CDC00-REG2038
Invertibility and Inversion for Systems over Rings and Applications to Delay-differential Systems

G. Conte
Dipartimento di Elettronica e Automatica, Universita' di Ancona
Via Brecce Bianche, 60131 Ancona, Italy
gconte@popcsi.unian.it

A. M. Perdon
Dipartimento di Matematica "V. Volterra", Universita' di Ancona
Via Brecce Bianche, 60131 Ancona, Italy
perdon@popcsi.unian.it

Abstract The notion of invertibility for systems with coefficients in a ring is investigate. An algebraic notion of relative degree is introduced and used to characterize the invertibility of a SISO system, while a geometric characterization of relative degree is given for the MIMO case. Application to the study of delay differential systems are presented.

1 INTRODUCTION

Systems with coefficients in a ring have can be viewed as a generalization of classical dynamical systems and have been introduced for modeling e.g. integer coefficients dynamical systems as well as for dealing with families of discrete-time, parameter-dependent systems. Their study is receiving an increasing attention as they prove to be an efficient tool for studying and solving design problems concerning continuous-time, delay-differential systems (see [7] and the references therein). The aim of this paper is to discuss and to investigate, by means of algebric and geometric methods, the concept of invertibility and, in particular, the related notions of relative degree and functional controllability for systems with coefficients in a ring. The main motivation for this study, besides a better understanding of system theoretic properties in the chosen framework, is in developing design and control tools for delay-diferential systems based on an algebraic and geometric approach which avoids the use of infinite dimensional state space models. System Invertibility is defined in Section 2, Where also an algebraic notion of Relative Degree for SISO systems is introduced. Examples show how the Relative Degree can be used for characterizing invertibility of delay-differential systems according to the chosen context.

In Section 3, Right Invertibility of MIMO system is analysed and related to the property of Functional Controllability. This allows to give a geometric characterization of Right Invertibility and a notion of Relative Degree for MIMO systems.

In Section 4 it is shown that the delay in invertible SISO delay-differential systems can be eliminated, working basically in the ring framework, by a suitable feedback. In addition, a procedure for constructing right inverses of MIMO delay-differential systems based on the solution of a Disturbance Decoupling Problem in the ring framework is illustrated. Such inverses can then be used, in case they turn out to be stable, for solving tracking and trajectory following problems.

2 SYSTEM INVERTIBILITY AND SISO RELATIVE DEGREE

Given a commutative ring R, by a dynamical system Σ over R we mean the object described by a set of equations of the form

$$\left\{ \begin{array}{rcl} x(t+1) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) \end{array} \right. \tag{1}$$

where x belongs to the free state module $\mathcal{X} = R^n$, u belongs to the free input module $\mathcal{U} = R^m$, y belongs to the free output module $\mathcal{Y} = R^p$ and A, B, C are matrices of suitable dimensions with entries in R.

Example 1 *The key example we have in mind is the following one. Consider the generic continuous time, delay-differential system Σ_d with real coefficients given by the equations*

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & A_0 x(t) + \sum_{k=1}^{r} A_i x(t-k\delta) + \\ & & B_0 u(t) + \sum_{k=1}^{r} B_k u(t-k\delta) \\ y(t) & = & C_0 x(t) + \sum_{k=1}^{r} C_k x(t-k\delta) \end{array} \right. \tag{2}$$

By introducing the delay operator Δ, defined for any time function $f(t)$ by $\Delta f(t) = f(t-\delta)$, we can formally associate to Σ_d a system Σ over the ring of polynomials $\mathcal{R}[\Delta]$ given by the equations

$$\left\{\begin{array}{rcl} x(t+1) & = & (A_0 + \sum_{k=1}^{r} A_k \Delta_k)x(t) + \\ & & (B_0 + \sum_{k=1}^{r} B_k \Delta^k)u(t) \\ y(t) & = & (C_0 + \sum_{k=1}^{r} C_k \Delta^k)x(t) \end{array}\right. \tag{3}$$

Note that the variables in equations (2) and in equations (3) have quite different meaning and the systems Σ_d and Σ are different objects, nevertheless they have the same signal flow graph. Then, many dynamic properties of the first system can be studied by analysing the second one. In case uncommensurable delays are present, a similar procedure gives rise to a system over a ring of polynomials in more than one variables.

