_{22}^0 y_{22} + Gu \\ &\qquad + \sum_{i=1}^{m} u^i \left(L_1^i y_1 + L_{21}^i y_{21} + L_{22}^i y_{22} \right) \end{aligned} \tag{7a}$$

$$\hat{x} \quad = \quad M z + J_{22} y_{22} \tag{7b}$$

where $z \in \mathbb{R}^\ell$ $(n = \ell + v)$ and $\hat{x} \in \mathbb{R}^n$, with

$$N(u) = N^0 + \sum_{i=1}^{m} u^i N^i. \tag{8}$$

The problem of the UIO design is reduced to find matrices N^i, L_1^i, L_{21}^i, L_{22}^i $(i = 0, \ldots, m)$, G, M and J_{22} of appropiate dimensions such that the reconstruction error $e = \hat{x} - x$ is exponentially stable irrespective of the initialisations $x(0)$ and $z(0)$, the control input u and the unmeasurable disturbance w.

The observer (7) can be expressed in terms of the measured output y of the initial SBS (1) as follows

$$\dot{z} = N(u)z + L^0 y + Gu + \sum_{i=1}^{m} u^i L^i y \tag{9a}$$

$$\hat{x} = Mz + Jy \tag{9b}$$

where (see (4) and (5))

$$L^i = \left[\, L_1^i \,\vdots\, L_{21}^i \quad L_{22}^i \,\right] \begin{bmatrix} I & 0 \\ 0 & V_3^T \end{bmatrix} V_1^T \;\; i = 0, \dots, m, \tag{10a}$$

$$J = \left[\, 0 \,\vdots\, 0 \quad J_{22} \,\right] \begin{bmatrix} I & 0 \\ 0 & V_3^T \end{bmatrix} V_1^T. \tag{10b}$$

The following assumption is made for the UIO design.

Assumption A1 The relation

$$\text{rank} \begin{bmatrix} E^T & {C_{22}^0}^T \end{bmatrix}^T = n \tag{11}$$

holds. ❍

3 Reduced order UIO design

Define the reconstruction error e as the difference between the estimated and actual states as follows

$$e = \hat{x} - x = Mz - (I - J_{22} C_{22}^0)x. \tag{12}$$

As the measurement matrix C_{22}^0 is of full row rank, there exists $T \in \mathbb{R}^{\ell \times r}$ such that $\det \left[\begin{smallmatrix} TE \\ C_{22}^0 \end{smallmatrix}\right] \neq 0$, and the observer matrices M and J_{22} can be computed as follows

$$\begin{bmatrix} TE \\ C_{22}^0 \end{bmatrix} \begin{bmatrix} M & J_{22} \end{bmatrix} = \begin{bmatrix} M & J_{22} \end{bmatrix} \begin{bmatrix} TE \\ C_{22}^0 \end{bmatrix} = I. \tag{13}$$

Inserting (13) in (7b) yields to

$$z = TE\hat{x}, \tag{14a}$$

$$y_{22} = C_{22}^0 \hat{x}. \tag{14b}$$

Defining the error $\bar{e}$ as follows

$$\bar{e} = z - TEx, \tag{15}$$

and using (12), (14) and (15), the reconstruction error e can be expressed in terms of the error $\bar{e}$ as follows

$$e = M\bar{e}. \tag{16}$$

Since the observer matrix M is of full column rank (see (13)), we have

$$\lim_{t \to \infty} e(t) = 0 \Longleftrightarrow \lim_{t \to \infty} \bar{e}(t) = 0 \tag{17}$$

irrespective of the initialisations $x(0)$ and $z(0)$, the control input u and the unmeasurable disturbance w. Then the convergence of the UIO (7) (or (9)) can be studied equivalently with the errors e or $\bar{e}(t)$.

The error $\bar{e}(t)$ has the following dynamics

$$\begin{aligned} \dot{\bar{e}} = {} & N(u)\bar{e} + \left(N^0 TE - TA^0 + L_{21}^0 C_{21}^0 + L_{22}^0 C_{22}^0\right) x \\ & + (G - TB)u + \left(L_1^0 - TD_{11}\right) y_1 - TD_{12} w_2 + \sum_{i=1}^{m} u^i L_1^i y_1 \\ & + \sum_{i=1}^{m} u^i \left(N^i TE - TA^i + L_{21}^0 C_{21}^i + L_{22}^i C_{22}^0\right) x \\ & + \sum_{i=1}^{m} u^i L_{21}^i \left(C_{21}^0 + \sum_{j=1}^{m} u^j C_{21}^j\right) x. \end{aligned} \tag{18}$$

Using (18), the solution of the reduced order UIO for SBS (6) (or (6)) is given by the following theorem.

