Controller synthesis for stabilizing periodic orbits in forced nonlinear systems

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Abstract

Delayed feedback controllers are an appealing tool for the stabilization of periodic orbits in nonlinear systems. Unfortunately, their inherent infinite dimensional structure prevents from the definition of reliable design procedures. This paper considers the use of finite dimensional linear time invariant controllers for the stabilization of periodic solutions in a general class of sinusoidally forced nonlinear systems. Such controllers — which can be interpreted as rational approximations of the delayed ones — provide an LMI-based synthesis technique, by mixing results concerning absolute stability of nonlinear systems and robustness of uncertain linear systems. In particular, the synthesis algorithm is able to select the controller maximizing a lower bound of the maximum amplitude of the forcing input, for which the corresponding periodic solutions are guaranteed to be stable. A single-mode CO_2 laser is employed to illustrate the main features of the developed synthesis technique.

1 Introduction

Periodic motions have been thoroughly investigated in nonlinear systems science since long time. More recently, a strong renewed interest has been observed in the subject of stability of periodic solutions due to its significant relations with chaos control problems (see [1] and references therein), where stabilization of periodic dynamics is often a satisfactory target. More precisely, the underlying key idea for controlling chaos is to stabilize one among the infinite unstable periodic orbits embedded in the chaotic attractor.

Among other approaches, delayed feedback controllers represent an appealing tool for stabilizing periodic solutions of nonlinear systems [2]-[6]. These linear time-invariant controllers ensure to maintain the periodic solutions of the uncontrolled system, an important property in chaos control applications [1]. However, since such controllers are not finite-dimensional, it is difficult to obtain a reliable design procedure, especially when there are several controller parameters to be designed. For example, the controller gain is tuned experimentally in [2], while the controller design involves a heavy computational burden in [3]-[6].

Motivated by the discussion above, the use of a general class of finite-dimensional controllers, which are rational approximations of the infinite dimensional delayed ones, have been considered more recently [7],[8]. In particular, the synthesis of controllers for the stabilization of periodic solutions in a class of sinusoidally forced Lur'e systems has been considered in [8], showing how Linear Matrix Inequalities (LMIs) can be employed for the design.

The present paper considers the same problem for a much larger class of sinusoidally forced nonlinear systems, containing meaningful systems which exhibit a rich nonlinear dynamics such as the single-mode CO_2 laser [9] and the Brusselator system [10]. An LMI-based synthesis technique is proposed by mixing results concerning absolute stability of nonlinear systems and robustness of uncertain linear systems. In particular, the designed controller maximizes a lower bound of the maximum amplitude of the sinusoidally forcing input, for which the corresponding periodic solutions are guaranteed to be stable. A single-mode CO_2 laser is used as an application example to illustrate the main features of the developed synthesis technique.

The paper is organized as follows. Section 2 formulates the problem and gives some preliminary results. The LMI-based synthesis technique is developed in Section 3. Section 4 presents in some detail an application example. Some brief comments end the paper in Section 5. The complete proofs of the results of the paper are reported in [11].

2 Problem formulation and preliminary results

Notation

$\mathbf{Z}$: set of positive integer numbers;
$\mathbf{R}$: set of real numbers;
$\mathbf{R}^n$: space of n-component real (column) vectors;
$\mathbf{R}^{r\times n}$: space of $(r \times n)$ real matrices;
$I_n \in \mathbf{R}^{n\times n}$: identity matrix of order n;
$x(t) \in \mathbf{R}^n$: n-valued time domain signal;
$X(s)$: Laplace transform of $x(t)$.

Consider the class of nonlinear systems Σ_P described by the following state-space representation:

$$\Sigma_P : \left\{ \begin{array}{lcl} \dot{x} & = & A\,x + B\,u + f(x) + E\,w \\ y & = & C\,x \end{array} \right. \tag{1}$$

where $x \in \mathbf{R}^n$ is the state vector, $w \in \mathbf{R}$ is the forcing input, $u \in \mathbf{R}$ is the control input, $y \in \mathbf{R}$ is the measured output, $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times 1}$, $C \in \mathbf{R}^{1\times n}$, $E \in \mathbf{R}^{n\times 1}$ are constant matrices, and $f : \mathbf{R}^n \to \mathbf{R}^n$ is a smooth vector function such that

$$f(0) = 0\ , \quad \left.\frac{\partial f}{\partial x}\right|_{x=0} = 0\ . \tag{2}$$

In the sequel, Σ_P will be referred to as the uncontrolled system when $u = 0$, and the unforced uncontrolled system when $u = w = 0$. In particular, we assume that the origin is a locally exponentially stable equilibrium point of the unforced uncontrolled system (i.e., A is Hurwitz).

