A Problem of Stochastic Impulse Control with Discretionary Stopping

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Abstract

We consider a stochastic control problem that has emerged in the economics literature as an investment model under uncertainty. This problem combines some of the features of stochastic impulse control with optimal stopping. The aim is to discover the form of the optimal strategy. The results that we establish are of an explicit nature.

1 Introduction

Problems that combine features of both stochastic optimal control and optimal stopping have attracted the interest of several researchers. Models of absolutely continuous control of the drift and discretionary stopping have been studied by Krylov [K], Beneš [B], Karatzas and Sudderth [KSu], and Karatzas and Wang [KW]. Models of combined singular stochastic control where the control effort takes the form of a finite variation process and discretionary stopping have been studied by Davis and Zervos [DZ], and Karatzas, Ocone, Wang and Zervos [KOWZ]. These two families of problems have been motivated by applications in target tracking where the controller has to steer a system close to a target and then decide on an engagement time, as well as by applications in finance. The latter ones include the classical consumption/investment problem for a small investor who can decide on the time of his "exit" from the market (see Karatzas and Wang [KW]), as well as the pricing of American contingent claims under constraints or with transaction costs.

In this paper, we consider a problem of stochastic impulse control combined with optimal stopping with a view to discovering the form of the optimal strategy. Note that the impulse control component of the control strategy is not of the standard form because the size of the jumps associated with each intervention strategy are not discretionary, but are constrained to follow the pattern $\ldots, 1, -1, 1, -1, \ldots$. This simplification makes the problem easier to analyse. However, it is offset by the extra complexity that is introduced by the additional control variable which is the discretionary stopping.

The motivation for this problem arises from the area of "real options" that has emerged in the economics literature in the past two decades. This area is concerned with the development of new stochastic models that can lead to more accurate pricing of investments in real assets by taking into account the value of managerial flexibility; the interested reader can consult the books by Dixit and Pindyck [DP], and Trigeorgis [T]. To fix ideas, consider an economic activity that is centred on a project that can operate in two modes, an "open" one and a "closed" one. Whenever the project is in its "open" operating mode, it yields a stream of profits or losses which is a functional of the uncertain prices of input and output commodities. Whenever the project is in its "closed" operating mode, it yields neither profits nor losses. The transition of the project from one of its operating modes to the other one forms a sequence of managerial decisions and is associated with certain fixed costs. The problem is to determine the switching strategy that maximises the expected present value of all profits and losses resulting from the project. Variants of this problem have been developed in the economics literature as models for the valuation of investments in real assets by Brennan and Schwartz [BS], Dixit [D], and Dixit and Pindyck [DP]. Such a problem has the features of stochastic impulse control, and explicit solutions have been obtained in the mathematics literature by Brekke and Øksendal [BØa, BØb], Lumley and Zervos [LZ], and Duckworth and Zervos [DuZ].

Suppose now that the option of totally abandoning the project at a discretionary time and at a certain fixed cost is added in the set of available managerial decisions. The resulting problem then combines some of the features of stochastic impulse control with discretionary stopping. In fact, such a model is a special case of the one developed by Brennan and Schwartz [BS], and is extensively discussed in Dixit and Pindyck [DP, Section 7.2]. However, these authors make little progress in solving the problem. The purpose of this paper is to solve completely the resulting optimisation problem under the assumption that the rate at which the project yields profits or losses is a standard Brownian motion. Such an assumption is probably crude as far as real life applications are concerned. However, it leads to explicit, non-trivial results that unveil the qualitative nature of the optimal strategy.