Given a dynamical system Σ of the form (1), Left Invertibility consists, from a general point of view, in the possibility of reconstructing univocally, by means of a dynamical process, the input that has produced a given output. On the other hand, Right Invertibility means the possibility of obtaining any output sequence with entries in $\mathcal{Y}$ by a suitable choice of the input. Working with coefficients in a ring, a weaker notion than Left Invertibility deserves also to be considered. More precisely, we can state the following definition.

Definition 1 *A dynamical system Σ of the form 1 is said* **Left Invertible** *if, for some k_0, there exists a system Σ' over R, with input module $\mathcal{Y}$ and output module $\mathcal{U}$, of the general form*

$$\left\{\begin{array}{l} z(t+1) = A'z(t) + B'y(t+k_0) \\ u(t) = C'z(t) + Dy(t+k_0) \end{array}\right. \tag{4}$$

having the following property: for any input/output pair $\{u(t), y(t)\}_{t\geq 0}$ of Σ, the system Σ', suitably initialized and fed with $\{y(t)\}_{t\geq 0}$, gives in response the sequence $\{u(t)\}_{t\geq 0}$.

Definition 2 *A dynamical system Σ of the form (1) is said* **Right Invertible** *if , for some k_0, there exists a system Σ' over R, with input module $\mathcal{Y}$ and output module $\mathcal{U}$, of the general form (4), having the following property: for any sequence $\{y(t)\}_{t\geq 0}, y(t) \in \mathcal{Y}$ the response $\{u(t)\}_{t\geq 0}$ of Σ' to $\{y'(t)\}_{t\geq 0}$, with $y'(t) = 0$ for $t < k_0$ and $y'(t) = y(t-k_0)$ for $t \geq k_0$, is such that $\{u(t), y'(t)\}_{t\geq 0}$ is an input/output pair for Σ.*

Definition 3 *A dynamical system Σ of the form (1) is said* **Injective** *if different input sequences $\{u(t)\}_{t\geq 0}$ and $\{u'(t)\}_{t\geq 0}$, for initial state $x(0) = 0$, produce respectively different output sequences $\{y(t)\}_{t\geq 0}$ and $\{y'(t)\}_{t\geq 0}$.*

Basic information about invertibility can be obtained, both in the case of linear and of nonlinear systems, by means of the so-called Inversion Algorithm ([16], [17]). For single input-single output (SISO) systems with coefficients in a ring, it is possible to extend in a strightforward way the classical Silverman Inversion Algorithm ([16]).To this aim, given a system Σ of the form (1) with $m = p = 1$ and writing accordingly $\Sigma = (A, b, c)$, let us evaluate recursively $y(t+k)$, for $k \geq 1$. Since $y(t+k) = cA^k x(t) + \sum_{i=0}^{k-1} cA^{k-i-1}bu(t+i)$, either $cA^{k-1}b = 0$ for all $k \geq 1$, or there exists k_0 (necessarily lesser than or equal to $dim\ \mathcal{X}$) such that $cA^{k-1}b = 0$ for $k < k_0$ and $cA^{k_0-1}b \neq 0$. In the last case the algorithm stops at step k_0, yielding

$$\begin{array}{l} y(t+1) = cAx(t) \\ y(t+2) = cA^2x(t) \\ \vdots \\ y(t+k_0) = cA^{k_0}x(t) + cA^{k_0-1}bu(t). \end{array} \tag{5}$$

It is clear now, that Σ is invertible if the coefficient $cA^{k_0-1}b$ is an invertible element of the ring R. In this case, an inverse Σ' for Σ is given by the following equations

$$\left\{\begin{array}{rcl} z(t+1) & = & (A + b(cA^{k_0-1}b)^{-1}cA^{k_0})z(t) + \\ & & +b(cA^{k_0-1}b)^{-1}cy(t+k_0) \\ u(t) & = & (cA^{k_0-1}b)^{-1}cA^{k_0}z(t) + \\ & & +(cA^{k_0-1}b)^{-1}cy(t+k_0) \end{array}\right. \tag{6}$$

In facts, initialised at $z(0) = x(0)$ Σ' gives, in response to any output sequence $\{y(t)\}_{t\geq 0}$ of Σ the input sequence $\{u(t)\}_{t\geq 0}$ of Σ that produced $\{y(t)\}_{t\geq 0}$. On the other hand, chosen any sequence $\{y(t)\}_{t\geq 0}$, the system Σ', initialized at $z(0) = 0$ and fed with $\{y'(t)\}_{t\geq 0}$, where $y'(t) = 0$ for $t < k_0$ and $y'(t) = y(t-k_0)$ for $t \geq k_0$, gives in response a sequence $\{u(t)\}_{t\geq 0}$ which, as input of Σ, produces the output $\{ytk)\}_{t\geq 0}$. Motivated by the above considerations and paralleling the case of linear systems with coefficients in a field, we can now state the following Definitions.