The above class includes many systems displaying a rich nonlinear dynamics which are not contained in the class of Lur'e systems recently investigated in [8] [1], such as the single mode CO_2 laser [9] and the Brusselator system [10].

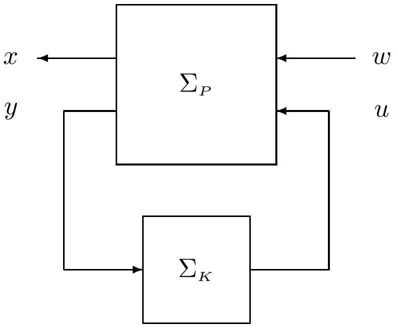

Figure 1: Controlled system Σ.

The problem investigated is in the spirit of [4],[6] and can be formulated as follows. Suppose that system Σ_P is subject to sinusoidal forcing inputs

$$w_\mu(t) = \mu \cos\left(\tfrac{2\pi}{T}t\right)\ ,\ \mu \geq 0 \tag{3}$$

of period $T > 0$ and amplitude μ. It can be proven that, if the origin is a hyperbolic equilibrium point, then for small values of μ the uncontrolled system displays T-periodic solutions $x_\mu(t) = x_\mu(t+T)$ nearby $x = 0$. Moreover, if the origin is locally asymptotically stable, then there exist locally asymptotically stable periodic solutions for some positive interval of μ (see [12] for more details). Our basic problem consists of synthesizing a linear time invariant controller Σ_K such that the controlled system Σ (see Fig. 1) possesses locally asymptotically stable T-periodic solutions for μ belonging to a larger interval (possibly the largest one according to some criteria).

Note that the above formulation requires that $x_\mu(t)$ is still a periodic solution of the controlled system Σ. As a consequence, the following constraints should be satisfied by the transfer function $K(s)$ of the controller Σ_K

$$K\left(jn\frac{2\pi}{T}\right) = 0 \qquad \text{for } \ n = 0, \pm 1, \pm 2, \ldots \tag{4}$$

These constraints are satisfied by delayed feedback controllers as those proposed in [2]-[6]. Although these controllers ensure the maintenance of the periodic solutions, their infinite dimensional structure makes it difficult to define reliable design procedures. In this paper, we consider controllers of the following structure

$$K(s) = F(s)\, s \prod_{n=1}^{N_0}\left(1 + \frac{T^2}{4n^2\pi^2}s^2\right), \tag{5}$$

where N_0 is a given positive integer and $F(s)$ is a rational function, with relative degree $2N_0 + 1$ at least, to be designed. It is clear that $K(s)$ satisfies constraints (4) for $|n| \leq N_0$, and therefore it can be seen as a finite dimensional approximation of delayed controllers. As a consequence, only slight modifications in the T-periodic solutions of the uncontrolled system are expected if N_0 is taken sufficiently large.

A standard approach in local stability analysis of $x_\mu(t)$ is to linearize the nonlinear system Σ around the solution. Let $\Delta_\mu(t)$ be a diagonal matrix of order p ($1 \leq p \leq n^2$) containing all the non-zero entries of the $n \times n$ matrix $\left.\frac{\partial f(x)}{\partial x}\right|_{x_\mu(t)}$. It is not difficult to prove that there exist suitable constant matrices $B_\Delta \in \mathbf{R}^{n\times p}$ and $C_\Delta \in \mathbf{R}^{p\times n}$ such that the linearization of Σ_P around $x_\mu(t)$ leads to the linear periodic system

$$\left\{ \begin{array}{lcl} \dot{x} & = & A\,x + B\,u + B_\Delta \Delta_\mu(t) C_\Delta\,x \\ y & = & C\,x\ . \end{array} \right. \tag{6}$$