The results of our analysis can be summarised informally as follows. Suppose that the switching costs are fixed. If the abandonment cost is very large (see case I in Theorem 6 and Figure 1), then it is optimal to perpetuate the project by switching it to its "closed" mode as soon as its output cash flow falls below a certain level, and by switching it to its "open" mode as soon as its potential output cash flow rises above a certain higher level. If the abandonment cost is very small (see case III of Theorem 6 and Figure 4), then abandonment is optimal, sooner or later. If the project is in its "closed" mode at time 0, then it is switched to its "open" mode as soon as its potential output cash flow exceeds a certain level. Once in it, the project should be kept in its "open" operating mode for as long as its output cash flow is above a given level, and should be abandoned as soon as its output cash flow falls below this level. For intermediate values of the abandonment cost, we have a rather unexpected combination of the two cases above (see case II of Theorem 6 and Figure 3). If the project starts from its "closed" mode, then it is never abandoned, and the situation resembles the case where the abandonment cost is very large. A similar scenario pertains to the case when the project is originally "open" and its output cash flow assumes sufficiently high levels. However, if the project is originally "open" and its output cash flow assumes very low values, then it is optimal to abandon the project immediately. The most interesting possibility arises when the project is originally "open" and its output cash flow assumes moderately low values. In this case, it is optimal to keep the project live and keep on accumulating losses until its output cash flow either falls below a certain level, on which event the project is totally abandoned, or rises above another level, on which event its operation enters the perpetual life cycle pertaining to the case of a large abandonment cost. As a result, the abandonment time of the project is either finite or infinite, and each of the two possibilities has positive probability.

2 Problem formulation

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a filtration $(\mathcal{F}_t)$ satisfying the usual conditions of right continuity and augmentation by P-negligible sets, and carrying a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion W. We denote by $\mathcal{Z}$ the family of all adapted, finite variation, càglàd processes Z with values in $\{0,1\}$, and by $\mathcal{S}$ the set of all $(\mathcal{F}_t)$-stopping times.

We consider a stochastic system that can operate in two modes, an "open" one and a "closed" one. The system's mode of operation can be changed at a sequence of $(\mathcal{F}_t)$-stopping times. These transition times constitute a decision strategy that we model by a process $Z \in \mathcal{Z}$. Specifically, given any time t, $Z_t = 1$ if the system is "open" at time t, whereas $Z_t = 0$ if the system is "closed" at time t. The stopping times at which the jumps of Z occur defined by

$$T_1 = \inf\{t \geq 0 : Z_t \neq z\}, \tag{1}$$

$$T_{n+1} = \inf\{t > T_n :\ Z_t \neq Z_{\tau_n+}\}, \tag{2}$$

are the intervention times at which the system's operating mode is changed. We denote by $z \in \{0,1\}$ the system's mode at time 0. We also assume that the operation of this system can be permanently abandoned at a $(\mathcal{F}_t)$-stopping time T, which is an additional decision variable. We define the set of all admissible strategies to be

$$\Pi_z = \{(Z,T) :\ Z \in \mathcal{Z},\ Z_0 = z,\ T \in \mathcal{S}\}.$$

Alternatively, we can define the set of admissible strategies to be the family of all pairs of sequences of $(\mathcal{F}_t)$-stopping times (T_n) such that $T_n \to \infty$, P-a.s., and $(\mathcal{F}_t)$ stopping times T. Such a definition would be more consistent with the impulse control literature. However, in view of (1)–(2) and the assumption that every $Z \in \mathcal{Z}$ is a finite variation process, the two definitions are equivalent.

We assume that, whenever the system is in its "open" operating mode, it yields payoff at a rate given by the state process X defined by

$$X_t = x + W_t, \qquad x \in \mathbb{R}.$$

On the other hand, we assume that, whenever the system is in its "closed" mode, it yields 0 payoff. Switching the system from its "closed" mode to its "open" one, and vice versa, is associated with certain fixed costs given by the constants $K_I, K_O > 0$, respectively. Also, complete termination of the system's operation is associated with a cost modelled by a constant $K > 0$. As a result, each admissible strategy $(Z,T) \in \Pi_z$ is associated with the expected payoff

$$J_{x,z}(Z,T) = E\left[\int_0^T e^{-rs} X_s Z_s\, ds - e^{-rT} K - \sum_{0 \leq s < T} e^{-rs} \left[K_I \left(\Delta Z_s\right)^+ + K_O \left(\Delta Z_s\right)^-\right]\right], \tag{3}$$

where $\Delta Z_t = Z_{t+} - Z_t$ and $(\Delta Z_t)^{\pm} = \max\{\pm \Delta Z_t, 0\}$. The objective is to maximise $J_{x,z}(Z,T)$ over Π_z. Accordingly we define the value function

$$v(x,z) = \sup_{(Z,T) \in \Pi_z} J_{x,z}(Z,T).$$

At this point, observe that the assumption that $K_I, K_O > 0$ ensures every switching strategy associated with a finite payoff can be modelled by a process in $\mathcal{Z}$, and the optimisation problem is well posed.