Theorem 1. *Under assumption A1, the system (7) (or (9)) is an exponential reduced order UIO for the SBS (6) (or (1)) if there exist matrices N^i, L_1^i, L_{21}^i, L_{22}^i $(i = 0, \dots, m)$, G, M, J_{22}, T and $Q = Q^T > 0$ of appropiate dimensions and a real $\mu > 0$ satisfying the following constraints for all admissible inputs $u \in \Omega$*

$$N^i TE - TA^i + L_{21}^0 C_{21}^i + L_{22}^i C_{22}^0 = 0, \;\; i = 0, \dots, m \tag{19a}$$

$$TD_{12} = 0 \tag{19b}$$

$$\begin{bmatrix} TE \\ C_{22}^0 \end{bmatrix} \begin{bmatrix} M & J_{22} \end{bmatrix} = \begin{bmatrix} M & J_{22} \end{bmatrix} \begin{bmatrix} TE \\ C_{22}^0 \end{bmatrix} = I \tag{19c}$$

$$G = TB \tag{19d}$$

$$L_1^0 = TD_{11} \tag{19e}$$

$$L_1^i = 0, \;\; i = 1, \dots, m \tag{19f}$$

$$L_{21}^i = 0, \;\; i = 1, \dots, m \tag{19g}$$

$$N^T(u)Q + QN(u) + \mu I < 0. \tag{19h}$$

■

Proof. Notice that the constraints (19f) and (19g) avoid to have products $u^i y_1$ and $u^i u^j x$ in dynamic equation (18). Since assumption A1 holds, then there exists a full row rank matrix $T \in \mathbb{R}^{\ell \times r}$ and full column rank matrices M and J_{22} verifying constraint (19c). Then relation (17) between the reconstruction error e (12) and the error $\bar{e}$ (15) holds. Inserting constraints (19a)-(19g) in (18) leads to the following error dynamics

$$\dot{\bar{e}} = N(u)\bar{e}. \tag{20}$$

Consider the following Lyapunov function candidate $V(\bar{e}) = \bar{e}^T Q \bar{e}$ where $Q = Q^T > 0$. The time derivative of this Lyapunov function along the trajectory of (20) is given by

$$\dot{V}(\bar{e}, u) = \bar{e}^T (N^T(u)Q + QN(u))\bar{e}. \tag{21}$$

If constraint (19h) holds, the error $\bar{e}$ is quadratically stable and we have

$$\dot{V}(\bar{e}, u) < -\mu \bar{e}^T \bar{e} \leqslant \frac{-\mu}{\lambda_{\max}(Q)} V(\bar{e}) \qquad \forall u \in \Omega. \tag{22}$$

Then the error $\bar{e}$ (15) is exponentially stable, i.e.

$$\|\bar{e}(t)\| \leqslant \sqrt{\frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)}} \|z(0) - TEx(0)\| \exp\!\left(\frac{-\mu t}{2\lambda_{\max}(Q)}\right) \tag{23}$$

for all admissible $u \in \Omega$. •

Notice that $\dot{V}(\bar{e}, u)$ is uniformly bounded with respect to u and the proposed observer works for non-uniformly observable systems as well as the control input u belongs to Ω, even if u is a non-universal input (see [9]).

3.1 Unknown inputs decoupling

In theorem 1, it is shown that the error $\overline{e}$ (15) is decoupled from the unmeasurable disturbances w if constraints (19a)-(19c) hold. Since $\det\left[\begin{smallmatrix} M & J_{22}\end{smallmatrix}\right] \neq 0$ (see (19c)), equation (19a) is equivalent to the following set of relations

$$N^i = TA^i M - L_{21}^0 C_{21}^i M \qquad i = 0, \ldots, m, \tag{24a}$$

$$L_{22}^i = TA^i J_{22} - L_{21}^0 C_{21}^i J_{22} \qquad i = 0, \ldots, m. \tag{24b}$$