Clearly, system (6) and the controller (5) yield the sought linearized system of Σ. It is straightforward to verify that such linearized system Σ_l can be put in the form of Fig. 2, which is a well-known structure in robust control theory [14]. Here, $v \in \mathbf{R}^p$, $z \in \mathbf{R}^p$, and $M(s)$ is the stable $(p+1) \times (p+1)$ rational transfer function matrix

$$M(s) = \left[\begin{array}{c|c} M_{11}(s) & M_{12}(s) \\ \hline M_{21}(s) & M_{22}(s) \end{array}\right], \tag{7}$$

where $M_{11}(s) = C_\Delta H(s)\, B_\Delta$, $M_{12}(s) = C_\Delta H(s)\, B$, $M_{21}(s) = CH(s)B_\Delta$, $M_{22}(s) = CH(s)B$, being $H(s) = (sI - A)^{-1}$.

The following standard stability result pertains to Σ_l [12].

Proposition 1

If system Σ_l is asymptotically stable, then the T-periodic solution $x_\mu(t)$ of system Σ is locally asymptotically stable.

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[1] System (1) reduces to a Lur'e system if $f(x) = Bn(Cx)$, where $n : \mathbf{R} \to \mathbf{R}$ is a scalar function.

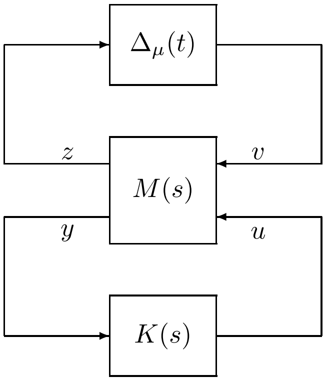

Figure 2: Linearized system Σ_l.

To investigate asymptotic stability of Σ_l, we have to show stability of the feedback interconnection of the periodic matrix gain $\Delta_\mu(t)$ and the $p \times p$ rational transfer function matrix

$$L(s) = M_{11}(s) + M_{12}(s)M_{21}(s)\left[\frac{K(s)}{1-K(s)M_{22}(s)}\right]. \quad (8)$$

The approach we pursue is based on classical absolute stability criteria. To this purpose, we first define the following sector condition for $\Delta_\mu(t)$.

Definition 1
The matrix gain $\Delta_\mu(t)$ belongs to sector $[\Delta_1, \Delta_2]$ if

- *i)* $\xi' \left[\Delta_2 - \Delta_\mu(t)\right]' \left[\Delta_\mu(t) - \Delta_1\right] \xi \geq 0$ $\forall\ t \in [0,T]\ ,\ \forall\ \xi \in \mathbf{R}^n\ ;$
- *ii)* $\big(\Delta_2 - \Delta_1\big)$ *is symmetric and positive definite.*

Since $\Delta_\mu(t)$ is diagonal, it obviously belongs to sector $[\underline{\Delta}_\mu, \overline{\Delta}_\mu]$ where

$$\begin{array}{rcl} \underline{\Delta}_\mu & \leq & \min\limits_{t\in[0,T]} \Delta_\mu(t) \\ \overline{\Delta}_\mu & \geq & \max\limits_{t\in[0,T]} \Delta_\mu(t)\ , \end{array} \quad (9)$$

being the min-max operations componentwise. Note that $\underline{\Delta}_\mu$ and $\overline{\Delta}_\mu$ can be trivially computed if the solutions $x_\mu(t)$ are known (see (9)). However, it is also sufficient to know upper and lower bounds on the components of $x_\mu(t)$.

Without loss of generality, assume that $\underline{\Delta}_\mu = 0$, i.e., $\Delta_\mu(t)$ satisfies the sector condition[2] $[0, \overline{\Delta}_\mu]$, and consider a (minimal) realization $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of $L(s)$. The following result can be derived from the application of the multivariable circle criterion in its LMI version [13], which represents a powerful and numerically efficient tool for investigating stability of feedback systems.

Proposition 2 *The solution $x_\mu(t)$ is locally asymptotically stable if the LMI*

$$\left[\begin{array}{c|c} \mathcal{A}'X + X\mathcal{A} & X\mathcal{B} + \overline{\Delta}_\mu\,\mathcal{C}' \\ \hline \mathcal{B}'X + \overline{\Delta}_\mu\,\mathcal{C} & -2I_p \end{array}\right] < 0 \quad (10)$$

holds for a symmetric and positive definite matrix X.