3 A verification theorem

The problem considered in the previous section combines features of both stochastic impulse control and optimal stopping. Therefore, we can expect that the value function v should satisfy the Hamilton-Jacobi-Bellman (HJB) equation which takes the form of the following pair of coupled quasi-variational inequalities

$$\max\left\{\tfrac{1}{2}w_{xx}(1,x) - rw(1,x) + x,\right.\\ \left. w(0,x) - w(1,x) - K_O,\ -w(1,x) - K\right\} = 0, \quad (4)$$

$$\max\left\{\tfrac{1}{2}w_{xx}(0,x) - rw(0,x),\right.\\ \left. w(1,x) - w(0,x) - K_I,\ -w(0,x) - K\right\} = 0. \quad (5)$$

The ideas behind the origins of these equations are the following. Suppose that, at time 0, the system is in its "open" operating mode. The controller's immediate decision consists of choosing between three actions. The first action is to totally terminate the system's operation at the cost of $-K$. Such a possibility gives rise to the inequality

$$v(1,x) \geq -K. \quad (6)$$

The second option is to pay the cost of K_O to switch the system to its "closed" operating mode, and then continue optimally. This possibility yields the inequality

$$v(1,x) \geq -K_O + v(0,x). \quad (7)$$

The third action is to leave the system in its "open" operating mode for a short time Δt, and then continue optimally. This action is associated with the inequality

$$v(1,x) \geq E\left[\int_0^{\Delta t} e^{-rs}X_s ds + e^{-r\Delta t}v(1,X_{\Delta t})\right].$$

Under the assumption that $v(1,\cdot)$ is sufficiently smooth, we may apply Itô's formula to the last term, and then divide by Δt before letting $\Delta t \downarrow 0$, to obtain

$$\tfrac{1}{2}v_{xx}(1,x) - rv(1,x) + x \leq 0. \quad (8)$$

Now, each of (6)–(8) can hold with strict inequality because the corresponding action may not be the best one. However, we expect that the three actions considered above form a complete repertoire of optimal tactics. Therefore, given any $x \in \mathbb{R}$, we expect that one of (6)–(8) should hold with equality. Combining all of these relationships, we can conclude that $v(1,\cdot)$ should satisfy (4). Using a similar reasoning, we can also conclude that $v(0,\cdot)$ should satisfy (5).

We can now prove conditions which are sufficient for optimality in our problem. It turns out that the value functions $v(1,\cdot)$, $v(0,\cdot)$ are C^1. However, to facilitate the discussion that will lead to the discovery of the optimal strategy, we are going to state the following result with weaker regularity assumptions on the value function than eventually needed.

Theorem 1 *Consider the control problem described in Section 2. Suppose that there exist continuous functions $w(1,\cdot), w(0,\cdot) : \mathbb{R} \to \mathbb{R}$ such that, given a finite number of points $a_1^1 < a_2^1 < \cdots < a_{N^1}^1$ and $a_1^0 < a_2^0 < \cdots < a_{N^0}^0$, $w(1,\cdot)$ (resp. $w(0,\cdot)$) is twice continuously differentiable and satisfies (4) (resp. (5)) at every point $x \in \mathbb{R} \setminus \{a_1^1, \cdots, a_{N^1}^1\}$ (resp. $x \in \mathbb{R} \setminus \{a_1^0, \cdots, a_{N^0}^0\}$). Also, suppose that, given any $z = 0, 1$, $w_x(z,\cdot)$ is bounded, and the limits*

$$w_x(z, a_{i^z}^z\pm) := \lim_{x \to a_{i^z}^z \pm} w_x(z,x),$$
$$w_{xx}(z, a_{i^z}^z\pm) := \lim_{x \to a_{i^z}^z \pm} w_{xx}(z,x),$$

exist, are finite, and satisfy

$$w_x(z, a_{i^z}^z-) \geq w_x(z, a_{i^z}^z+), \qquad \forall i^z = 1,2,\ldots,N^z. \quad (9)$$