Define the following non-singular matrix

$$\begin{bmatrix} R \\ C_{22}^0 \end{bmatrix} = \begin{bmatrix} I & \Phi \\ 0 & I \end{bmatrix} \begin{bmatrix} TE \\ C_{22}^0 \end{bmatrix} \tag{25}$$

where $\Phi \in \mathbb{R}^{\ell \times v}$ is arbitrary and $R \in \mathbb{R}^{\ell \times n}$ of full row rank. Then equations (19b) and (25) can be written as

$$\begin{bmatrix} T & \Phi \end{bmatrix} \begin{bmatrix} E & D_{12} \\ C_{22}^0 & 0 \end{bmatrix} = \begin{bmatrix} R & 0 \end{bmatrix}. \tag{26}$$

The solutions N^i, L_{21}^0, L_{22}^i $(i = 0, \ldots, m)$, M and J_{22} of constraints (19a)-(19c) depend on the solutions T and Φ of equation (26). The existence of these solutions is given by the following lemma.

Lemma 1. *There exist solutions T and Φ of equation (26) if and only if*

$$\operatorname{rank} \begin{bmatrix} E & D_{12} \\ C_{22}^0 & 0 \end{bmatrix} = n + \operatorname{rank} D_{12}. \tag{27}$$

■

Proof. The proof is straightforward and is omitted. •

Notice that condition (27) implies assumption A1.

By using (19c) and (25), M and J_{22} are given by

$$M = \begin{bmatrix} R \\ C_{22}^0 \end{bmatrix}^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix}, \tag{28a}$$

$$J_{22} = \begin{bmatrix} TE \\ C_{22}^0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{28b}$$

From relation (26), we have

$$\begin{bmatrix} T & \Phi \end{bmatrix} \begin{bmatrix} E \\ C_{22}^0 \end{bmatrix} = R, \tag{29}$$

and the solution (29) is given by

$$\begin{bmatrix} T & \Phi \end{bmatrix} = R \begin{bmatrix} E \\ C_{22}^0 \end{bmatrix}^{\dagger} + Z \left(I - \begin{bmatrix} E \\ C_{22}^0 \end{bmatrix} \begin{bmatrix} E \\ C_{22}^0 \end{bmatrix}^{\dagger} \right) \tag{30}$$

where Z is an arbitrary matrix of appropriate dimensions and $A^{\dagger}$ is a generalised inverse of matrix A satisfying $AA^{\dagger}A = A$ [25]. The generalised inverse in (30) can be chosen as follows (see assumption A1 or relation(27))

$$\begin{bmatrix} E \\ C_{22}^0 \end{bmatrix}^{\dagger} = \Psi \begin{bmatrix} E^T & {C_{22}^0}^T \end{bmatrix} \tag{31}$$

where

$$\Psi = (E^T E + {C_{22}^0}^T C_{22}^0)^{-1}, \tag{32}$$

and matrix T is given by

$$T = R\Psi E^T + Z\varphi \tag{33}$$

where

$$\varphi = \begin{bmatrix} I - E\Psi E^T \\ -C_{22}^0 \Psi E^T \end{bmatrix}. \tag{34}$$

Inserting (33) in (19b), we obtain

$$Z\varphi D_{12} = -R\Psi E^T D_{12}. \tag{35}$$

There exists a solution Z of equation (35) if matrix φD_{12} is of full column rank, i.e. if condition (27) in lemma 1 holds. This can be seen with the following relation

$$\begin{aligned} \operatorname{rank} \begin{bmatrix} E & D_{12} \\ C_{22}^0 & 0 \end{bmatrix} &= \operatorname{rank} \begin{bmatrix} \Psi \left[\begin{smallmatrix} E^T & {C_{22}^0}^T \end{smallmatrix}\right] \\ \left(I - \left[\begin{smallmatrix} E \\ C_{22}^0 \end{smallmatrix}\right] \Psi \left[\begin{smallmatrix} E^T & {C_{22}^0}^T \end{smallmatrix}\right] \right) \end{bmatrix} \\ &\qquad \times \begin{bmatrix} E & D_{12} \\ C_{22}^0 & 0 \end{bmatrix} \\ &= \operatorname{rank} \begin{bmatrix} I & \Psi E^T D_{12} \\ 0 & \varphi D_{12} \end{bmatrix}. \end{aligned} \tag{36}$$

The solution Z is given by

$$Z = -R\Psi E^T D_{12} \left(\varphi D_{12}\right)^{\dagger} + \overline{Z} \left(I - \varphi D_{12} \left(\varphi D_{12}\right)^{\dagger} \right) \tag{37}$$

where $\overline{Z}$ is an arbitrary matrix of appropriate dimensions.