Clearly, the above proposition requires that the matrix $\mathcal{A}$ is Hurwitz, i.e., system Σ_l is internally stable when $\Delta_\mu(t) = 0$. In the opposite case, (10) has no solutions even if the sector is empty, i.e., $\overline{\Delta}_\mu = 0$. Taking into account that A is Hurwitz by assumption, the next result is simply a restatement of the well-known parameterization of stabilizing controllers [14].

Proposition 3 *The matrix $\mathcal{A}$ is Hurwitz if and only if the controller transfer function has the form*

$$K(s) = \frac{Q(s)}{1 + M_{22}(s)Q(s)} \quad (11)$$

where $Q(s)$ is any proper stable rational function.

Let us define the class

$$\begin{array}{rl} \mathcal{K} = & \Big\{K(s) = \frac{Q(s)}{1+M_{22}(s)Q(s)}\ ,\ Q(s) \text{ stable}\ , \\ & Q\big(j\frac{2\pi n}{T}\big) = 0 \text{ for } n = 0, \ldots, \pm N_0\Big\} \end{array} \quad (12)$$

of controllers in (11) such that constraints (5) hold. Note that, for any controller $K(s) \in \mathcal{K}$, the LMI (10) is trivially satisfied when $\mu = 0$, since $\Delta_\mu(t) = 0$ and thus $\overline{\Delta}_\mu = 0$. Therefore, for any controller it is possible to determine the maximum μ such that Proposition 2 holds. We are interested in selecting the optimal one according to the following problem.

Problem. Let $\overline{\Delta}_\mu$, $\mu \geq 0$, be given and consider the class $\mathcal{K}$ of stabilizing controllers in (12). Then, solve the following optimization problem

$$\mu^* = \sup_{K(s)\in\mathcal{K}} \mu \quad (13)$$

such that condition (10) holds.

Remark 1. The quantity μ^* represents the largest amplitude of the forcing inputs $w_\mu(t)$ (see (3)) such that the corresponding periodic solutions $x_\mu(t)$ are ensured to be stable by a controller belonging to $\mathcal{K}$. Such an optimal controller is denoted by $K^*(s)$.

We conclude this section by showing how an a-priori upper bound on μ^* can be obtained. It is well known that the existence of the solution of the LMI (10) is equivalent to the strict positive realness of the transfer function matrix $[I_p - \overline{\Delta}_\mu \cdot L(s)]$. Therefore, taking into account (12), we have the following result.

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[2] This condition can always be achieved by employing standard pole-shifting techniques such that $\overline{\Delta}_\mu$ and A are replaced by $(\overline{\Delta}_\mu - \underline{\Delta}_\mu)$ and $(A + B_\Delta \underline{\Delta}_\mu C_\Delta)$, respectively.

Proposition 4 *Let $\overline{\Delta}_\mu$, $\mu \geq 0$, be given. Compute*

$$\overline{\mu} \; = \sup \mu \qquad \textit{subject to} \tag{14}$$

$$2I_p - \overline{\Delta}_\mu \left[M_{11}\Big(jn\frac{2\pi}{T}\Big) + M_{11}'\Big(-jn\frac{2\pi}{T}\Big)\right] > 0 \tag{15}$$

for $n = 0, 1, \ldots, N_0$. Then,

$$\mu^* \leq \overline{\mu}. \tag{16}$$

3 Controller synthesis

In this section, we provide a solution to the considered problem of computing the largest stability bound μ^* and the corresponding optimal controller $K^*(s)$. The main difficulty concerns the fact that a state-space realization (A_Q, B_Q, C_Q, D_Q) of the rational function $Q(s)$ affects in general the matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}$ of the realization of $L(s)$. As a consequence, condition (10) would be no longer an LMI in the joint variables X, A_Q, B_Q, C_Q, and D_Q.