Definition 4 *Given the single input-single output system $\Sigma = (A, b, c)$ with coefficients in the ring R, assume that there exists k_0 such that $cA^{k-1}b = 0$ for $k < k_0$ and $cA^{k_0-1}b \neq 0$. Then, we say that Σ has a finite relative degree equal to k_0. If, in addition, $cA^{k_0-1}b$ is an invertible element of R, we say that the relative degree is pure. Alternatively, if $cA^{k-1}b = 0$ for all $k \geq 1$, we say that Σ has no finite relative degree.*

The following Examples and Proposition explain why the notion of pure relative degree is important in the framework of systems with coefficients in a ring

Example 2 *Let us consider the system Σ described over $\mathcal{R}[\Delta]$, the ring of real polynomials in one inde-*

terminate, by the equations

$$\left\{\begin{array}{rcl} x_1(t+1) & = & \Delta x_1(t) - x_2(t) \\ x_2(t+1) & = & x_2(t) + u(t) \\ y(t) & = & (1+\Delta)x_1 \end{array}\right. \tag{7}$$

Evaluation of $y(t+k)$ for $k \geq 1$ yields

$$\begin{array}{rcl} y(t+1) & = & (1+\Delta)\Delta x_1 - (1+\Delta)x_2 \\ y(t+2) & = & (1+\Delta)\Delta^2 x_1(t) - (1+\Delta)^2 x_2(t) \\ & & -(1+\Delta)u(t) \end{array}$$

The relative degree of the system Σ is therefore $k = 2$. Since the polynomial $p(\Delta) = 1 + \Delta$ is not invertible in the ring, the relative degree is not pure. If we consider Σ as a system over the ring $\mathcal{R}_{rea} = S^{-1}\mathcal{R}[\Delta]$, where

$$S = \{p(\Delta) \in \mathcal{R}[\Delta] \text{ such that } p(0) \neq 0\},$$

namely the ring of rational functions in one indeterminate over the reals whose denominator has non zero constant part, then $1+\Delta$ is an invertible element, since $\dfrac{1}{1+\Delta}$ belongs to the ring $\mathcal{R}_{rea}$. In this case the relative degree $k = 2$ of Σ is pure. It turns out that an inverse of Σ can be constructed only if one admits systems with coefficients in $\mathcal{R}_{rea}$.

Proposition 1 *Let Σ be a SISO dynamical system of the form 1 over the ring R. Then, Σ is*
i) *injective if and only if it has a finite relative degree;*
ii) *left and right invertible if its relative degree is pure.*

Example 3 *Let us consider the system Σ defined over the ring of real polynomials in one indeterminate $\mathcal{R}[\Delta]$ by the equations*

$$\left\{\begin{array}{rcl} x_1(t+1) & = & u(t) \\ x_2(t+1) & = & \Delta x_2(t) + \Delta u(t) \\ x_3(t+1) & = & x_2(t) \\ y(t) & = & x_3(t) \end{array}\right. \tag{8}$$

Evaluation of $y(t+k)$ for $k \geq 1$ gives

$$\begin{array}{rcl} y(t+1) & = & x_2(t) \\ y(t+2) & = & \Delta x_2(t) + \Delta u(t) \end{array}$$

The relative degree of the system Σ is $k = 2$, but Δ is not an invertible element of $\mathcal{R}[\Delta]$. Hence Σ is injective but not invertible. The same conclusion hold also if we consider Σ as a system over the ring $\mathcal{R}_{rea}$.