3.2 Exponential stability

We use an LMI approach to determine matrices N^i and Q such that (19h) in theorem 1 holds. Consider each u^i in (18) (or (20)) as an "structured uncertainty". Notice that the definition of the "uncertainty set" Ω in relation (2) can leads to some conservatism to solve (19h) (see [26]) since, in the general case, we have $|u_{\min}^i| \neq |u_{\max}^i|$ with $|u_{\min}^i| \neq 1$ and $|u_{\max}^i| \neq 1$. To overcome this conservatism, each u^i can be rewritten as follows

$$u^i(t) = \alpha^i + \sigma^i \varepsilon^i(t) \tag{38}$$

where $\alpha^i \in \mathbb{R}$ and $\sigma^i \in \mathbb{R}$ are given by (for $i = 1, \ldots, m$)

$$\alpha^i = \frac{1}{2}(u_{\min}^i + u_{\max}^i), \; \sigma^i = \frac{1}{2}(u_{\max}^i - u_{\min}^i), \tag{39}$$

$\alpha^0 = 1$ and $\sigma^0 = 0$. The new "uncertain" variable is $\varepsilon \in \overline{\Omega} \subset \mathbb{R}^m$ where the polytope $\overline{\Omega}$ is defined as

$$\overline{\Omega} := \{ \varepsilon \in \mathbb{R}^m \mid \varepsilon_{\min}^i = -1 \leqslant \varepsilon^i \leqslant \varepsilon_{\max}^i = 1 \text{ for } i = 1, \ldots, m \}. \tag{40}$$

By using relations (38)-(40), the dynamics of the error $\overline{e}$ can be rewritten as follows

$$\begin{aligned} \dot{\overline{e}} = {} & \sum_{i=0}^{m} \alpha^i N^i \overline{e} + \sum_{i=1}^{m} \sigma^i \varepsilon^i N^i \overline{e} + (G - TB)u + \left(L_1^0 - TD_{11}\right) y_1 \\ & + \sum_{i=1}^{m} u^i L_1^i y_1 - TD_{12} w_2 + \sum_{i=1}^{m} u^i L_{21}^i \left(C_{21}^0 + \sum_{j=1}^{m} u^j C_{21}^j \right) x \\ & + \sum_{i=0}^{m} \alpha^i \left(N^i TE - TA^i + L_{21}^0 C_{21}^i + L_{22}^i C_{22}^0 \right) x \end{aligned}$$

$$+\sum_{i=1}^{m}\sigma^i\varepsilon^i\left(N^iTE-TA^i+L_{21}^0C_{21}^i+L_{22}^iC_{22}^0\right)x. \quad (41)$$

By inserting (33) and (37) in (24a), we obtain

$$\mathcal{N}=\sum_{i=0}^{m}N^i=\mathcal{A}-\mathcal{K}\mathcal{C}=T\left(\sum_{i=0}^{m}A^iM\right)-L_{21}^0\left(\sum_{i=0}^{m}C_{21}^iM\right)(42)$$

where the matrices $\mathcal{A}$, $\mathcal{K}$ and $\mathcal{C}$ are given by

$$\begin{aligned}\mathcal{A} &= \sum_{i=0}^{m}\alpha^iR\Psi E^T\left(I-D_{12}(\varphi D_{12})^\dagger\varphi\right)A^iM, && (43a)\\ \mathcal{K} &= \begin{bmatrix}K_1^0 & K_2^0 & K_2^1 & \dots & K_2^m\end{bmatrix}, && (43b)\\ \mathcal{C} &= \Big[\ \textstyle\sum_{i=0}^{m}\alpha^iM^T{C_{21}^i}^T \quad \alpha^0M^T{A^0}^T\widetilde{\varphi}^T \quad \alpha^1M^T{A^1}^T\widetilde{\varphi}^T \\ &\qquad \dots \quad \alpha^mM^T{A^m}^T\widetilde{\varphi}^T\ \Big]^T && (43c)\end{aligned}$$

with

$$\widetilde{\varphi}=\left(I-\varphi D_{12}(\varphi D_{12})^\dagger\right)\varphi \qquad (44)$$