Then, given any initial condition $(z,x) \in \{0,1\} \times \mathbb{R}$,
(a) $v(z,x) \leq w(z,x)$, and
(b) if

$$a_1^1, \ldots, a_{N^1}^1 \in \mathbb{R} \setminus \operatorname{int} \overline{\{x \in \mathbb{R} :\ \tfrac{1}{2}w_{xx}(1,x) - rw(1,x) + x = 0\}}, \quad (10)$$

$$a_1^0, \ldots, a_{N^0}^0 \in \mathbb{R} \setminus \operatorname{int} \overline{\{x \in \mathbb{R} :\ \tfrac{1}{2}w_{xx}(0,x) - rw(0,x) = 0\}}, \quad (11)$$

and there exists $Z^* \in \mathcal{Z}$ such that

$$\tfrac{1}{2}w_{xx}(Z_t^*, X_t) - rw(Z_t^*, X_t) + X_t Z_t^* = 0,$$
$$[w(1,X_t) - w(0,X_t) - K_I]\,(\Delta Z_t^*)^+ = 0,$$
$$[w(0,X_t) - w(1,X_t) - K_O]\,(\Delta Z_t^*)^- = 0,$$

for all $t \leq T^*$, P-a.s., where

$$T^* = \inf\{t \geq 0 :\ w(Z_t^*, X_t) = -K\},$$

then $v(z,x) = w(z,x)$, and the optimal strategy is (Z^*, T^*).

Remark 1 The requirements (10)–(11) of part (b) of the theorem state that the points where C^1 regularity fails should not belong to the interior of the "continuation" region, but can be allowed in the interior of the closure of the "switching" or "stopping" regions. Also, discontinuities of the first derivative should satisfy (9). There is no asymmetry here: had the optimisation problem been a minimisation one, we would have to consider the reverse inequalities in (9).

4 The solution of the control problem

We now solve completely the problem formulated in Section 2 by finding a solution of the HJB equations (4)–(5) which satisfies the requirements of the verification theorem in the previous section. To simplify the notation, we write w_I and w_O in place of $w(1,\cdot)$ and $w(0,\cdot)$, respectively, throughout this section.

A first possibility arises if abandonment is not part of the optimal scenario. In such a case, we should switch the system from its "closed" to its "open" mode whenever the state process X exceeds a level specified by a constant α, and we should switch the system from its "open" to its "closed" mode whenever the state process X falls below a level given by a constant β. Clearly, such a strategy is well defined only if $\beta < \alpha$. It can be depicted by Figure 1. If such a strategy is indeed

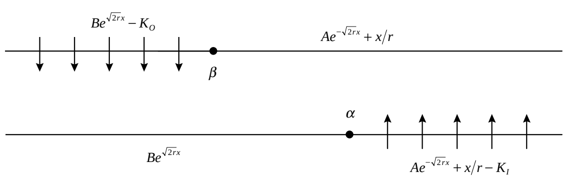

Figure 1: The "no-abandonment" case.

optimal, the value function should be composed of the functions w_I, w_O given by

$$w_I(x) = \begin{cases} Be^{\sqrt{2r}x} - K_O, & \text{if } x \le \beta, \\ Ae^{-\sqrt{2r}x} + x/r, & \text{if } x > \beta, \end{cases} \tag{12}$$

$$w_O(x) = \begin{cases} Be^{\sqrt{2r}x}, & \text{if } x < \alpha, \\ Ae^{-\sqrt{2r}x} + x/r - K_I, & \text{if } x \ge \alpha, \end{cases} \tag{13}$$

respectively. To specify the parameters A, B, α, β, we postulate that w_I, w_O are C^1 at the free boundary points β, α, respectively. This requirement implies

$$A = -\frac{\beta + rK_O - 1/\sqrt{2r}}{2r} e^{\sqrt{2r}\beta}, \tag{14}$$

$$B = \frac{\beta + rK_O + 1/\sqrt{2r}}{2r} e^{-\sqrt{2r}\beta}, \tag{15}$$

$$\left(\alpha - rK_I - 1/\sqrt{2r}\right) e^{\sqrt{2r}\alpha} = \left(\beta + rK_O - 1/\sqrt{2r}\right) e^{\sqrt{2r}\beta}, \tag{16}$$

$$\left(\alpha - rK_I + 1/\sqrt{2r}\right) e^{-\sqrt{2r}\alpha} = \left(\beta + rK_O + 1/\sqrt{2r}\right) e^{-\sqrt{2r}\beta}. \tag{17}$$

The next lemma is concerned with the solvability of (16)–(17) and with necessary and sufficient conditions under which the functions w_I, w_O given above satisfy the HJB equations (4)–(5).