Remark 1 *The basic idea which leads to the the definition of relative degree in this Section is conceptually similar to that already employed by Germani and coworkers in [9] and [10], where a notion of Delay Relative Degree for nonlinear delay-differential systems was introduced and studied. Actually, when applied to a linear delay-differential system Σ_d, the procedure employed for computing the Delay Relative Degree coincides with the steps of the Silverman Algorithm applied to the associated system Σ with coefficients in a ring. Nevertheless, the Delay Relative Degree of Σ_d turns out to be defined only in case Σ has a pure relative degree, in the sense of Definition 1, as a system over $\mathcal{R}_{rea}$. For example, comparing with the above Remark, one can see that the delay-differential system Σ_d given by the equations*

$$\left\{\begin{array}{rcl} \dot{x}_1(t) & = & u(t) \\ \dot{x}_2(t) & = & x_2(t-h) + u(t-h) \\ \dot{x}_3(t) & = & x_2(t) \\ y(t) & = & x_3(t) \end{array}\right. \tag{9}$$

has not a finite Delay Relative Degree in the sense of [9]. Moreover, the notion described in Definition 1 can also be applied, by considering an associated system over a suitable ring, to delay-differential systems with noncommensurable delays.

Let $\Sigma = (A, b, c)$ have relative degree k_0 and write $cA^{k-1}b = m_k$ for $k \geq k_0$. Then, the transfer function $T(z)$ of Σ is given by $T(z) = \sum_{k=k_o}^{\infty} m_k z^{-k}$. This and the similarity with the case of systems with coefficients in a field suggest the following Definition.

Definition 5 *The system $\Sigma = (A, b, c)$ with coefficients in the ring R is said to have a formal zero at infinity of order k_0 if it has relative degree k_0. In other terms, the formal zero structure of Σ at infinity is said to be $\{k_0\}$.*

3 RIGHT INVERTIBILITY, FUNCTIONAL CONTROLLABILITY AND MIMO RELATIVE DEGREE

For systems with coefficients in a ring having more than one input and one output the situation concerning invertibility is more complicated than that regarding SISO systems and a characterization based on the Silverman Inversion Algorithm is not easy. The difficulties arise from the fact that, essentially, the aim of the classical Algorithm is to select recursively a set of independent rows in the matrices $CB, CAB, \ldots, CA^{n-1}B$, where $n = dim\ \mathcal{X}$. The set of independent row vectors so obtained gives rise to a matrix which is full row rank, but, having entries in R, may not be invertible. In such case, the construction of an inverse cannot be performed. In this Section we will focus on Right Invertibility, for which a geometric characterization can be given, and will see that, in case invertion is possible, a notion of MIMO finite relative degree can be introduced. To begin with, let us first remark that in case a given system Σ of dimension n over the ring R is right invertible, it can follow any arbitrary output trajectory after, at most, n time instants. In other terms, right invertibility is equivalent to the discrete time concept of Functional Controllability as described, for instance, in [2]. In particular, a right invertible system can follow trajectories $y(t)$ whose components $y_j(t), j = 1, \ldots, m$ are identically zero for all $j \neq i$ and arbitrarily chosen for $j = i$ and $t \geq n$. It may however happen that output reference trajectories can be exactly followed after $t < n$ instants. We can therefore introduce the notion of relative degree of the i-th output as follows.

Definition 6 *Assume that the system (1) is functionally controllable. Then the relative degree of the i-th output component is the minimum r_i such that any arbitrary sequence $w = \{a_1, a_2, \ldots\}$ can be obtained at the i-th output channel after r_i instants, while mantaining all the other output components equal to zero.*

A definition of relative degree for the overall system (1), which agrees with the one given in Section 2 for SISO systems, can now be given as follows.

Definition 7 *Assume that the system (1) is functionally controllable. Then its relative degree is the vector $r_\Sigma = (r_1, r_2, \ldots, r_p)$, where r_i is the relative degree of the i-th output.*

Example 4 *Consider the system*

$$\left\{ \begin{array}{rcl} x_1(t+1) & = & u_1(t) \\ x_2(t+1) & = & x_3(t) + u_1(t) \\ x_3(t+1) & = & u_2(t) \\ y_1(t) & = & x_1(t) \\ y_2(t) & = & x_2(t) \end{array} \right. \tag{10}$$

Taking as initial conditions $x(0) = 0$ and $u(t) = 0$ for $t \leq 0$, we have that any sequence $w = \{a_1, a_2, \ldots\}$ can be obtained at the output y_1 for $t \geq 2$ while mantainig y_2 identically zero by the control input $u_1(1) = 0, u_1(1+t) = a_t$ for $t \geq 1$ and $u_2(t) = -a_t$ for $t \geq 1$. Then the relative degree of the first output is $r_1 = 2$. Analogously we can see that any sequence $w = \{b_1, b_2, \ldots\}$ can be obtained at the output y_2 for $t \geq 2$ while mantainig y_1 identically zero by the control input $u_1(t) = 0$ for $t \geq 1$ and $u_2(t) = b_t$ for $t \geq 1$. Then the relative degree of the second output is $r_2 = 2$, and the relative degree of the overall system is $r_\Sigma = (2, 2)$.