and

$$L_{21}^0=K_1^0. \qquad (45)$$

By using (33), (37), (42) and (43a)-(43c), the matrix T given by (33) can be rewritten as

$$T=R\Psi E^T\left(I-D_{12}(\varphi D_{12})^\dagger\varphi\right)+\overline{Z}\widetilde{\varphi} \qquad (46)$$

where

$$\begin{aligned}\overline{Z}=&\left(\mathcal{N}+L_{21}^0\left(\sum_{i=0}^{m}C_{21}^iM\right)-R\Psi E^T\left(I-D_{12}(\varphi D_{12})^\dagger\varphi\right)\right.\\ &\left.\times\left(\sum_{i=0}^{m}A^iM\right)\right)\left(\widetilde{\varphi}\left(\sum_{i=0}^{m}A^iM\right)\right)^\dagger. \qquad (47)\end{aligned}$$

Inserting (19a)-(19g) and (42) in (41) yields to

$$\dot{\overline{e}}=(\mathcal{N}+H_1\Delta(\varepsilon)H_2)\overline{e} \qquad (48)$$

where H_1, $\Delta(\varepsilon)\in\mathbb{R}^{m\ell\times m\ell}$ and $H_2\in\mathbb{R}^{m\ell\times\ell}$ are given by

$$\begin{aligned}H_1&=\begin{bmatrix}\sigma^1N^1 & \cdots & \sigma^mN^m\end{bmatrix} && (49a)\\ \Delta(\varepsilon)&=\operatorname{bdiag}(\varepsilon^iI) \qquad i=1,\dots,m && (49b)\\ H_2&=\begin{bmatrix}I & \dots & I\end{bmatrix}^T && (49c)\end{aligned}$$

and bdiag(.) denotes a bock-diagonal matrix. From (40), the matrix $\Delta(\varepsilon)$ is bounded as

$$\left\|\Delta(\varepsilon)\right\|\leqslant 1. \qquad (50)$$

By using constraints (19a)-(19g) and relations (42)-(50), the exponential stability of the error $\overline{e}$ (or e) is guaranteed if inequality (19h) in theorem 1 holds, as can be shown in the following lemma.

Lemma 2. *If constraints (19a)-(19g) in theorem 1 holds, then the system (7) (or (9)) is an exponential reduced order UIO for the SBS (6) (or (1)) if there exist* $Q=Q^T>0$, $S=S^T>0$, $\mathcal{Y}$ *and* $\mu>0$ *such that the following LMI*

$$\mathcal{S}=\left[\begin{array}{c:c}\mathcal{A}^TQ+Q\mathcal{A}-\mathcal{C}^T\mathcal{Y}-\mathcal{Y}^T\mathcal{C}+H_2^TSH_2+\mu I & \bullet\\ \hdashline \widetilde{\mathcal{A}}^TQ-\widetilde{\mathcal{C}}^T\mathcal{Y} & -S\end{array}\right]<0 \quad (51)$$

holds where "•" is the transpose of the off-diagonal part and

$$\begin{aligned}\mathcal{Y}&=\mathcal{K}^TQ, && (52a)\\ \widetilde{\mathcal{A}}&=R\Psi E^T\left(I-D_{12}(\varphi D_{12})^\dagger\varphi\right)[\sigma^1A^1M\dots\sigma^mA^mM], && (52b)\end{aligned}$$

$$\widetilde{\mathcal{C}}=\begin{bmatrix}\sigma^1C_{21}^i & \sigma^2C_{21}^2 & \dots & \sigma^mC_{21}^m\\ 0 & 0 & \dots & 0\\ \sigma^1\widetilde{\varphi}A^1M & 0 & \dots & 0\\ 0 & \sigma^2\widetilde{\varphi}A^2M & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \dots & 0 & \sigma^m\widetilde{\varphi}A^mM\end{bmatrix}. \quad (52c)$$

■

Proof. Assume that constraints (19a)-(19g) in theorem 1 holds. Then, by replacing u^i by ε^i and using relations (38)-(40), the error dynamics (41) can be written as (48) and relation (50) holds with $\Delta(\varepsilon)$ given by (49b).