Lemma 2 *There exists a unique pair of points $\alpha = \alpha(r, K_I, K_O)$ and $\beta = \beta(r, K_I, K_O)$ which satisfies (16)–(17). Point β is the unique solution of*

$$f(\beta) := \frac{\beta + rK_O + 1/\sqrt{2r}}{\beta + rK_O - 1/\sqrt{2r}} \exp\left(-\sqrt{2r}(2\beta + rK_O - rK_I)\right) = -1, \tag{18}$$

and satisfies

$$-rK_O - \frac{1}{\sqrt{2r}} < \beta < -rK_O, \tag{19}$$

whereas

$$\alpha = -\beta - rK_O + rK_I > \beta. \tag{20}$$

The functions w_I, w_O defined by (12), (13), respectively, where α and β are as above and $A, B > 0$ are given by (14), (15) , are non-decreasing, C^1 for all $x \in \mathbb{R}$ and C^2 for all $x \in \mathbb{R} \setminus \{\beta\}$, $x \in \mathbb{R} \setminus \{\alpha\}$, respectively, and satisfy

$$\max\left\{\tfrac{1}{2}w_I''(x) - rw_I(x) + x, w_O(x) - K_O - w_I(x)\right\} = 0, \quad \forall x \in \mathbb{R} \setminus \{\beta\},$$

$$\max\left\{\tfrac{1}{2}w_O''(x) - rw_O(x),\, w_I(x) - K_I - w_O(x),\, -K - w_O(x)\right\} = 0, \quad \forall x \in \mathbb{R} \setminus \{\alpha\}.$$

Moreover, $w_I(x) \ge -K$ if and only if $K \ge K_O$.

If the condition $K \ge K_O$ is not satisfied, we expect that abandonment becomes part of the optimal scenario. Now, assuming that the optimal strategy has a continuous qualitative character, we should expect that, as K_O rises above K, abandonment should become optimal if the system is "open" and the state process X assumes sufficiently small values. The obvious modification of the strategy studied above, can be depicted by Figure 2. Such a possibility involves 5 parameters and 3 free boundary points, so we cannot impose a C^1 fit at all of the free boundary points. By an obvious

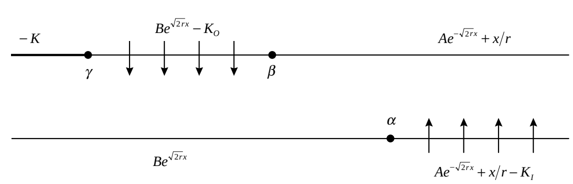

Figure 2: An obvious modification of the "no-abandonment" case.

symmetry argument, we can conclude that the value function is C^1 at the points α, β and C^0 at the point γ. However, by elementary considerations, we can see that the value function is non-decreasing in x. Therefore, if the optimal strategy identifies with the one depicted by Figure 2, we must have $w_I(\gamma-) = 0 < w_I(\gamma+)$, which is unacceptable in the light of Remark 1. Alternatively, we can postulate that the value function is C^1 at γ and β (resp. α), and C^0 at α (resp. β). However, such a possibility would impose a discontinuity of the first derivative of the candidate value functions inside the interior of the "continuation" region, which is again contradicting the conclusions of Remark 1. It turns out that a strategy having the form depicted by Figure 2 cannot be optimal. However, the idea that the optimal strategy should possess a character which depends continuously on the problem's data leads us to the conclusion that we should look for a further modification of this strategy. Such a modification can be obtained by inserting a "do-not-abandon-or-switch-off" region around γ, and can be depicted by Figure 3. If