We can now point out, as illustrated in [14] in the context of continuous time linear systems with coefficients in a field, that functional controllability of a given system Σ is equivalent to solvability of a specific Disturbance Decoupling Problem for a suitable extension of Σ. To show this, given the system Σ of the form (1), let us consider the extended system Σ_e of dimension $n + p(n+1)$ defined by the equations

$$\left\{ \begin{array}{rcl} x_e(t+1) & = & A_e x_e(t) + B_e u(t) + D_e q(t) \\ y(t) & = & C_e x_e(t) \end{array} \right. \tag{11}$$

where q is viewed as a disturbance and the matrices A_e, B_e, C_e, D_e are defined as follows: $A_e = \begin{bmatrix} A & 0 & \ldots & 0 \\ 0 & S & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & S \end{bmatrix}$, $B_e = \begin{bmatrix} B \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $C_e = \begin{bmatrix} C_1 & Q & 0 & \ldots & 0 \\ C_2 & 0 & Q & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_p & 0 & 0 & \ldots & Q \end{bmatrix}$, C_i being the i-th rows of C, S being a matrix of dimension $(n+1) \times (n+1)$, Q being a row vector of dimension $n+1$ and R being a column vector of dimension $(n+1)$, defined respectively by $S = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$, $Q^T = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $R = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$. The system $\Sigma_e = (A_e, B_e, C_e, D_e)$, is the parallel composition of system Σ together with n chains of delay operators, and each new outputs e_i, for $i = 1, \ldots, p$, is the sum of the output y_i of Σ and the output of one chain of delay operators. It is not difficult to see that Σ is functionally controllable, namely after n instant it can reproduce at the output any arbitrary sequence, starting from zero initial conditions, if fed by a suitable input, if and only if the disturbance q can be decoupled from the output of Σ_e. Denoting by V_e^* the maximum (A_e, B_e)-invariant submodule of the extended system Σ_e contained in $Ker\ C_e$ we can then state the following Theorem (see [3].

Theorem 1 *The system Σ defined by equations (1) over a ring R is functionally controllable if and only if $Im\ D_e \subseteq V_e^* + Im B_e$.*

Recent results ([1]) on the Disturbance Decoupling Problem provide feasible procedures for checking he condition of the above Theorem and for computing possible feedback solutions. Assuming that Σ is functionally controllable, the following Algorithm can be used in order to compute its relative degree.

MIMO Relative Degree Computation

Step 1 Modify the i-th column $D_i$of the matrix D_e by substituting to the column vector R the column vector R_k whose components are all equal to zero, except the k-th one, which is equal to 1.
Step 2 Find the least k such that the Disturbance De-

coupling Problem concerning Σ_e with D_e modified as above is solvable.

4 APPLICATIONS

In the first part of this Section we assume that the coefficient ring R is a Principal Ideal Domain. Let us assume that the SISO system $\Sigma = (A, b, c)$ with coefficients in the ring R has pure relative degree k_0 and write $cA^{k_0}b = m_{k_0}$. This implies that the matrix $\begin{bmatrix} c \\ cA \\ \dots \\ cA^{k_0-1} \end{bmatrix}$ is full row rank, since the product

$$\begin{bmatrix} c \\ cA \\ \dots \\ cA^{k_0-1} \end{bmatrix} [A^{k_0-1}bA^{k_0-2}b \dots b] =$$

$$\begin{bmatrix} m_{k_0} & 0 & \dots & \dots & 0 \\ a_{11} & m_{k_0} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{k_01} & \dots & \dots & a_{k_0k_0} & m_{k_0} \end{bmatrix}$$

is a square invertible matrix. Moreover, the set of row vectors $\{cA^k\}_{k=0,\dots,k_0-1}$ can be completed to a basis of $\mathcal{X}$. Then, if we write

$$z_{k+1} = cA^k \quad \textit{for } k = 0, \dots, k_0 - 1,$$

in the new bases $\{z_1, \dots, z_{k_0}, z_{k_0+1}, \dots, z_n\}$ the system Σ takes the form

$$\begin{cases} z_1(t+1) = z_2(t) \\ \vdots \\ z_{k_0}(t+1) = A_0 z(t) + m_{k_0}u(t) \\ z_{k_0+1}(t+1) = A_1 z(t) \\ \vdots \\ z_n(t+1) = A_{n-k_0}z(t) \\ y(t) = z_1(t) \end{cases} . \tag{12}$$