To overcome this drawback, we consider the following class of controllers

$$\mathcal{K}_N = \left\{ K(s) = \frac{Q_0(s)Q_R(s)}{1 + M_{22}(s)Q_0(s)Q_R(s)} \right\}, \tag{17}$$

where

$$Q_0(s) = \left(\frac{s}{s+\lambda}\right) \prod_{n=1}^{N_0} \left[\frac{1 + s^2\left(\frac{T}{2n\pi}\right)^2}{(s+\lambda)^2} \right], \tag{18}$$

for any positive λ, is a stable transfer function which ensures fulfillment of constraints (5), and

$$Q_R(s) = \sum_{i=0}^{N} \frac{q_i}{(s+\lambda)^i}\ , \tag{19}$$

is a truncated Ritz series [15] depending on the parameter vector $q = [q_0, q_1, \ldots, q_N] \in \mathbf{R}^{N+1}$. Obviously, it is evident by construction that

$$\mathcal{K}_N \subseteq \mathcal{K}_{N+1} \subseteq \mathcal{K} \quad \forall\, N \in \mathbf{Z}.$$

Now we show that, for any given N and λ, condition (10) is still an LMI in the joint variables X and q. To proceed, we choose a convenient state-space realization (A_R, B_R, C_R, D_R) of $Q_R(s)$

$$A_R = \begin{bmatrix} -\lambda & 0 & 0 & \cdots & 0 \\ 1 & -\lambda & 0 & \cdots & 0 \\ 0 & 1 & -\lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -\lambda \end{bmatrix}, \quad B_R = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$C_R = [\ q_1, q_2, \ldots, q_N\ ]\,, \qquad D_R = q_0\ ,$$

such that the closed-loop transfer function

$$L(s) = M_{11}(s) + Q_R(s)Q_0(s)M_{12}(s)M_{21}(s) =$$

$$= \left[C_\Delta + \underbrace{Q_R(s)Q_0(s)I_p}_{(A_Q,B_Q,C_Q,D_Q)} C_\Delta H(s)BC \right] H(s)B_\Delta \tag{20}$$

admits the state-space realization

$$\mathcal{A} = \left[\begin{array}{c|c|c} A & 0 & 0 \\ \hline B\,C & A & 0 \\ \hline 0 & B_Q C_\Delta & A_Q \end{array}\right], \quad \mathcal{B} = \left[\begin{array}{c} B_\Delta \\ \hline 0 \\ \hline 0 \end{array}\right],$$

$$\mathcal{C} = \left[\ C_\Delta \;|D_Q C_\Delta\; |C_Q\ \right],$$

where

$$A_Q = \left[\begin{array}{c|c|c} \left[\begin{array}{c|c} A_0 & 0 \\ \hline B_R C_0 & A_R \end{array}\right] & \cdots & 0 \\ \hline \vdots & \ddots & \vdots \\ \hline 0 & \cdots & \left[\begin{array}{c|c} A_0 & 0 \\ \hline B_R C_0 & A_R \end{array}\right] \end{array}\right]$$

$$B_Q = \left[\begin{array}{c|c|c} \left[\begin{array}{c} B_0 \\ \hline B_R D_0 \end{array}\right] & \cdots & 0 \\ \hline \vdots & \ddots & \vdots \\ \hline 0 & \cdots & \left[\begin{array}{c} B_0 \\ \hline B_R D_0 \end{array}\right] \end{array}\right]$$

$$C_Q = \left[\begin{array}{c|c|c} \left[\begin{array}{c|c} D_R C_0 & C_R \end{array}\right] & \cdots & 0 \\ \hline \vdots & \ddots & \vdots \\ \hline 0 & \cdots & \left[\begin{array}{c|c} D_R C_0 & C_R \end{array}\right] \end{array}\right]$$

$$D_Q = \left[\begin{array}{c|c|c} D_R D_0 & \cdots & 0 \\ \hline \vdots & \ddots & \vdots \\ \hline 0 & \cdots & D_R D_0 \end{array}\right]$$

are block-diagonal matrices corresponding to a state-space realization of $Q_R(s)Q_0(s)I_p$, and $(A_0,\ B_0\ C_0\ D_0)$ denotes any minimal realization of $Q_0(s)$. Note that vector $q = [D_R | C_R]$ now enters the matrix $\mathcal{C} = \mathcal{C}(q)$ linearly, whereas it does not affect $\mathcal{A}$ and $\mathcal{B}$.

The following result can be easily proven.