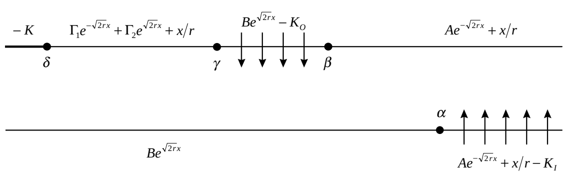

Figure 3: The case where abandonment becomes part of the optimal tactics.

this case is indeed optimal the value functions should be given by the functions w_I, w_O defined by

$$w_I(x) = \begin{cases} -K, & \text{if } x \le \delta, \\ \Gamma_1 e^{-\sqrt{2r}x} + \Gamma_2 e^{\sqrt{2r}x} + x/r, & \text{if } \delta < x < \gamma, \\ Be^{\sqrt{2r}x} - K_O, & \text{if } \gamma \le x \le \beta, \\ Ae^{-\sqrt{2r}x} + x/r, & \text{if } x > \beta, \end{cases} \tag{21}$$

$$w_O(x) = \begin{cases} Be^{\sqrt{2r}x}, & \text{if } x < \alpha, \\ Ae^{-\sqrt{2r}x} + x/r - K_I, & \text{if } x \ge \alpha. \end{cases} \tag{22}$$

The parameters A, B, Γ_1, Γ_2, α, β, γ, δ can then be specified by the requirement that w_I, w_O are C^1 at the free boundary points α, β, γ, δ. We can verify that this requirement implies that α, β, A, B, should satisfy (14)–(17),

$$\Gamma_1 = -\frac{\gamma + rK_O - 1/\sqrt{2r}}{2r} e^{\sqrt{2r}\gamma}, \tag{23}$$

$$\Gamma_2 = B - \frac{\gamma + rK_O + 1/\sqrt{2r}}{2r} e^{-\sqrt{2r}\gamma}, \tag{24}$$

and γ, δ should satisfy the system of equations

$$F_1(\gamma,\delta) := \left(\delta + rK - 1/\sqrt{2r}\right) e^{\sqrt{2r}\delta} - \left(\gamma + rK_O - 1/\sqrt{2r}\right) e^{\sqrt{2r}\gamma} = 0, \tag{25}$$

$$F_2(\gamma,\delta) := \left(\delta + rK + 1/\sqrt{2r}\right) e^{-\sqrt{2r}\delta} + 2rB - \left(\gamma + rK_O + 1/\sqrt{2r}\right) e^{-\sqrt{2r}\gamma} = 0. \tag{26}$$

The next lemma is concerned with the solvability of (25)–(26) as well as with necessary and sufficient conditions under which the functions w_I, w_O considered above satisfy the HJB equations (4)–(5).

Lemma 3 *Let $\alpha = \alpha(r, K_I, K_O)$, $\beta = \beta(r, K_I, K_O)$ and $A, B > 0$ be as in Lemma 2. The system of equations (25)–(26) has a unique solution $\gamma = \gamma(r, K, K_I, K_O)$, $\delta = \delta(r, K, K_I, K_O)$ such that $\delta < \gamma < \beta$ if and only if*

$$K_* \vee 0 < K < K_O, \tag{27}$$

where $K_* = K_*(r, K_I, K_O) < K_O$ is defined by

$$K_* = -\frac{1}{r\sqrt{2r}} \ln\left[-\frac{\sqrt{2r}}{2}\left(\beta + rK_O - 1/\sqrt{2r}\right) \times \exp\left(\sqrt{2r}(\beta + 1/\sqrt{2r})\right)\right]. \tag{28}$$

If $K_* > 0$ and $K = K_*$, then $\gamma = \beta$, $\delta = -rK - 1/\sqrt{2r}$, $\Gamma_1 = A$, and $\Gamma_2 = 0$. If (27) is true, then the functions w_I, w_O defined by (21), (22), respectively, where $\Gamma_1, \Gamma_2 > 0$ are given by (23)–(24), are non-decreasing, C^1 for all $x \in \mathbb{R}$ and C^2 for all $x \in \mathbb{R} \setminus \{\delta, \gamma, \beta\}$, $x \in \mathbb{R} \setminus \{\alpha\}$, respectively, and satisfy the HJB equations (4)–(5).