Applying the state feedback defined by $u(t) = m_{k_0}^{-1}(-A_0 z(k) + v(t))$, the system reduces to the following simple form

$$\begin{cases} z_1(t+1) = z_2(t) \\ \vdots \\ z_{k_0}(t+1) = v(t) \\ z_{k_0+1}(t+1) = A_1 z(t) \\ \vdots \\ z_n(t+1) = A_{n-k_0}z(t) \\ y(t) = z_1(t) \end{cases} \tag{13}$$

consisting of a reachable and observable subsystem and of an autonomous, unobservable one. We can summarize what seen above in the following Proposition.

Proposition 2 *Let $\Sigma = (A, b, c)$ have pure relative degree k_0 equal to its dimension n. Then, Σ is reachable and observable and there exist a change of basis and a state feedback which brings the system Σ into the following canonical form*

$$\begin{cases} z_1(t+1) = z_2(t) \\ \vdots \\ z_n(t+1) = v(t) \\ y(t) = z_1(t) \end{cases} \tag{14}$$

Applying the above result to delay-differential systems, we get the following Proposition.

Proposition 3 *Let Σ_d be a SISO delay differential system with a finite number of commensurable delays and let the associated system Σ over $\mathcal{R}[\Delta]$ have pure relative degree $k_0 = dim\ \mathcal{X}$. Then, there exists a change of basis and a state feedback in the delay differential framework which transform Σ_d in a system without delays.*

Example 5 *Let us consider the system Σ_d defined by the equations*

$$\begin{cases} \dot{x}_1(t) &= x_1(t-h) - x_3(t) + u(t-h) \\ \dot{x}_2(t) &= x_1(t) + u(t) \\ \dot{x}_3(t) &= x_2(t) \\ y(t) &= x_1(t) + x_2(t-h) \end{cases} . \tag{15}$$

In this case computation of the Relative Degree for the associated system Σ yields

$$\begin{aligned} y(t+1) &= x_3(t) \\ y(t+2) &= x_2(t) \\ y(t+3) &= x_1(t) + u(t) \end{aligned} .$$

Therefore the pure relative degree is $k_0 = 3 = dim\ \mathcal{X}$ By the change of basis $\xi = Px$ given by tha matrix

$$P = \begin{bmatrix} c \\ cA \\ cA^2 \\ cA^3 \end{bmatrix} = \begin{bmatrix} 1 & -\Delta & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

the system Σ assumes the following form

$$\begin{cases} \xi_1(t+1) &= \xi_2(t) \\ \xi_2(t+1) &= \xi_3(t) + u(t) \\ \xi_3(t+1) &= \xi_1(t) + \Delta\xi_3(t) \\ & \quad +u(t) \\ y(t) &= \xi_1(t) \end{cases} \tag{16}$$

Applying the feedback law $u(t) = -\xi_1(t) + \Delta\xi_3(t) + v(t)$ we obtain a system without delays

$$\begin{cases} \xi_1(t+1) &= \xi_2(t) \\ \xi_2(t+1) &= v(t) \\ \xi_3(t+1) &= \xi_3(t) \\ y(t) &= \xi_1(t) \end{cases}$$

The results concerning Right Invertibility of MIMO systems given in Section 3 can be used for constructing, by solving a Disturbance Decoupling Problem, a right inverse in the delay-differential framework. This fact is better illustratd by the following Example.

Example 6 *Consider the delay-differential system Σ_d described by the equations*

$$\left\{\begin{array}{rcl} \dot{x}_1(t) & = & x_1(t-\delta)+x_2(t)+u_1(t) \\ \dot{x}_2(t) & = & -x_2(t)+u_2(t) \\ y_1(t) & = & x_1(t-\delta)-x_2(t)+x_2(t-\delta) \\ y_2(t) & = & x_1(t)+x_2(t) \end{array}\right. , \tag{17}$$

and the associated system Σ over the ring of real polynomials $\mathcal{R}[\Delta]$, described by the equations

$$\left\{\begin{array}{rcl} x_1(t+1) & = & \Delta x_1(t)+x_2(t)+u_1(t) \\ x_2(t+1) & = & -x_2(t)+u_2(t) \\ y_1(t) & = & \Delta x_1(t)+(\Delta-1)x_2(t) \\ y_2(t) & = & x_1(t)+x_2(t) \end{array}\right. \tag{18}$$