Let $\overline{V}(\overline{e})=\overline{e}^T\overline{Q}\overline{e}$ be a Lyapunov function candidate with $\overline{Q}=\overline{Q}^T>0$). The error $\overline{e}$ is exponentially stable for all ε in the polytope $\overline{\Omega}$ if the time derivative of $\overline{V}(\overline{e})$ along the trajectory of (48) satisfies

$$\begin{aligned}\dot{\overline{V}}(\overline{e},\varepsilon)+\overline{\mu}\overline{e}^T\overline{e}=&\,\overline{e}^T\left(\mathcal{N}\overline{Q}+\overline{Q}\,\mathcal{N}^T\right.\\ &\left.+H_1\Delta(\varepsilon)H_2\overline{Q}+\overline{Q}H_2^T\Delta(\varepsilon)H_1^T+\overline{\mu}I\right)\overline{e}<0 \quad (53)\end{aligned}$$

where $\overline{\mu}>0$ is a given real.

Now defining $\overline{Q}=Q^{-1}$ and choosing a real $\mu>0$ such that $\mu I>\overline{\mu}Q^2$, inequality (53) can be rewritten as the following LMI

$$\mathcal{S}=\left[\begin{array}{c:c}Q\mathcal{N}+\mathcal{N}^TQ+H_2^TSH_2+\mu I & QH_1\\ \hdashline H_1^TQ & -S\end{array}\right]<0 \quad (54)$$

by sing the Schur lemma [26], where $S=S^T>0$ must be chosen such that [26]

$$S\Delta(\varepsilon)=\Delta(\varepsilon)S \qquad (55)$$

in order to take the structure of $\Delta(\varepsilon)$ into account. From the structure of $\Delta(\varepsilon)$ given by eq-deltar, relation (55) holds for all matrices $S=S^T>0$ of appropriate dimensions.

Using and the notations introduced in (42)-(49c) and (52), LMI (54) is equivalent to LMI (51). If inequality (51) holds, it is easy to se that constraint (19h) in theorem 1 is verified and the error dynamics (20) is exponentially stable. Since LMI (51) has an affine dependence on variable Q and $\mathcal{Y}$, then the solution and the feasibility of this LMI can be performed with convex optimisation methods [26]. •

4 Synthesis of the reduced order UIO

The design procedure of the exponential reduced order UIO (7) (or 9)) for the SBS (6) (or (1)) can be summarised as follows.

Exponential reduced order UIO

- **(a)** Find V_1, V_2 and V_3 to transform the SBS (6) into (1) (see (4) and (5)).
- **(b)** Check condition (27) in lemma 1.
- **(c)** Set $L_1^i = 0$ and $L_{21}^i = 0$ $(i = 1, \ldots, m)$.
- **(d)** Compute Ψ, φ and $\tilde{\varphi}$ with (32), (34) and (44).
- **(e)** Choose R such that $\det \begin{bmatrix} R \\ C_{22}^0 \end{bmatrix} \neq 0$ and compute M with (28a).
- **(f)** Compute α_i and σ_i $(i = 0, \ldots, m)$ with (39), and $\mathcal{A}$, $\mathcal{C}$, H_2, $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{C}}$ with (43a), (43c), (49c), (52b) and (52c).
- **(g)** Choose $\mu > 0$, and compute the solutions Q and $\mathcal{Y}$ of LMI (51).
- **(h)** Check if $Q = Q^T > 0$, then go to step ***(i)***, else go to step ***(g)*** or to step ***(e)***.
- **(i)** Compute $\mathcal{K}$ given in (43b) with (52a) and $\mathcal{N}$ with (42).
- **(j)** Compute $\overline{Z}$, T and J_{22} with (47), (46) and (28b).
- **(k)** Compute G, L_1^0, N^i and L_{22}^i $(i = 0, \ldots, m)$ with (19d), (19e), (24a) and (24b), and set $L_{21}^0 = K_1^0$.
- **(l)** To obtain the UIO (9) from the UIO (7), compute L^i $(i = 0, \ldots, m)$ and J with (10a) and (10b).

5 Conclusion

The objective of this paper is the design of exponential reduced order UIO for SBS with bilinear measurement and subjected to bounded control inputs and unmeasurable disturbances. No assumption is made on the size of the matrix E. The existence condition of this UIO (see lemma 1) yields to an observation error dynamics which is decoupled from the unmeasurable disturbances. Considering the bounded control inputs as "structured uncertainties", the exponential stability of the reconstruction error is guaranteed by solving an LMI associated to a robust stabilisation problem (see lemma 2). To reduce the conservatism inherent to the robust control theory, a control inputs change of coordinates has been made.

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