The optimality of the case considered in the previous lemma depends crucially on the parameter K_*. If $K_* \le 0$ for every admissible choice of the problem's data, then our solution is complete. However, it turns out that this is not in general the case.

Lemma 4 *Given any values of the parameters $r, K_I > 0$, the function $K_*(r, K_I, \cdot)$ is well defined, at least C^1, and strictly increasing on $]-K_I, \infty[$, and satisfies $\lim_{K_O \to \infty} K_* = \infty$ and $\lim_{K_O \downarrow 0} K_* < 0$.*

In Lemma 3, we proved that if $K_* > 0$ and $K = K_*$, then $\gamma = \beta$, so the "switch-from-open-to-closed" region disappears, and the optimal strategy can be depicted by Figure 4. For $K < K_*$, we can expect that it is not optimal to switch the system from its "open" to its "closed" operating mode at any time, so that the optimal strategy can again be depicted by Figure 4. If this strategy is indeed optimal, the value function

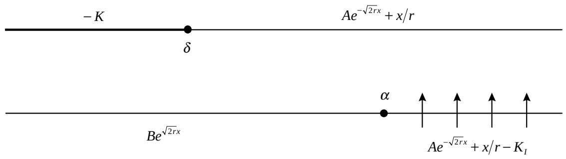

Figure 4: The case where switching the system to its "closed" mode is never optimal.

should be given in terms of the functions

$$w_I(x) = \begin{cases} -K, & \text{if } x \le \delta, \\ Ae^{-\sqrt{2r}x} + x/r, & \text{if } x > \delta, \end{cases} \tag{29}$$

$$w_O(x) = \begin{cases} Be^{\sqrt{2r}x}, & \text{if } x < \alpha, \\ Ae^{-\sqrt{2r}x} + x/r - K_I, & \text{if } x \ge \alpha. \end{cases} \tag{30}$$

Again, we require that w_I, w_O are C^1 at the free boundary points δ, α, respectively. C^1 fit at δ yields

$$A = \frac{1}{r\sqrt{2r}} \exp\left(-\sqrt{2r}(rK + 1/\sqrt{2r})\right), \tag{31}$$

$$\delta = -rK - \frac{1}{\sqrt{2r}}, \tag{32}$$

whereas C^1 fit at α yields

$$B = \frac{\alpha - rK_I + 1/\sqrt{2r}}{2r} e^{-\sqrt{2r}\alpha}, \tag{33}$$

$$f(\alpha) := \frac{\sqrt{2r}}{2}\left(\alpha - rK_I - 1/\sqrt{2r}\right) \times \exp\left(\sqrt{2r}(\alpha + rK + 1/\sqrt{2r})\right) = -1. \tag{34}$$

The next lemma is concerned with the solvability of (34) and with necessary and sufficient conditions under which this case is optimal.

Lemma 5 *Equation (34) has a unique solution* $\alpha = \alpha(r, K, K_I)$ *such that* $\alpha > -rK - 1/\sqrt{2r}$. *For this value of* α, *and for* $A, B > 0$ *and* δ *given by (31), (33), (32), respectively, the functions* w_I, w_O *defined by (29), (30), respectively, are non-decreasing,* C^1 *for all* $x \in \mathbb{R}$ *and* C^2 *for all* $x \in \mathbb{R} \setminus \{\delta\}$, $\mathbb{R} \setminus \{\alpha\}$, *respectively. Moreover, assuming that* $K_* > 0$, *they satisfy the HJB equations (4)–(5) if and only if* $0 < K \le K_*$.

We can now state the main result of the paper.

Theorem 6 *Consider the stochastic optimisation problem defined in Section 2. The value function* v *is* C^1 *and non-decreasing in* x, *and is given by* $v(1, \cdot) = w_I$ *and* $v(0, \cdot) = w_O$, *where:*
(I) *If* $K_O \le K$, w_I, w_O *are given by Lemma 2 (see Figure 1).*
(II) *If* $K_* < K < K_O$, *where* $K_* < K_O$ *is given by (28),* w_I, w_O *are given by Lemma 3 (see Figure 3).*
(III) *If* $K_* > 0$ *and* $K < K_*$, w_I, w_O *are given by Lemma 5 (see Figure 4).*

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