A solution to the Disturbance Decoupling Problem concerning Σ_e as described in Section 3is, in this case, given e.g. by the feedback

$$\left\{\begin{array}{rcl} u_1(t) & = & \Delta x_1(t)+x_2(t)+q_1(t)-(\Delta-1)q_2(t) \\ u_2(t) & = & -x_2(t)-q_1(t)+\Delta q_2(t) \end{array}\right. \tag{19}$$

Going back to the delay-differential framework, we obtain the system Σ' defined by the equations

$$\left\{\begin{array}{rcl} \dot{x}_1(t) & = & 2x_1(t-\delta)+2x_2(t)+q_1(t) \\ & & -q_2(t-\delta)+q_2(t) \\ \dot{x}_2(t) & = & -2x_2(t)-q_1(t)+q_2(t-\delta) \\ u_1(t) & = & x_1(t-\delta)+x_2(t)+q_1(t) \\ & & -q_2(t-\delta)+q_2(t) \\ u_2(t) & = & -x_2(t)-q_1(t)+q_2(t-\delta) \end{array}\right. , \tag{20}$$

which is a right inverse of Σ.

Construction of right inverses as illustrated above can provide a tool for solving tracking and trajectory following problems in the delay-differential framework. More work for studying the existence of stable inverses and for characterizing this by means of a suitable notion of minimum phase, possibly in the ring framework, is required and will be the object of future research.

References

[1] J. Assan, J.F. Lafay and A.M. Perdon, Computation of maximal pre-controllability submodules over a Noetherian ring, *System and Control letters (1999)*

[2] G. Basile , G. Marro, Controlled and conditioned invariants in linear system theory, *Prentice Hall, Englewood Cliffs, (1992)*

[3] G. Conte and A.M. Perdon, The disturbance decoupling problem for systems over a ring. *SIAM Journal on Control and Optimization, 33 (1995).*

[4] G. Conte and A.M. Perdon, Noninteracting control problems for delay-differential systems via systems over rings, *European J. of Automation, vol. 31, n. 6* (1997)

[5] G. Conte and A.M. Perdon, The block decoupling problem for systems over a ring. *IEEE Trans. AC 43 (1998)*

[6] G. Conte, A.M. Perdon and A. Lombardo, Dynamic feedback decoupling problem for delay-differential systems via systems over rings. *Mathematics and Computers in Simulation, 1462 (1998).*

[7] G. Conte and A.M. Perdon, System over rings: geometric theory and applications, *Proc. IFAC Workshop on Time Delay Systems, Grenoble, France (1998)*

[8] G. Conte, C.H. Moog and A.M. Perdon, Nonlinear control systems, An Algebraic setting, *Springer LN in Contr. and Information Sc. Springer vol. 242* (1999)

[9] A. Germani, C. Manes and P. Pepe, Linearization of input-output mapping for nonlinear delay systems via static state feedback, *Proc. CESA IMACS Multiconf. on Computational Eng. in Sys. Appl., vol. I, Lille, France (1996)*

[10] A. Germani, C. Manes and P. Pepe, Linearization and decoupling of nonlinear delay systems, *Proc. American Control conference,Philadelphia, Pennsylvania (1998)*

[11] A. Isidori, Nonlinear control systems, 2^{nd} edition, *Communication and Control Engineering Series, Springer (1989)*

[12] T. Kailath, Linear systems, *Prentice Hall (1980)*

[13] M. Malabre and R. Rabah, Structure at infinity, model matching and disturbance rejection for linear systems with delays. *Kybernetika, 29, No.5* (1993)

[14] L. Marconi and G. Marro, On the relative degree of multivariable linear systems in geometric terms, *Proc. 37th IEEE Conference on Decision and Control , Tampa, Florida (1998)*

[15] P. Picard, Sur l'observabilité et la commande des systemes linéaires a retards modélisés sur un anneau, *These de Doctorat, Un. de Nantes, Nantes* (1996)

[16] L. Silverman, Inversion of multivariable linear systems, *IEEE Trans. Aut. Control, 14, (3)* (1969)

[17] S. N. Singh, A modified algorithm for invertibility in nonlinear systems, *IEEE Trans. AC-26 (1981) 595-598*