Theorem 1 *Let $\overline{\Delta}_{\hat{\mu}}$, $\hat{\mu} \leq \overline{\mu}$, λ be given, and suppose that the LMI*

$$\left[\begin{array}{c|c} \mathcal{A}'X + X\mathcal{A} & X\mathcal{B} + \overline{\Delta}_{\hat{\mu}}\, \mathcal{C}'(q) \\ \hline \mathcal{B}'X + \overline{\Delta}_{\hat{\mu}}\, \mathcal{C}(q) & -2I_p \end{array}\right] < 0 \tag{21}$$

holds for a symmetric and positive definite matrix X and $q = q_{\hat{\mu}} \in \mathbf{R}^{N+1}$.

Then, the controller $K(s) \in \mathcal{K}_N$ with $q = q_{\hat{\mu}}$, stabilizes asymptotically the periodic solution $x_{\hat{\mu}}(t)$.

Moreover, if the following condition holds

$$\overline{\Delta}_\mu \leq \overline{\Delta}_{\hat{\mu}} \qquad \forall\ \mu \leq \hat{\mu}\ , \tag{22}$$

then the periodic solutions $x_\mu(t)$, $\mu \leq \hat{\mu}$, are also asymptotically stable.

Remark 2. Assumption (22) is made for simplifying the developments because in most systems a larger amplitude μ often implies a larger sector $\overline{\Delta}_\mu$ (at least for not very large values of μ). However, if condition (22) does not hold, it is always possible to redefine $\overline{\Delta}_{\hat{\mu}} := \sup\limits_{\mu\in[0,\hat{\mu}]} \overline{\Delta}_\mu$ such that the theorem is still valid (see also [7]).

The above theorem is the basic step for solving the considered problem (13), i.e., finding the largest stability bound μ^* and the corresponding optimal controller $K^*(s)$. Let $\mu^{(N)}$ denote the maximum value of $\hat{\mu}$ such that Theorem 1 holds, i.e.,

$$\mu^{(N)} = \sup_{q\in\mathbf{R}^{N+1}} \mu \tag{23}$$

such that the LMI (21) is satisfied for some symmetric positive definite matrix X and some $q \in \mathbf{R}^{N+1}$. It is evident that $\mu^{(N)}$ can be computed via the solution of a one-parameter family of LMI feasibility problems. The next result pertains to $\mu^{(N)}$.

Theorem 2
Consider the sequence $\mu^{(N)}$, $N = 0, 1, \ldots$. Then,

$$\lim_{N\to\infty} \mu^{(N)} = \mu^* \ .$$

Proof
The sequence $\mu^{(N)}$ is non-decreasing, since, for any fixed λ, $\mathcal{K}_N \subseteq \mathcal{K}_{N+1}$ for all N. Moreover, $\forall\ \epsilon > 0$, there exists $\overline{N} \in \mathbf{Z}$ such that $\mu^{(N)} > \mu^* - \epsilon$, $\forall\ N > \overline{N}$. In particular, it is sufficient to prove that the rational function

$$Q^*(s) = \frac{K^*(s)}{1 - K^*(s)M_{22}(s)}$$

can be arbitrarily approximated by $Q_{\scriptscriptstyle R}(s)Q_{\scriptscriptstyle 0}(s)$ (see (17)-(19)) for large values of N.
The complete proof can be found in [11].

To summarize, the above theorem states that the sought optimal controller $K^*(s)$ can be approximated by a sequence of controllers $K^{(N)}(s)$ belonging to the class $\mathcal{K}_N$ for increasing values of N.

4 Application example

A *single-mode CO$_2$ laser* can exhibit a cascade of period doubling bifurcations leading to chaos when the cavity losses are modulated by a sinusoidal signal of increasing amplitude. Here, we consider the control-relevant model described in [7], which can be written in the general form of Σ_P in Eq. (1) with

$$A = \begin{bmatrix} 0 & k_0 & 0 \\ -\eta & -\Gamma-\eta & \gamma_R \\ 0 & \beta & -\alpha \end{bmatrix}, B = E = \begin{bmatrix} k_0 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}',$$

$$f(x) = \eta \begin{bmatrix} 0 \\ x_1 + (x_2+1)(1-\exp(x_1)) \\ 0 \end{bmatrix},$$

and w is the forcing term belonging to the class of sinusoidal signals (3) of period $T = 10^{-5}$s. The remaining parameters are set as follows

$$\begin{array}{l} k_0 = 3.18 \times 10^7 \mathrm{s}^{-1}\ ,\ \gamma_R = 7.0 \times 10^5 \mathrm{s}^{-1}\ , \\ \Gamma = 7.05 \times 10^6 \mathrm{s}^{-1}\ ,\ \eta = 9.129 \times 10^4 \mathrm{s}^{-1}\ , \\ \alpha = 6.767 \times 10^5 \mathrm{s}^{-1}\ ,\ \beta = 6.626 \times 10^6 \mathrm{s}^{-1}\ . \end{array}$$

For the linearized model (6) we easily compute

$$B_\Delta = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix},\ C_\Delta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\Delta_\mu(t) = \begin{bmatrix} k_1(t) & 0 \\ 0 & k_2(t) \end{bmatrix},$$

with

$$k_1(t) = \left.\frac{\partial f_2(x)}{\partial x_1}\right|_{x_\mu(t)} = \eta\,[1 - (x_{2\mu}(t) + 1)\exp(x_{1\mu}(t))]\,,$$

$$k_2(t) = \left.\frac{\partial f_2(x)}{\partial x_2}\right|_{x_\mu(t)} = \eta\,[1 - \exp(x_{1\mu}(t))]\,.$$

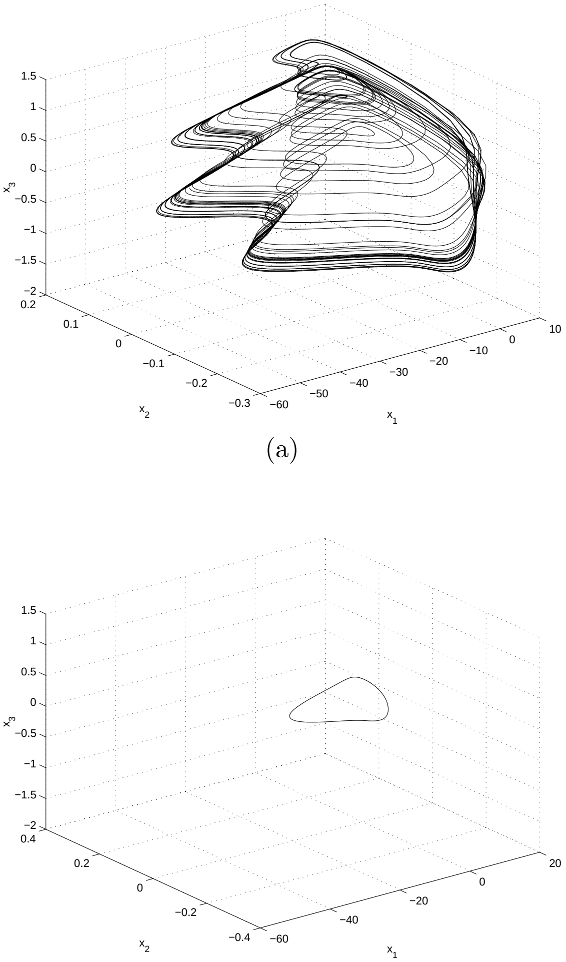
To proceed with the synthesis algorithm, we first need to compute the range of forcing amplitude where Theorem 1 admits at least one solution. The above range starts from $\mu = 0$ and is bounded by $\overline{\mu} = 0.0074$ given by (14). However, since the LMI (10) for the uncontrolled system (i.e., when $L(s) = M_{11}(s)$) is feasible for any $\mu \in [0, 0.0057]$, then we can proceed with the controller synthesis starting at $\mu = 0.0057$. It is easy to show that the sequence of controllers $K^{(N)}(s)$ obtained by solving the one-parameter family of LMI feasibility problems introduced by Theorem 2, rapidly converges to the optimal controller $K^*(s)$, while the largest stability bound μ^* appears to be very close to its upper bound $\overline{\mu} = 0.0074$.
For instance, selecting $N_{\scriptscriptstyle 0} = 1,\ \lambda = 2\pi/T = 6.28 \cdot 10^5$, we achieve $\mu^{(1)} = 0.0072$ for

$$q = [-0.156,\ 0.025 \cdot \lambda]\ , \tag{24}$$

whereas $\mu^{(2)} = 0.0073$ is obtained for

$$q = [-0.156,\ 0.055 \cdot \lambda,\ -0.028 \cdot \lambda^2]\ . \tag{25}$$

To conclude the example, Fig. 3 (a) shows a chaotic state-space trajectory of the uncontrolled system for $\mu = 0.2$. For comparison, the corresponding controlled periodic solution is given in Fig. 3 (b), where the controller $K^{(1)}(s)$ with q given in (24) has been adopted. Finally, we recall that Theorem 1 and Theorem 2 provide only a sufficient condition for the stability of $x_\mu(t)$. However, this application example shows that the performance of the controller designed via the proposed approach is satisfactory, since stabilization of periodic orbits with suppression of chaotic dynamics is achieved even for large values of the forcing amplitude.



(a)

(b)

Figure 3: State-space trajectories for the uncontrolled (a) and controlled (b) CO_2 laser ($\mu = 0.2$).

5 Conclusion

The synthesis of finite dimensional linear controllers for the stabilization of periodic solutions in a general class of single-input single-output sinusoidally forced nonlinear systems is considered. For any controller within this class, exploiting the multivariable circle criterion, a lower bound of the maximum amplitude of the forcing input, for which the corresponding periodic solutions are guaranteed to be stable, is given. An LMI based synthesis technique is proposed for determining the optimal controller, i.e., the one maximizing such a lower bound. In particular, this controller is obtained by means of a truncated Ritz series. Finally, an application example concerning a CO_2 laser is developed to illustrate the main features of the developed synthesis technique. Based on the analysis of several other application examples, it is believed that controllers obtained via the proposed approach, though optimizing only a lower bound, can be successfully applied in chaos control applications.

References

[1] G. Chen (Ed.), *Controlling Chaos and Bifurcations in Engineering Systems*, CRC Press, Boca Raton, FL, 1999.

[2] K. Pyragas, “Continuous control of chaos by self-controlling feedback,” *Phys. Lett. A*, 170, pp. 421-428, 1992.

[3] J.E.S. Socolar, D.W. Sukow and D.J. Gauthier, “Stabilizing unstable periodic orbits in fast dynamical systems,” *Phys. Rev. E*, vol. 50, pp. 3245-3248, 1994.

[4] M. Basso, R. Genesio and A. Tesi, “Stabilizing periodic orbits of forced systems via generalized Pyragas controllers, ” *IEEE Trans. Circ. and Syst. - I*, 44, pp. 1023-1027, 1997.

[5] G. Chen and X. Yu, “On time delayed feedback control of chaotic systems,” *IEEE Trans. Circ. and Syst. - I*, 46, pp. 767-772, 1999.

[6] M. Basso, R. Genesio, L. Giovanardi, A. Tesi and G. Torrini, “On optimal stabilization of periodic orbits via time-delayed feedback control,” *Internat. J. of Bifurcation and Chaos*, 8, pp. 1699–1706, 1998.

[7] M. Basso, R. Genesio, L. Giovanardi and A. Tesi, “Frequency domain methods for chaos control,” in *Controlling Chaos and Bifurcations in Engineering Systems*, G. Chen (Ed.), CRC Press, Boca Raton, FL, pp. 179-203, 1999.

[8] M. Basso, L. Giovanardi and A. Tesi, “LMI-based synthesis for controlling periodic solutions in a class of nonlinear systems,” *Proc. of the 2000 IEEE International Symposium on Circuits and Systems*, Geneva, CH, 2000.

[9] M. Stanghini, M. Basso, R. Genesio, A. Tesi, M. Ciofini and R. Meucci, “A new three-equation model for the CO_2 laser,” *IEEE J. Quantum Electron.*, 32, pp. 1126-1131, 1996.

[10] A. Holden and M. Muhamad, “A graphical zoo of strange and peculiar attractors,” in *Chaos*, A. Holden (Ed.), University Press, Manchester, 1986.

[11] M. Basso, R. Genesio and A. Tesi, “Controller synthesis for stabilizing periodic orbits in forced nonlinear systems,” *Research Report* 01/2000, Dipartimento di Sistemi e Informatica, Università di Firenze, 2000.

[12] H. Khalil, *Nonlinear Systems*, Macmillan, New York, 1992.

[13] S. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, Siam, Philadelphia, 1994.

[14] K. Zhou, *Essentials of Robust Control*, Prentice Hall, New Jersey, 1998.

[15] S. Boyd and C. H. Barrat, *Linear Controller Design*, Prentice Hall, Englewood Cliffs, NJ, 1991.